Distance Pebbling on Directed Cycle Graphs

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1 Introduction

Suppose that $G$ is a graph, and imagine that we have placed pebbles at some of the vertices of $G$. Traditionally, a pebbling move on $G$ is defined as follows. If a vertex $v$ of $G$ contains at least two pebbles, then we may remove two pebbles from $v$, and add one pebble to a vertex adjacent to $v$. (If we have a directed graph, then we may only add a pebble to a vertex $w$ if there is an edge which goes from $v$ to $w$.) The pebbling number of $G$ is then defined as the smallest number $s$ with the property that if the initial configuration contains at least $s$ pebbles, then no matter how they are configured it is always possible to find a sequence (possibly empty) of pebbling moves which places a pebble on any specified vertex. There are many variations of the pebbling number which can be studied, and [3] is an excellent reference.

For our purposes, we make a cosmetic change in the definition of a pebbling move. Instead of removing two pebbles from $v$ and then adding an entirely new pebble to the graph, we will remove one pebble from $v$ and then move another pebble from $v$ to an adjacent vertex. While the two definitions are obviously equivalent in terms of which configurations of pebbles can arise on the graph, our definition allows us to follow a particular pebble through a sequence of pebbling moves. For a positive integer $d$, we define the distance $d$ pebbling number of $G$ to be the smallest number $s$ such that if the initial configuration contains at least $s$ pebbles, then there must exist a pebble which can be moved to a vertex which is at a distance of at least $d$ from its

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starting point. This can also be thought of as finding the target pebbling number $\pi^-(G, D)$ (see the introduction to [3]), where $D$ is a particular set of configurations of pebbles on $G$.

In this article, we will determine the distance $d$ pebbling numbers for a directed cycle graph with $n$ vertices. Let us denote this number by $P_n(d)$. Then we have the following theorem.

**Theorem 1.** Suppose that $d < n$, and write $n = dq + r$, where $q, r$ are integers with $0 \leq r \leq d - 1$. Then we have $P_n(d) = (2^d - 1)q + 2^r$.

It is not too hard to see that $P_n(d)$ must be at least as large as the bound in the theorem. Label the vertices of the graph consecutively as $v_0, \ldots, v_{n-1}$. Put $2^d - 1$ pebbles on each of $v_0, v_d, v_{2d}, \ldots, v_{(q-1)d}$, and put $2^r - 1$ pebbles on $v_{qd}$. Then it is not hard to see that none of the pebbles on $v_0$ can be moved to $v_d$, none of the pebbles on $v_d$ can be moved to $v_{2d}$, and so on. Moreover, none of the pebbles on $v_{qd}$ can be moved to $v_0$. Our initial configuration involves $(2^d - 1)q + 2^r - 1$ pebbles, but no pebble can be moved $d$ vertices from its starting point. Hence we must have $P_n(d) \geq (2^d - 1)q + 2^r$. To complete the proof of the theorem, it suffices to show that in any initial configuration of exactly $(2^d - 1)q + 2^r$ pebbles, there does exist a pebble which can be moved $d$ vertices. We will show that this is indeed the case in Sections 3 and 4.

We note that the same techniques that are used in this article can be used to prove an interesting result about solutions of homogeneous additive equations over the field $\mathbb{Q}_2$ of 2-adic numbers. If a positive integer $n$ is given, then the techniques in this article can be used to find the smallest number $s$ of variables which guarantees that the equation

$$a_1x_1^n + a_2x_2^n + \cdots + a_sx_s^n = 0$$

always has a nontrivial 2-adic solution regardless of the (2-adic integer) values of the coefficients. In fact, the bound itself is entirely analogous to the one in our theorem here. The interested reader may refer to the article [4], which can be thought of as a companion paper to this one.
2 Preliminary Lemmata

In this section, we give the preliminary lemmata needed to prove our formula for the value of $P_n(d)$. One of our key tools is the following combinatorial lemma due to Davenport & Lewis [1].

**Lemma 2.** Let $a_0, a_1, \ldots, a_{n-1}$ be real numbers, and put $a_{j+n} = a_j$ for all $j$. Let

$$a_0 + a_1 + \cdots + a_{n-1} = s.$$ 

Then there exists a number $r$ such that

$$a_r + \cdots + a_{r+t-1} \geq ts/n \quad \text{for} \quad t = 1, \ldots, n.$$ 

For our purposes, we can interpret this result as follows. Given an initial configuration of $s$ pebbles on the graph, we wish to select a vertex to call $v_0$ and then label the vertices consecutively as $v_0, v_1, \ldots, v_{n-1}$. Let $m_i$ represent the number of pebbles at the vertex $v_i$. Then Lemma 2 says that we may select $v_0$ so that we have $m_0 + \cdots + m_{t-1} \geq ts/n$ for $t = 1, \ldots, n$.

We now prove a lemma about the greatest integer function, which generalizes [2, Lemma 4.14]. In the proof of Theorem 1, we only need the special case where $b_i = 2$ for each $i$. But it is just as easy to prove the lemma in full generality, and so we do so here.

**Lemma 3.** Suppose that $a_1, a_2, \ldots$ are nonnegative integers and that $b_1, b_2, \ldots$ are positive integers. Define the numbers $g_i = g_i(a, b)$ recursively by

$$g_1 = \left\lceil \frac{a_1}{b_1} \right\rceil$$

and

$$g_{i+1} = \left\lceil \frac{g_i + a_{i+1}}{b_{i+1}} \right\rceil,$$

where $\lceil \cdot \rceil$ represents the greatest integer function (i.e., $[x]$ returns the greatest integer less than or equal to $x$). Then for all $i$, we have

$$g_i = \left\lceil \frac{a_1}{b_1 \cdots b_i} + \frac{a_2}{b_2 \cdots b_i} + \frac{a_3}{b_3 \cdots b_i} + \cdots + \frac{a_i}{b_i} \right\rceil.$$
For example, when \( i = 2 \) this lemma says that
\[
\left[ \frac{a_1}{b_1} + a_2 \right] \cdot \frac{b_2}{b_2} = \left[ \frac{a_1}{b_1} + \frac{a_2}{b_2} \right],
\]
and when \( i = 3 \) it says that we have
\[
\left[ \left[ \frac{a_1}{b_1} + a_2 \right] + a_3 \right] \cdot \frac{b_3}{b_3} = \left[ \frac{a_1}{b_1 b_3} + \frac{a_2}{b_2 b_3} + a_3 \right].
\]

**Proof.** The lemma is obviously true for \( i = 1 \). Suppose by way of induction that it is true for a specific number \( i \). Then we have
\[
g_{i+1} \leq \frac{g_i + a_{i+1}}{b_{i+1}} < g_i + 1.
\]
Our inductive hypothesis then leads to
\[
b_{i+1}g_{i+1} - a_{i+1} \leq \left[ \frac{a_1}{b_1 \cdots b_i} + \frac{a_2}{b_2 \cdots b_i} + \frac{a_3}{b_3 \cdots b_i} + \cdots + \frac{a_i}{b_i} \right] < b_{i+1}(g_{i+1}+1) - a_{i+1}.
\]
Since these upper and lower bounds are both integers, this implies that we have
\[
b_{i+1}g_{i+1} - a_{i+1} \leq \frac{a_1}{b_1 \cdots b_i} + \frac{a_2}{b_2 \cdots b_i} + \frac{a_3}{b_3 \cdots b_i} + \cdots + \frac{a_i}{b_i} < b_{i+1}(g_{i+1}+1) - a_{i+1},
\]
which gives
\[
g_{i+1} \leq \frac{a_1}{b_1 \cdots b_i + b_{i+1}} + \frac{a_2}{b_2 \cdots b_i b_{i+1}} + \frac{a_3}{b_3 \cdots b_i b_{i+1}} + \cdots + \frac{a_i}{b_i b_{i+1}} + \frac{a_{i+1}}{b_{i+1}} < g_i + 1.
\]
Since \( g_{i+1} \) is an integer, the last inequality immediately implies that
\[
g_{i+1} = \left[ \frac{a_1}{b_1 \cdots b_i b_{i+1}} + \frac{a_2}{b_2 \cdots b_i b_{i+1}} + \cdots + \frac{a_i}{b_i b_{i+1}} + \frac{a_{i+1}}{b_{i+1}} \right].
\]
This completes the proof of the lemma.
We now use Lemma 3 to prove a lemma and corollary which give conditions guaranteeing that we can move a pebble from a vertex \( v_j \) on the directed cycle graph to another vertex \( v_{j+k} \).

**Lemma 4.** Consider a directed cycle graph with \( n \) consecutive vertices \( v_0, v_1, \ldots \), where the subscripts are meant to be interpreted modulo \( n \). Let \( m_i \) be the number of pebbles at vertex \( i \), and fix a vertex \( v_j \) and a positive integer \( k < n \). Define numbers \( g_1, g_2, \ldots \) inductively by

\[
g_1 = \left\lfloor \frac{m_j}{2} \right\rfloor \quad \text{and} \quad g_i = \left\lfloor \frac{g_{i-1} + m_{j+i-1}}{2} \right\rfloor \quad \text{for } i \geq 2.
\]

Then we can move a pebble from \( v_j \) to \( v_{j+k} \) if we have \( g_i \geq 1 \) for \( 1 \leq i \leq k \).

**Proof.** We begin with a few simple (and perhaps overly pedantic) observations. We can move a pebble from \( v_j \) to \( v_{j+k} \) if and only if we can move it first to \( v_{j+1} \), then to \( v_{j+2} \), and so on until it is eventually at \( v_{j+k} \). Next, suppose that we can move a pebble from a vertex \( v_j \) to \( v_{j+1} \). If we can subsequently move any pebble from \( v_{j+1} \) to \( v_{j+2} \), then we can arrange for this pebble to be the one which came from \( v_j \). Hence, we can move a pebble from \( v_j \) to \( v_{j+2} \) if and only if we can move a pebble from \( v_j \) to \( v_{j+1} \), and then move a pebble from \( v_{j+1} \) to \( v_{j+2} \). (In other words, we only need to determine whether moving a pebble is possible, and don’t have to keep track of which pebbles are being moved.) By induction, we can move a pebble from \( v_j \) to \( v_{j+k} \) if and only if we can first move a pebble from \( v_j \) to \( v_{j+1} \), then move a pebble from \( v_{j+1} \) to \( v_{j+2} \), and so on, eventually being able to move a pebble from \( v_{j+k-1} \) to \( v_{j+k} \).

Now, when we begin making pebbling moves, we have \( m_j \) pebbles stationed at \( v_j \). Then the number of pebbles we can move to \( v_{j+1} \) is \( \left\lfloor \frac{m_j}{2} \right\rfloor = g_1 \), since this is the number of disjoint pairs of pebbles at \( v_j \). So we can move a pebble from \( v_j \) to \( v_{j+1} \) if \( g_1 \geq 1 \). After moving as many pebbles as possible from \( v_j \) to \( v_{j+1} \), the number of pebbles at \( v_{j+1} \) will be \( g_1 + m_{j+1} \). The maximum number of pebbles we can then move from \( v_{j+1} \) to \( v_{j+2} \) will be \( \left\lfloor \frac{g_1 + m_{j+1}}{2} \right\rfloor = g_2 \). Hence we can move a pebble from \( v_j \) to \( v_{j+2} \) if \( g_1 \geq 1 \) and \( g_2 \geq 1 \). Continuing in this manner, we see that we can move a pebble from \( v_j \) to \( v_{j+k} \) if \( g_i \geq 1 \) for \( 1 \leq i \leq k \). This completes the proof of the lemma.

\( \square \)
**Corollary 5.** With all variables as in Lemma 4, we can move a pebble from $v_j$ to $v_{j+k}$ if we have

\[
\begin{align*}
m_j & \geq 2 \\
m_j + 2m_{j+1} & \geq 4 \\
m_j + 2m_{j+1} + 4m_{j+2} & \geq 8 \\
& \vdots \\
m_j + 2m_{j+1} + 4m_{j+2} + \cdots + 2^{k-1}m_{j+k-1} & \geq 2^k.
\end{align*}
\]

**Proof.** We will show that the condition $g_i \geq 1$ is equivalent to the $i$-th inequality in the system. We have

\[
\begin{align*}
g_i &= \left[ \frac{g_{i-1} + m_j + i - 1}{2} \right] \\
&= \left[ \left[ \frac{g_{i-2} + m_j + i - 2}{2} \right] + m_j + i - 1 \right] \\
& \quad \vdots \\
&= \left[ \frac{m_j}{2^i} + \frac{m_{j+1}}{2^{i-1}} + \frac{m_{j+2}}{2^{i-2}} + \cdots + \frac{m_{j+i-1}}{2} \right],
\end{align*}
\]

where the last equality is true by Lemma 3.

Hence we have $g_i \geq 1$ if and only if we have

\[
\frac{m_j}{2^i} + \frac{m_{j+1}}{2^{i-1}} + \frac{m_{j+2}}{2^{i-2}} + \cdots + \frac{m_{j+i-1}}{2} \geq 1.
\]

This is equivalent to having

\[
m_j + 2m_{j+1} + 4m_{j+2} + \cdots + 2^{i-1}m_{j+i-1} \geq 2^i,
\]

which is indeed the $i$-th inequality. This completes the proof of the corollary. \qed
We finish this section with two straightforward lemmata. These are also\textsuperscript{1} Lemmata 3.1 and 3.2 of [4]. We repeat the proofs here for completeness.

**Lemma 6.** Suppose that $n$, $N$, and $a$ are positive integers such that $aN/n > 2^a - 1$. Then we have $kN/n > 2^k - 1$ for all $k$ satisfying $0 < k < a$.

*Proof.* It is sufficient to show that $g(x) = \frac{2^x - 1}{x}$ is an increasing function for $x > 0$. We have $g'(x) = \frac{x2^x \ln(2) - 2^x - 1}{x^2}$. The Mean Value Theorem implies that there exists $c$ with $0 < c < x$ such that $\frac{2^x - 1}{x} = 2^c \ln(2) < 2^x \ln(2)$. Thus $x2^x \ln(2) - (2^x - 1) > 0$ and it follows that $g'(x) > 0$.

\[\square\]

**Lemma 7.** Suppose that $d, q$ are positive integers with $d \geq 2$, that $r$ is an integer with $0 \leq r < d$, and set $n = dq + r$. Moreover, suppose that $m_0, \ldots, m_{d-2}$ are integers such that

$$m_0 + \cdots + m_{k-1} \geq \frac{k((2^d - 1)q + 2^r)}{n}$$

for $1 \leq k \leq d - 1$. Then we have

$$m_0 + \cdots + m_{k-1} > 2^k - 1$$

for $1 \leq k \leq d - 1$.

*Proof.* By Lemma 6 with $N = (2^d - 1)q + 2^r$, it suffices to prove the conclusion for $k = d - 1$. Therefore we only need to show that

$$\frac{(d - 1)((2^d - 1)q + 2^r)}{n} > 2^{d-1} - 1.$$  

Some algebra shows that this is true if and only if we have

$$2^{d-1}(dq - 2q - r) + (2^r(d - 1) + q + r) > 0. \quad (1)$$

To see that (1) holds, we simply note that

$$2^{d-1}(dq - 2q - r) + 2^r(d - 1) + q + r$$

$$\geq 2^r(dq - 2q - (d - 1)) + 2^r(d - 1) + q + r$$

$$= 2^r(d - 2)q + q + r$$

$$> 0.$$  

This completes the proof of the lemma.

\[\square\]

\textsuperscript{1}The statements and proofs of these lemmata were suggested by the anonymous referee of [4]. This both strengthened the lemmata from previous drafts and simplified their proofs.
3 Proof of the Theorem - The Easy Cases

In this section, we prove Theorem 1 when \( d \leq 2 \), and also when \( n \) is divisible by \( d \). The proof of the remaining cases is somewhat more complex and will be given in the next section.

Lemma 8. We have \( P_n(1) = n + 1 \) for any \( n \), as in Theorem 1.

Proof. This case of Theorem 1 is trivial. As shown in the remarks after the statement of the theorem, we only need to show that \( P_n(1) \leq n + 1 \). If there are \( n + 1 \) pebbles on a graph with \( n \) vertices, then some vertex has two pebbles, and a pebbling move can be made, allowing a pebble to move a distance of 1 from its starting position. Hence \( P_n(1) \leq n + 1 \), and the proof is complete.

\( \Box \)

Lemma 9. If \( d \mid n \), with \( n = dq \), then we have \( P_n(d) = (2^d - 1)q + 1 \), as in Theorem 1.

Proof. As above, it suffices to show that \( (2^d - 1)q + 1 \) is an upper bound for \( P_n(d) \). Suppose that there are exactly \( (2^d - 1)q + 1 \) pebbles on the graph. As discussed in the remarks after Lemma 2, we can label the vertices consecutively as \( v_0, v_1, \ldots, v_{n-1} \) in such a way that if \( m_i \) represents the number of pebbles at the vertex \( v_i \), then we have

\[
m_0 + m_1 + \cdots + m_{k-1} \geq \frac{k((2^d - 1)q + 1)}{n} \tag{2}
\]

for \( k = 1, 2, \ldots, n \). We wish to show that it is always possible to move a pebble from \( v_0 \) to \( v_d \). Thus, with \( j = 0 \) in Corollary 5, we must show that

\[
m_0 + 2m_1 + \cdots + 2^{k-1}m_{k-1} \geq 2^k
\]

for \( 1 \leq k \leq d \). We’ll do even a little bit better than this, showing that

\[
m_0 + m_1 + \cdots + m_{k-1} \geq 2^k
\]

for these \( k \). By (2), and noting that \( m_0 + \cdots + m_{k-1} \) must be an integer, it suffices to show that

\[
\frac{k((2^d - 1)q + 1)}{n} > 2^k - 1
\]
for each \( k \) in question.

By Lemma 6, we only need to prove the inequality when \( k = d \). However, this is trivial since when \( k = d \) we have

\[
\frac{k((2d - 1)q + 1)}{n} = 2^d - 1 + \frac{1}{q} > 2^d - 1.
\]

This completes the proof of the lemma.

\[\square\]

**Lemma 10.** Suppose that \( d = 2 \) and write \( n = 2q + r \) with \( 0 \leq r \leq 1 \). Then we have \( P_n(2) = 3q + 2r \), as in Theorem 1.

**Proof.** If \( r = 0 \), then we are done by Lemma 9. Assume then that \( r = 1 \) and that there are exactly \( 3q + 2 \) pebbles on the graph. Using Lemma 2, label the vertices in the same way as in the proof of Lemma 9. We will show that it is always possible to move a pebble from \( v_0 \) to \( v_2 \). It is enough to show that \( m_0 \geq 2 \) and \( m_0 + m_1 \geq 4 \). By Lemma 2, we have

\[
m_0 \geq \frac{3q + 2}{2q + 1} > 1.
\]

Since \( m_0 \) is an integer, this implies that \( m_0 \geq 2 \), as desired. Similarly, Lemma 2 yields

\[
m_0 + m_1 \geq \frac{2(3q + 2)}{2q + 1} = \frac{6q + 4}{2q + 1} > 3,
\]

and again we obtain the desired conclusion since \( m_0 + m_1 \) is an integer.

This shows that \( P_n(2) \leq 3q + 2r \). As above, since we know that \( P_n(2) \geq 3q + 2r \), we must have equality, completing the proof of the lemma.

\[\square\]

## 4 Completion of the Proof of the Theorem

In this section, we prove Theorem 1 in the case where \( d \geq 3 \) and \( d \nmid n \). Suppose that \( n = dq + r \) with \( 1 \leq r \leq d - 1 \), and let \( N = (2^d - 1)q + 2r \). As before, we only need to prove that \( P_n(d) \leq N \), and so we suppose that there are exactly \( N \) pebbles on the graph. Label the vertices consecutively
as \(v_0, \ldots, v_{n-1}\) so that if \(m_i\) represents the number of pebbles on vertex \(v_i\), then we have
\[
m_0 + \cdots + m_{k-1} \geq \frac{kN}{n}
\]
for all \(1 \leq k \leq n\).

By Lemma 7, noting that \(m_0 + \cdots + m_{k-1}\) is an integer, we have \(m_0 + \cdots + m_{k-1} \geq 2^k\) for \(1 \leq k \leq d - 1\). If in addition we have \(m_0 + \cdots + m_{d-1} \geq 2^d\), then by Corollary 5 we can move a pebble from \(v_0\) to \(v_d\), and we are done. Hence we may assume that \(m_0 + \cdots + m_{d-1} \leq 2^d - 1\). This implies that
\[
m_d + \cdots + m_{n-1} \geq N - (2^d - 1) = (2^d - 1)(q - 1) + 2^r.
\]
By Lemma 2 we can now relabel the vertices \(v_d, \ldots, v_{n-1}\) as \(w_0, \ldots, w_{n-d-1}\) such that both of the following properties hold.

1. The ordered tuple \((w_0, \ldots, w_{n-d-1})\) is a cyclic permutation of the ordered tuple \((v_d, \ldots, v_{n-1})\).
2. If we write \(m_i^*\) for the number of pebbles at the vertex \(w_i\), then we have
\[
m_0^* + \cdots + m_{k-1}^* \geq \frac{k(N - 2^d + 1)}{n - d} = \frac{k ((2^d - 1)(q - 1) + 2^r)}{d(q - 1) + r}
\]
for \(1 \leq k \leq n - d\).

Lemma 7 again ensures (assuming that \(q-1 \geq 1\)) that \(m_0^* + \cdots + m_{k-1}^* \geq 2^k\) for \(1 \leq k \leq d - 1\). Suppose that \(w_0 = v_i\) for some \(i\) with \(n - d + 1 \leq i \leq n - 1\). Then Corollary 5 shows that we can move a pebble from \(w_0\) to \(v_0\), and then as above we can move this pebble from \(v_0\) to \(v_d\). Since the distance from \(w_0\) to \(v_{d-1}\) is at least \(d\) vertices, we are finished. Hence we can assume that \(i \leq n - d\). That is, we can assume that the path from \(w_0\) to \(w_{d-1}\) does not include any of the vertices \(v_0, \ldots, v_{d-1}\), and this implies that the vertices \(w_0, \ldots, w_{d-1}\) are consecutive as we move around the cycle graph. Corollary 5 now shows that we can move a pebble from \(w_0\) to \(w_{d-1}\). Again, if we have \(m_0^* + \cdots + m_{d-1}^* \geq 2^d\), then we could move a pebble from \(w_0\) even further, which would be a distance of at least \(d\). Hence we may assume that \(m_0^* + \cdots + m_{d-1}^* \leq 2^d - 1\).
We can now repeat the above argument with the remaining vertices to show that either some pebble can be moved a distance of $d$ vertices, or else there must exist a third set of $d$ consecutive vertices, disjoint to the two sets we have already found, which contains a total of at most $2d - 1$ pebbles and such that we can move a pebble from the first vertex to the last. Continuing in this manner, we can in fact show that either some pebble can be moved a distance of at least $d$ vertices or else there must exist $q$ such mutually disjoint sets. This leaves us with $r$ vertices for which we have not yet accounted, and these vertices will contain at least $2^r$ pebbles among them.

Again, we can relabel these final remaining vertices consecutively as $v_{x_0}, \ldots, v_{x_{r-1}}$ in such a way that if $m_i^{**}$ denotes the number of pebbles on $v_{x_i}$, then we have $m_0^{**} + \cdots + m_{k-1}^{**} \geq k \cdot 2^r / r \geq 2^k$ for $1 \leq k \leq r$. Now fix $k$ to be the smallest number so that $v_{x_{k+1}} \neq v_{x_k+1}$, i.e., so that the vertex following $v_{x_k}$ on the graph is not one of the $v_{x_i}$. (If the vertices $v_{x_0}, \ldots, v_{x_{r-1}}$ are all consecutive, then we take $k = r - 1$.) Then the vertex $v_{x_k+1}$ is the first vertex in one of the sets described in the previous paragraph. Corollary 5 shows that we can move a pebble from $v_{x_0}$ to $v_{x_k+1}$, and then the way we constructed our sets in the previous paragraph shows that we can further move this pebble from $v_{x_k+1}$ to $v_{x_k+d}$. Hence this pebble can be moved a total of at least $d$ vertices. This completes the proof of the theorem.

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References


