Chapter 1

Review

1.1 Lines

Lines have more than one form of equation; here we'll recall a few of them.

Definition. In x and y coordinates, here are two forms of the equation of a line:

$$\begin{aligned} y &= mx + b && \text{slope-intercept} \quad (m = \text{slope, } b = y\text{-intercept}) \\ y &= m(x - x_0) + y_0 && \text{point-slope} \quad (m = \text{slope, } (x_0, y_0) = \text{given point}) \end{aligned}$$

There are a two important variations:

- A vertical line has an equation of the form $x = C$, where C is a constant.
- A horizontal line has an equation of the form $y = C$, where C is a constant.

Example 1. Find the equation of the line that goes through the point $(-2, 9)$ and has slope $4/5$.

Example 2. Find the equation of the line that goes through the following points

$$(-1, 7) \text{ and } (5, 8)$$

1.2 Fractions

We used fractions in the previous subsection when we calculated slopes of lines, but we didn't really practice manipulating fractions much. All we really needed to know there was that a fraction $\frac{a}{b}$ represented a number, and/or a ratio. Now we will recall how to manipulate fractions a little bit. Here are the rules for adding, subtracting, multiplying and dividing fractions[1]:

$$\begin{aligned}\frac{a}{b}+\frac{c}{d} &= \frac{ad+bc}{bd}\\ \frac{a}{b}-\frac{c}{d} &= \frac{ad-bc}{bd}\\ \frac{a}{b}\cdot\frac{c}{d} &= \frac{ac}{bd}\\ \frac{a}{b}\div\frac{c}{d} &= \frac{a}{b}\cdot\frac{d}{c}\end{aligned}$$

Note that in general, the following are not equal, do not make the mistake that they are:

$$\begin{aligned}\frac{1}{x+y} &\neq \frac{1}{x}+\frac{1}{y}\\ \frac{a}{x+a} &\neq \frac{1}{x+1}\end{aligned}$$

Example 1. Simplify the following (in each case get a single fraction, with no compound fractions)

(a) $\frac{2}{3}+\frac{3}{4}$

(b) $\frac{2}{3}\cdot\frac{3}{4}$

(c) $\frac{2}{3}+\frac{1}{x}$

(d) $x\left(\frac{1+\frac{1}{x}}{x+\frac{1}{x}}\right)$

[1] If you like, you can take these rules as the definitions, or even as some made up formulas that tell us how to play a game. However, I don't really recommend this: I think most people want to know *why* these are the right rules, *where* they came from, and *what* they mean. However, we don't have time or space to deal with that here.

1.3 Rules of exponents

$$a^{-b} \text{ means } \underline{\hspace{2cm}}$$
$$a^{1/b} \text{ means } \underline{\hspace{1.6cm}}$$
$$(a^n)^m = \underline{\hspace{2cm}}$$
$$a^n a^m = \underline{\hspace{2.4cm}}$$
$$\frac{a^n}{a^m} = \underline{\hspace{2.4cm}}$$
$$(ab)^n = \underline{\hspace{2.2cm}}$$

Note that in general the following are not equal, do not make the mistake that they are

$$(x+y)^2 \neq x^2 + y^2 \quad \text{(unless } x = 0 \text{ or } y = 0\text{)}$$
$$\sqrt{x+y} \neq \sqrt{x} + \sqrt{y} \quad \text{(unless } x = 0 \text{ or } y = 0\text{)}$$

Example 1. Using the above properties, simplify the following.

(a) $(-2)^5$
(b) $\dfrac{x^{17}}{x^{22}}$
(c) $4^{-3/2}$
(d) $\sqrt{36x^4}$

Example 2. Simplify the following

$$\left(\frac{(-2x^{-4}y^{6})^{-8}}{(3x^{3}y^{-3})^{-2}}\right)^{-2}$$

so that your final answer has no fractions, and each base, 2, 3, x and y, appears only once.

1.4 Factoring quadratics

To *factor* is to write something as the product of two things. For example, we can factor 12 as $3 \cdot 4$. We can factor $x^2 + x$ as $x(x + 1)$. Our main use of factoring is to solve equations where one side is equal to 0. The reason for this is: if you have two things multiplied together, and the result is 0, then one of the things you multiplied must be zero. Thus, if we can factor an equation with 0 on one side, we can break it into pieces (the things that are being multiplied) and one of these pieces equals 0. Note: this only works when one side of the equation is 0, not when the one side equals 1 or 4 or any other number. Some types of equations have special tricks for factoring them, namely quadratics; we'll review those below.

Example 1. Factor $3x^3 + 2x^2$ and use this to solve $3x^3 + 2x^2 = 0$.

A **quadratic** equation is one of the form $ax^2 + bx + c = 0$. Sometimes these are fairly easy to factor. If $a = 1$ then you can try factoring it this way

$$x^2 + bx + c = (x \pm ?)(x \pm ?)$$

where you try to fill in the question marks with two things that multiply together to give you c and add together to give you b.

If you can't factor a quadratic this way, then you can always use the quadratic formula:

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 2. Solve $3x^2 - 14x - 5 = 0$.

1.5 Function Notation

Definition. Here's the most common **function notation**: "$f(x)$". In this, f is the name of a function, x is the input, and whatever $f(x)$ equals is the output. Sometimes we replace "f" with some familiar function such as natural log: $\ln(x)$, or sine: $\sin(x)$. Sometimes we set $f(x)$ equal to a formula, and the formula is how you calculate the output, such as $f(x) = x^2$. Finally, sometimes f is defined by a graph.

Example 1. Let f be the function given by $f(x) = x^2$.

(a) What is $f(2)$?

(b) In the equation "$f(x) = x^2$", what is the input? What is the output?

(c) In the following description of this function, fill the blank in (with a verb): "f is the function that takes an input and ______________ it."

(d) In the expression "$f(3x)$", what is the input? What do you get when you take this input and do to it the verb that you filled in the blank with in the previous part?

(e) In the expression "$f(3 + x)$", what is the input? What do you get when you take this input and do to it the verb that you filled in the blank with in the previous part?

Example 2. Let $f(x) = x^2 + x + 1$.

(a) Find $f(1)$
(b) Solve $f(x) = 1$
(c) Find $f(h)$
(d) Find $f(x+1)$

1.6 Piecewise functions

Example 1. Let $f(x)$ be defined by the following formulas, each applying to just one piece of $f(x)$.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ -x^2 & \text{if } 0 < x \leq 3 \\ e^{x-2} & \text{if } 3 < x. \end{cases}$$

Find $f(-1)$, $f(2)$ and $f(4)$, and make a graph of $f(x)$.

1.7 Function Combinations

We can combine old functions to make new functions. We can add, subtract, multiply, divide, and plug one inside of another. The notation $f(g(x))$ means that $g(x)$ is the input for $f(x)$. We illustrate these combinations by example.

Example 1. Let $f(x) = 3x^2 + 1$ and $g(x) = \sin(2x)$.
(a) Find a formula for $f(x) + g(x)$.
(b) Find $f(1) - g(1)$
(c) Find a formula for $f(x)/g(x)$
(d) Find $f(g(2))$.
(e) Find a formula for $f(g(x))$.

1.8 Atomic Functions

Complicated functions are usually made by combining simple ones. The simplest ones are "atomic" in the sense that we can't break them down any more. Here is a list of all the atomic functions that you (should) have seen before (as well as one that's *not* atomic):

Powers of x:

- Things like x, x^2, x^3, ...
- Things like $\sqrt{x}$, $\sqrt[3]{x}$, ...
- Things like $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{\sqrt[5]{x}}$.

Polynomials:

- Sums of powers of x, using only whole number powers, and together with *coefficients*, real numbers that are multiplied the powers of x
- Special cases: things like 5, like x, like x^2
- Typcial cases: things like $3x+5$, $-5.7x^3+7x^2-100$, or $x-5x^2+3.1x^{10}$.
- The general expression:

$$a_n x^n + \ldots a_1 x + a_0$$

  where each a_i is a coefficient.

Exponential and logarithm:

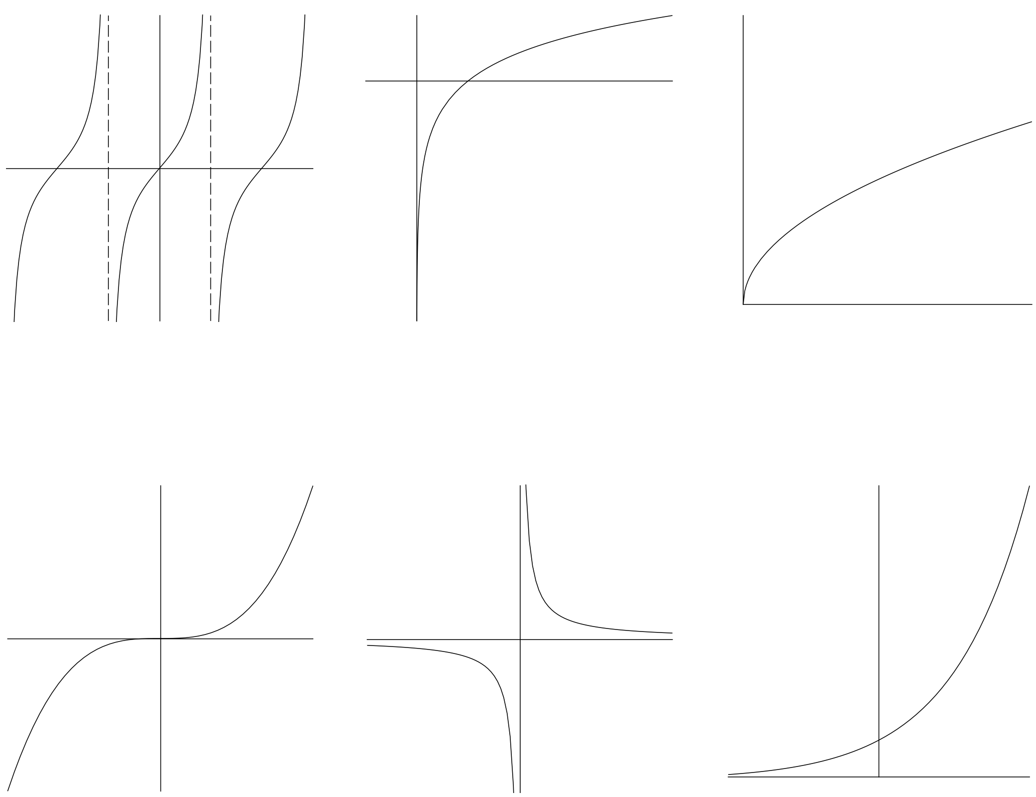
- Exponential Functions: things like 2^x, 10^x, $(1/2)^x$, e^x. In general a^x where $a > 0$.
- Logarithmic Functions: things like $\log_2(x)$, $\log_{10}(x)$, $\ln(x)$.

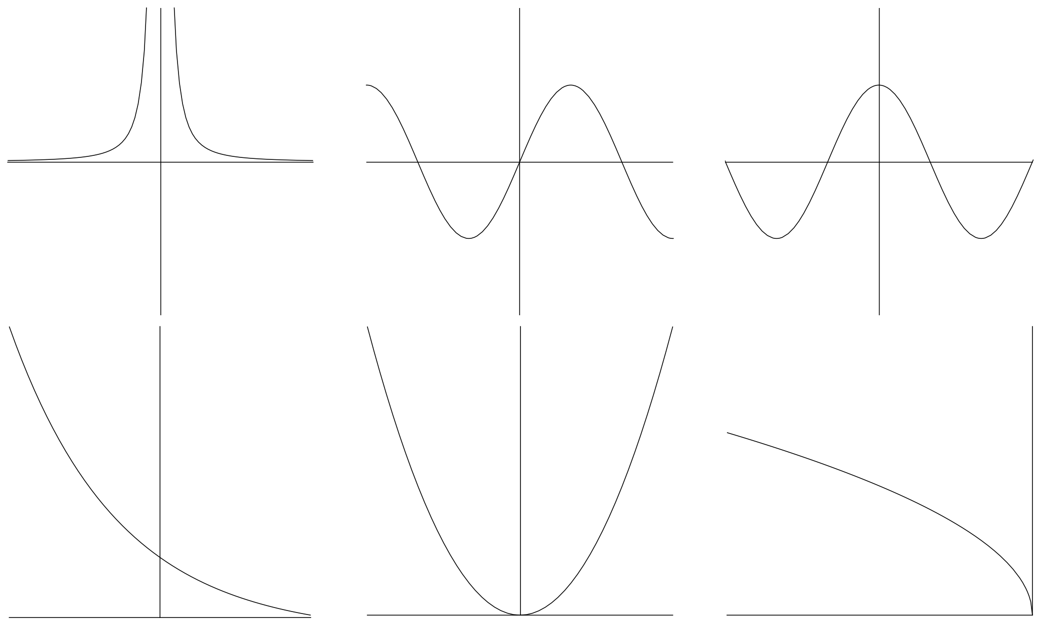
Trigonometric Functions

- θ is always in radians
- $\sin(\theta)$, $\cos(\theta)$, $\tan(\theta)$
- $\sec(\theta)$, $\csc(\theta)$, $\cot(\theta)$

Example 1. Match each of the functions below with it's graph.

(a) $y = x^2$
(b) $y = x^3$
(c) $y = \frac{1}{x}$
(d) $y = \frac{1}{x^2}$
(e) $y = \sqrt{x}$
(f) $y = \sqrt{-x}$
(g) $y = e^x$
(h) $y = (1/2)^x$
(i) $y = \ln(x)$
(j) $y = \sin(x)$
(k) $y = \cos(x)$
(l) $y = \tan(x)$





1.9 Logarithmic functions

Definition. By definition, the function $\ln(x)$ is the inverse function of e^x. Here are three equivalent ways to say the same thing:

1. $\ln(x)$ is the number that, when we raise e to this number, we get x (i.e. $\ln(x)$ answers a question: $e^? = x$),
2. $\ln(a) = b$ means the same thing as __________ (i.e. the inputs and the outputs are reversed),
3. $\ln(e^x) =$ ____ and $e^{\ln(x)} = x$ (i.e. $\ln(x)$ and e^x cancel each other).

Example 1. Simplify $\ln(e^{2x+1})$.

Example 2. Solve

$$e^{x-1} = 2$$

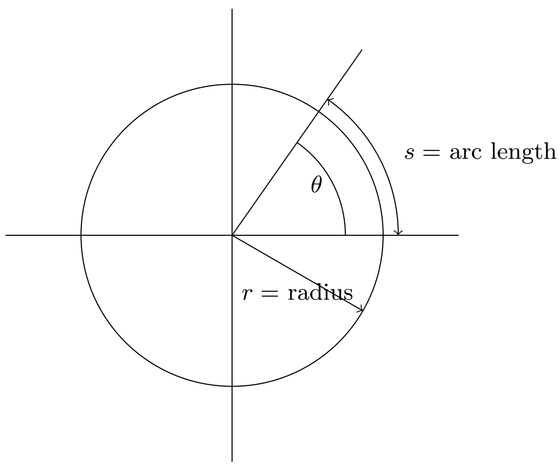
using $\ln(x)$.

Fact. Properties of logarithms.[2]

1. $\ln(ab) = \ln(a) + \ln(b)$
2. $\ln\left(\dfrac{a}{b}\right) = \ln(a) - \ln(b)$
3. $\ln(a^b) = b\ln(a)$

1.10 Trigonometric functions

Definition. Let θ be an angle placed in a circle of radius r, and let s be the arc-length of the circle contained in θ, as pictured:



Then the radian measure of θ is

$$\theta \text{ rad} = \underline{\qquad\qquad}$$

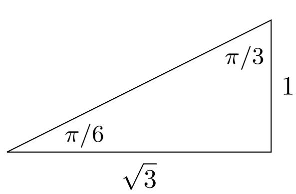
Example 1. (a) Use a circle of radius 1 to calculate the radian measure of $\theta = 360°$.

(b) Multiply or divide your answer to the previous part, to find radian equivalents of the following angles

$$0°, 30°, 45°, 60°, 90°, 180°, 270°.$$

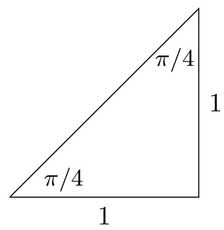
[2]Property (1) comes from $e^a \cdot e^b = e^{a+b}$. In other wonrds, for e^x, multiplying the outputs is the same as adding the inputs. For $\ln(x)$, it's the reverse. Property (2) comes from $\dfrac{e^a}{e^b} = e^{a-b}$. In other words, for e^x, dividing the outputs is the same as subtracting the inputs. For $\ln(x)$, it's the reverse. Property (3) comes from $(e^a)^b = e^{ab}$. In other words, for e^x, raising the output to the power b is the same as multiplying the input by b. For $\ln(x)$, it's the reverse.

Example 2. (a) Fill in the missing side of the triangle, and then fill in the table of trig values:



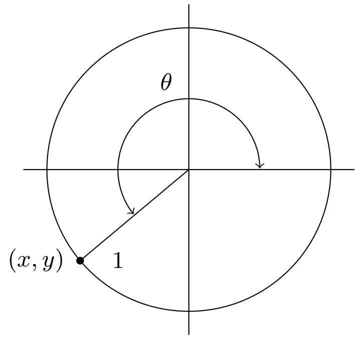
$\sin(\pi/6)$	$=$	$\sin(\pi/3)$	$=$
$\cos(\pi/6)$	$=$	$\cos(\pi/3)$	$=$
$\tan(\pi/6)$	$=$	$\tan(\pi/3)$	$=$

(b) Fill in the missing side of the triangle, and then fill in the table of trig values:



sin(45)	=
cos(45)	=
tan(45)	=

Definition. Given any angle θ, we place θ in a circle of radius 1.



Then we have the following definitions of sin, cos and tan.

$$\sin(\theta) = y \qquad (= \frac{y}{1} = \frac{\text{opp}}{\text{hyp}})$$
$$\cos(\theta) = x \qquad (= \frac{x}{1} = \frac{\text{adj}}{\text{hyp}})$$
$$\tan(\theta) = \frac{y}{x} \qquad (= \frac{\text{opp}}{\text{adj}})$$

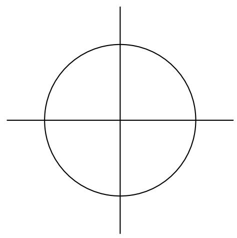
Example 3. Fill in the following chart:

	0	$\pi/2$	π	$3\pi/2$	2π
sin					
cos					
tan					

Example 4. Summarize the previous examples by filling in the following chart. See if you can find an easy mnemonic pattern to fill it in (i.e. something that you can remember without having to go back and check all the triangles).

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin(\theta)$					
$\cos(\theta)$					

Fact. The picture below shows which trig functions are positive in each quadrant



Example 5. Fill in the following chart:

	$16\pi/3$
sin	
cos	
tan	

Example 6. Combining (1) The standard trig values between 0 and $\pi/2$, (2) examples where we've found trig values in other quadrants, and (3) the rule about which trig functions are positive in which quadrants, we should now be able to fill in the values for sine and cosine all the way around the unit circle. Fill in the following chart, and again look for simple mnemonic patterns.

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π	$7\pi/6$	$5\pi/4$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$7\pi/4$	$11\pi/6$
sin																
cos																
tan																

Chapter 2

Limits

2.1 Tangent and velocity problems

The problems in this section are at the heart of Calculus and lead directly to the main idea in Calculus, limits.

Example 1. The height of a thrown ball is given by the following function:

$$p(t) = -4.9t^2 + 3.5t + 2.$$

Find (approximate) the velocity at $t = 2.3$.

There are two conclusions we will make from this example (and partly by looking ahead and knowing what Calculus is about): Sometimes we can make a sequence of approximations that appear to be getting closer and closer to the correct answer; A difference quotient may be an important example of a thing that we can approximate like this.

In fact, both of these observations are extremely profound and important. The first one becomes, in its finial from, the concept of a mathematical limit. The second one becomes the derivative.

Finally, note that even the partial solution we have of this problem is a monumental step forward in the history of science. The very idea that you could quantify physical processes, and do mathematics with these quantities, was incomprehensible to almost everyone, until Galileo. After Galileo it took another 100 years before people understood that you could start with one quantified process, position in this case, and do something to it to calculate its rate of change.

Example 2. Find the equation of the tangent line at $x = 0.8$ for the function $f(x) = \sin(x)$, and make a graph showing $f(x)$ and the tangent line.

The problems in this section are at the heart of Calculus and lead directly to the main idea in Calculus, limits. But even more important than the problems themselves is the technique we use to solve them. Generalizing this technique leads directly to limits and derivatives.

2.2 Limits

Recall one of the main conclusions of the previous section: that sometimes we do a calculation, have one of the inputs of the calculation get closer and closer to a single number, and see that outputs are getting closer and closer to a single number. If we take this idea, and make a formal definition out of it, we get the technical concept of limit.

Definition. We write

$$\lim_{x\to a} f(x) = L \text{ to mean that } \left\{ \begin{array}{l} \text{the} \qquad\qquad\qquad \text{of} \qquad\qquad \text{become} \\ \qquad\qquad\qquad\qquad\quad \text{to} \qquad\qquad \text{as the} \qquad\quad \text{-} \\ \text{values become} \qquad\qquad\qquad\qquad\qquad \text{to the} \\ \text{number } a. \end{array} \right\}$$

Note two things about this definition. (1) The phrase "infinitely close" is vague. But it can be made precise using a somewhat technical mathematical description involving finite, but arbitrarily small, distances (δ's and ε's). (2) When we have x infinitely close to a, we do *not* let x equal a.

There are a few of basic variations on the definition of limit.

Definition. We write

$$\lim_{x\to a^+} f(x) = L \text{ to mean that } \left\{ \begin{array}{l} \text{the} \qquad\qquad\qquad \text{of} \qquad\qquad \text{become} \\ \qquad\qquad\qquad\qquad\quad \text{to} \qquad\qquad \text{as the} \qquad\quad \text{-} \\ \text{values become} \qquad\qquad\qquad\qquad\qquad \text{to the} \\ \text{number } a\text{, but with} \qquad\qquad\qquad . \end{array} \right\}$$

We write

$\lim_{x\to a^-} f(x) = L$ to mean that $\left\{\begin{array}{l}\text{the} \qquad\qquad\qquad\qquad \text{of} \qquad\qquad \text{become} \\ \qquad\qquad\qquad\qquad\qquad \text{to} \qquad\qquad \text{as the} \qquad\quad \text{-} \\ \text{values become} \qquad\qquad\qquad\qquad\qquad\qquad \text{to the} \\ \text{number } a\text{, but with} \qquad\qquad\qquad\qquad .\end{array}\right\}$

We write

$\lim_{x\to a} f(x) = \infty$ to mean that $\left\{\begin{array}{l}\text{the} \qquad\qquad\qquad\qquad \text{of} \qquad\qquad \text{become} \\ \qquad\qquad\qquad\qquad\qquad \text{as the} \qquad\quad \text{-values become} \\ \qquad\qquad\qquad\qquad\qquad\quad \text{to the number } a.\end{array}\right\}$

We can also combine these concepts to have

$$\lim_{x\to a^+} f(x) = \infty,\ \lim_{x\to a^-} f(x) = \infty,\ \lim_{x\to a} f(x) = -\infty,\ \lim_{x\to a^+} f(x) = -\infty,\ \lim_{x\to a^-} f(x) = -\infty.$$

Example 1. Example 1 from section 2.1 involved a limit. Can you figure out what we took the limit of and what it equalled?

Example 2. Example 2 from Section 2.1 involved a limit. Can you figure out what we took the limit of and what it equalled?

Example 3. Find $\lim_{x\to 3} \frac{\sin(x-3)}{x-3}$.

Example 4. Find the following limit, if it exists.

$$\lim_{x \to 1} \sin\left(\frac{1}{x-1}\right)$$

Example 5. (Stewart, 6e, 2.2#7) Look at the graph given in the book, and determine what the indicated limits are.

Example 6. (Stewart, 6e, 2.2#15) Make up a graph of a function $f(x)$ that has the following properties: (a) $\lim_{x \to 3^+} f(x) = 4$, (b) $\lim_{x \to 3^-} f(x) = 2$, (c) $\lim_{x \to -2} f(x) = 2$, (d) $f(3) = 3$, (e) $f(-2) = 1$.

Example 7. Find $\lim_{x \to 2} \dfrac{1}{(x-2)^2}$.

Example 8. Find the following limits, if they exist:

$$\lim_{x\to 3^+} \frac{x|6-2x|}{3-x}, \quad \lim_{x\to 3^-} \frac{x|6-2x|}{3-x}, \quad \lim_{x\to 3} \frac{x|6-2x|}{3-x}.$$

2.3 Algebraic approach to limits

Now we start to learn how to find limits algebraically. This starts with the simplest possible limits, and then builds these up to more complicated examples.

Fact. If C is a constant, then $\lim_{x\to a} C =$ ____ .

Fact. $\lim_{x\to a} x =$ ____

Fact. If $f(x)$ and $g(x)$ are any functions with $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both existing then we have

1. $\lim_{x\to a}[f(x) \pm g(x)] =$ ____________________
2. $\lim_{x\to a}[f(x)g(x)] =$ ____________________
3. $\lim_{x\to a} \dfrac{f(x)}{g(x)} =$ (as long as the ____________).

Comments. Using these facts we can already do a simple limit.

Example 1. Find $\lim_{x\to 4}(3x^2 + 2x + 5)$, algebraically, using the limit laws, showing all possible steps.

You can generalize the same argument as in Example 1, to any polynomial

Fact. If $p(x)$ is any polynomial then

In other words **JUST**

Theorem 1. If $f(x)$ is any combination (i.e. sum, product, fraction, composition, etc.) of basic functions (i.e. powers of x, exponentials, trig functions, inverses, etc.) and $x = a$ is in the domain of $f(x)$, then

$$\lim_{x\to a} f(x) = \_\_\_\_\_\_$$

Comments. In other words **JUST** (as long as it's defined).

Example 2. Find $\lim_{x \to 3} \frac{5x^2 + 2x}{\sin\left(\frac{\pi}{2}x\right)}$.

If we can't just plug it in then we use the following result.

Theorem 2 (Squeeze Theorem)**.** Let $g(x) \leq f(x) \leq h(x)$ for all x near $x = a$ (except possibly at $x = a$). Suppose

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L.$$

Then

$$\lim_{x \to a} f(x) = L$$

□

Example 3. Make up graphs that illustrate the Squeeze Theorem.

Example 4. Find $\lim_{x \to 0} x^2 \sin(1/x) + 1$ using the Squeeze Theorem and graph the results.

Example 5. (Section 2.3#38) Find $\lim_{x \to 0^+} \sqrt{x}\, e^{\sin(\pi/x)}$.

Corollary (Corollary of Squeeze Theorem)**.** If $h(x) = f(x)$ for all x near $x = a$ (except possibly at $x = a$) and $\lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.

□

Example 6. Find $\lim_{x\to 2} \dfrac{x^2 + x - 6}{x - 2}$.

Example 7. Find $\lim_{x\to -4} \dfrac{x^2 + 5x + 4}{x^2 + 3x - 4}$.

Here are some rules of thumb for algebraic manipulation of limits:

- If the limit of $\dfrac{f(x)}{g(x)}$ involves division by 0, then

  .

- If the limit of $\dfrac{f(x)}{g(x)} - \dfrac{h(x)}{p(x)}$ involves division by 0, then

  .

- If the limit of $\dfrac{\sqrt{f(x)} - \sqrt{g(x)}}{h(x)}$ involves division by 0, then rationalize the numerator:

  $$\frac{\sqrt{f(x)} - \sqrt{g(x)}}{h(x)} \cdot \frac{}{} = \frac{}{}$$

  then

Example 8. Find $\displaystyle\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h}$.

Example 9. Find $\displaystyle\lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right)$

Example 10. Find $\lim_{x\to 0} \frac{\sin(x)}{x}$

2.5 Continuity

Definition. Let $f(x)$ be any function and $x = a$ a number in the domain of $f(x)$. We say that $f(x)$ is **continuous at** $x = a$ if ________________________ .

Comments. In this language, the plugging in theorem says that all combinations of our basic functions are continuous everywhere they defined.

Example 1. Make up two pictures of how a function could *fail* to be continuous at a point. What does continuity mean about holes in the graph?

Definition. If $f(x)$ is any function defined on an interval $[a, b]$, we say that $f(x)$ is **continuous on the interval** if it is continuous at every point in the interval.

Comments. By the discussion above, we can think of this as meaning that $f(x)$ has no holes anywhere in the interval. From this viewpoint, the plugging in theorem says that all combinations of our basic functions have no holes on any interval they are defined on.

Example 2. Let $g(x)$ be defined by the graph in section 2.2, #7. List the points where $g(x)$ is not continuous.

Example 3. Show[1] that the function

$$f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

is not continuous at $x = 3$.

[1] The word "show" here, and elsewhere means that you have to provide more than just an answer. You need to record (write) some of your reasons for this answer.

Example 4. Find a and b so that the function defined by

$$f(x) = \begin{cases} x+1 & \text{if } x \leq -2 \\ ax^2+bx & \text{if } -2 < x < 1 \\ -x+2 & \text{if } x \geq 1 \end{cases}$$

is continuous everywhere, and graph your results

Theorem 1 (IMV: Intermediate Value Theorem)**.** If $f(x)$ is continuous on the interval $[a, b]$, then the graph of $f(x)$ (for $a \leq x \leq b$) crosses every y-value between $f(a)$ and $f(b)$. In other words, for each N between $f(a)$ and $f(b)$, there exists a solution x, with $a \leq x \leq b$, such that $f(x) = N$.

Example 5. Make up a graph illustrating the theorem.

Example 6. Show that 1071 has 5th root, and find integers that are upper and lower bounds for this 5th root.

Example 7. Find a better approximation of $\sqrt[5]{1071}$ by repeatedly using the IMV.

Example 8. Show that the equation $x + \sin(x) = e$ has a solution and find upper and lower bounds.

2.6 Limits at infinity

Definition. We write

$$\lim_{x\to\infty} f(x) = L \text{ to mean that } \left\{\begin{matrix}\text{the} & \qquad\qquad\qquad \text{of} & \qquad\qquad \text{become} \\ & \qquad\qquad\qquad\qquad \text{as the} & \qquad \text{-values} \\ \text{become} & \qquad\qquad\qquad\qquad . & \end{matrix}\right\}$$

This limit goes by another name: $f(x)$ has a horizontal asymptote, on the right, of $y = L$.

We have obvious variations

$$\lim_{x\to-\infty} f(x) = L,\ \lim_{x\to\infty} f(x) = \infty,\ \lim_{x\to\infty} f(x) = -\infty,\ \lim_{x\to-\infty} f(x) = -\infty$$

Example 1. Find $\displaystyle\lim_{x\to\infty} \frac{1}{x}$ and discuss why this limit makes sense.

Fact. Here are all the basic functions you know with horizontal asymptotes (as well as two functions that *don't* have them).

- $\lim\limits_{x\to\pm\infty} \dfrac{1}{x^p} =$ ____ for any real number $p > 0$.
- $\lim\limits_{x\to-\infty} e^x =$ ____
- $\lim\limits_{x\to\infty} \tan^{-1}(x) =$ ________ , $\lim\limits_{x\to-\infty} \tan^{-1}(x) =$ __________
- $\lim\limits_{x\to\infty} \ln(x) =$ ______ , $\lim\limits_{x\to\infty} e^x =$ ______ , $\lim\limits_{x\to\infty} \sqrt{x} =$ ______

Now we generalize two of the above facts

Fact. If $\lim\limits_{x\to a} f(x) = L$ (with $L \neq \pm\infty$) and $\lim\limits_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \_\_\_\_$$

Note: the same result holds if we replace $x \to a$ with $x \to a^+$, $x \to a^-$, $x \to \infty$ or $x \to -\infty$.

Shorthand mnemonic: " $\overline{\qquad\qquad\qquad}$ ". Be careful here, we are not treating ∞ as a real number, but merely writing something which is shorthand for the correct statement involving limits.

Example 2. Find the horizontal asymptote of $\dfrac{3x^8 - 100x^3 + 17.304}{7x^8 - x + 100,000}$.

Fact (Horizontal Asymptotes of Rational Functions)**.**

$$\lim_{x\to\infty} \frac{a_n x^n + \dots \quad \text{(a poly. of degree } n)}{b_m x^m + \dots \quad \text{(a poly. of degree } m)} = \begin{cases} \_\_\_ & \text{if } n < m \\ \_\_\_\_\_ & \text{if } n = m \\ \_\_\_\_ & \text{if } n > m \end{cases}$$

Rule. To find

$$\lim_{x\to\infty} \frac{\text{sum of powers of } x}{\text{sum of powers of } x}$$

we divide the top and the bottom by the biggest simplified power of x that is on the bottom. ("Simplified" means that we take something like $\sqrt{x^2 + \dots}$ and use x, not x^2.)

Example 3. Find $\displaystyle\lim_{x\to\infty} \frac{3x^2 + x + 1/x}{\sqrt{9x^4 + 1/x}}$

Example 4. Find $\lim_{x \to \infty} \dfrac{x^2 - 7x}{\sqrt{x^5 + 1}}$.

Example 5. Find $\lim_{x \to \infty} \cos(x/\pi)$.

Example 6. Find $\lim_{x\to\infty}(\sqrt{4x^2+x+1}-2x)$.

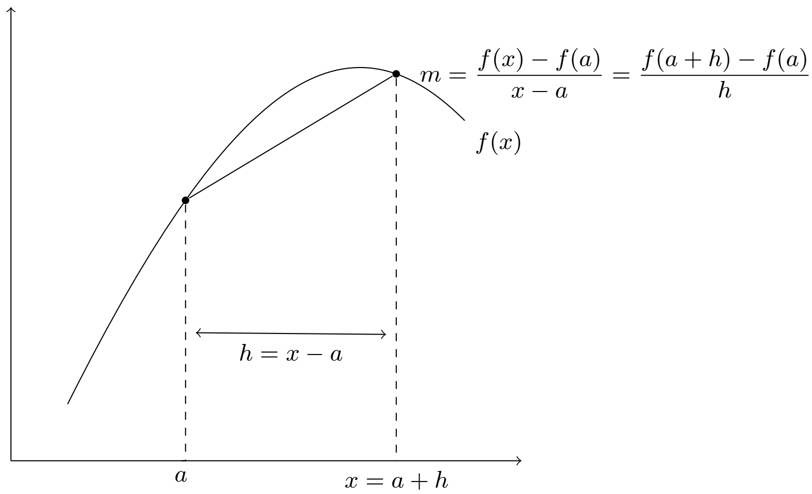
Example 7. Find $\lim_{x\to\infty}(-28x^{11} + 1000x^{10} + 1)$.

2.7 Tangents and velocities revisited

Definition. The **derivative of** $f(x)$ **at the point** $x = a$ is a *number* denoted by $f'(a)$ and defined as a limit

$$f'(a) = \underline{\qquad\qquad\qquad\qquad} = \underline{\qquad\qquad\qquad\qquad}$$

note that the two limits are equal: just let $h = x - a$. The fractions used here are called **difference quotients**, and they equal the slope of a secant line through the points $(a, f(a))$ and $(x, f(x))$ (or $(a + h, f(a + h))$), as pictured



Definition. The **tangent line** to a function $f(x)$ at a point $x = a$ is given by

________________________________ .

Example 1. Find the equation of the tangent line at $x = 5$ of $f(x) = 2x^2 - x + 3$.

Definition. If $f(t)$ equals position as a function of time, then the **velocity** at $t = a$ (i.e. the instantaneous velocity) is given by

$$v(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

note that the two limits are equal: just let $h = t - a$.

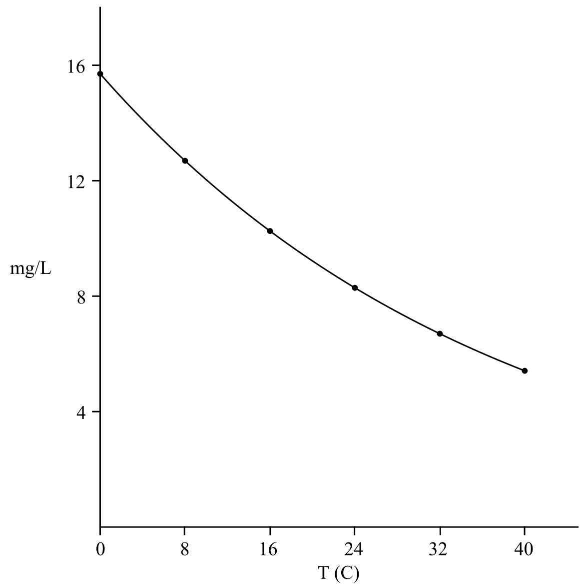
Example 2. Repeat the calculations of Example 1, Section 2.1, using limits and algebra.

Interpretations:

- If we are looking at a graph of $f(x)$ then $f'(a)$ equals the slope of the tangent line at $x = a$.
- If $f(t)$ is a position function with $t =$ time, then $f'(a)$ equals the velocity at $t = a$.
- If $f(x)$ equals any real quantity whatsoever with, then $f'(a)$ equals the rate of change of this quantity, with units given by $\dfrac{\text{units of } f(t)}{\text{units of } x}$.

Example 3. (Stewart 2.7#49) The quantity of oxygen that can dissolve in water depends on the temperature of the water. (So thermal pollution influences the oxygen content of water.) The graph shows how oxygen solubility S varies as a function of the water temperature T.

(a) What is the meaning of the derivative $S'(T)$? What are its units?
(b) Estimate the value of $S'(16)$ and interpret it.



Example 4. Find the derivative of $f(x) = \dfrac{1}{x}$ at $x = 4$.

Example 5. Find a formula for the derivative of $f(x) = \sqrt{x}$ at $x = a$ where a is an arbitrary unknown constant.

Example 6. Find a formula for $f'(a)$ with $f(x) = \dfrac{1}{\sqrt{x+1}}$ and a an arbitrary unknown constant.

2.8 The derivative as a function

Definition. The **derivative** of $f(x)$ is the *function* $f'(x)$ defined as follows

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Note: this differs from the definition in section 2.7 in that we don't have the phrase "at the point $x = a$". As a result, in this definition $f'(x)$ is a *function*, not a number.

Comments. We interpret $f'(x)$ as giving a formula for the slope of the tangent line, the velocity, etc.

Example 1. Find $f'(x)$ for $f(x) = \dfrac{1}{x^2}$.

Notation for derivative: $f'(x)$ or y' (with $y = f(x)$) or $\frac{d}{dx}f(x)$ or $\frac{df}{dx}$ or $\frac{dy}{dx}$ (with $y = f(x)$).

Last three = *Leibniz* notation.

Leibniz notation advantages:

- You don't have to name $f(x)$,
- This notation suggests a ratio,
- this notation makes explicit the role of x in taking the derivative,
- this notation makes the chain rule (coming later) look nice,
- this notation reminds us that $\frac{d}{dx}$ is an operator (i.e. it is a thing that we apply to functions),
- this notation reads as "take the derivative (with respect to x) of ...",
- this notation is easier to see than $'$.

Example 2. Write the answer to Example 1 in different notation.

At this point we turn to graphical derivatives.

Example 3. Do problem #6 in 2.8.

Example 4. Do problem #41 in 2.8

Chapter 3

Rules for Derivatives

3.1 Shortcuts for powers of x, e^x, constants, sums, and differences

Now we start to fill in shortcuts. Recall that we already know $\frac{d}{dx}\frac{1}{x^2} = -\frac{2}{x^3}$.

Example 1. Find $\frac{d}{dx}x$.

Example 2. Find $\frac{d}{dx}x^2$.

Example 3. Find $\dfrac{d}{dx}\sqrt{x}$

Rule. We generalize the previous examples in the following rule:

$$\boxed{\text{Power rule: } \frac{d}{dx}x^n = nx^{n-1} \text{ for any real number } n}$$

We've proven this rule for $n = 1, 2, 1/2, -1/2$ (and in the homework probably you've done $n = 3$). We will prove it for all n when we do logarithmic differentiation.

Example 4. Find the equation of the tangent line at $x = 1$ of $f(x) = x^{3.7}$.

Rule.

$$\boxed{\text{Constant multiple rule: } \quad \frac{d}{dx}C \cdot f(x) = C \cdot f'(x), \text{ where } C \text{ is a constant}}$$

□

The next rule follows from the previous one and a special case of the Power Rule.

Rule.

Constant rule: $\dfrac{d}{dx}C = 0,$ where C is a constant

Rule.

Sum and Difference rule: $\dfrac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

□

Example 5. Find when $f(x) = x^3 - 20x^2 + 4x$ has a horizontal tangent line.

Definition. We define the number e as the number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

(once we prove that such a number exists).

□

Rule.

$$\boxed{\text{Exponential rule: } \frac{d}{dx}e^x = e^x}$$

□

Example 6. Find the equation of the tangent line at $x = 0$ where $f(x) = 3x^{3/2} - 5e^x$.

Example 7. Find the derivative of $\sqrt[5]{32x} + 5\sqrt{x^5}$.

Example 8. Find the equation of the normal line at $x = -2$ for $f(x) = 2x^2 + 3x + 1$.

3.2 Product and Quotient Rules

Rule (Product rule:)**.**

$$\boxed{(f \cdot g)' = f' \cdot g + f \cdot g'}$$

Rule.

$$\frac{d}{dx}[f(x)g(x)] = \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right]$$

Figure 3.2 shows one way you can understand this rule.

□

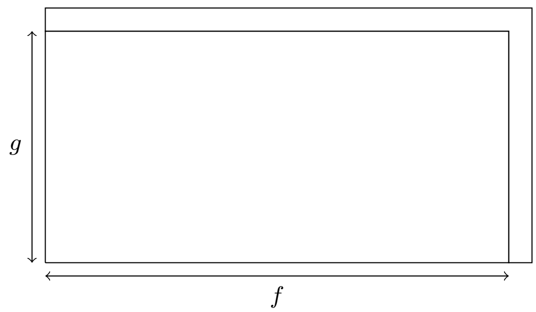
Example 1. Find $\frac{d}{dx}(3x + e^x)(x^2 - 4e^x)$.

Example 2. Find $\frac{d}{dx}(x + e^x)\left(\frac{1}{x} + x\right)$.

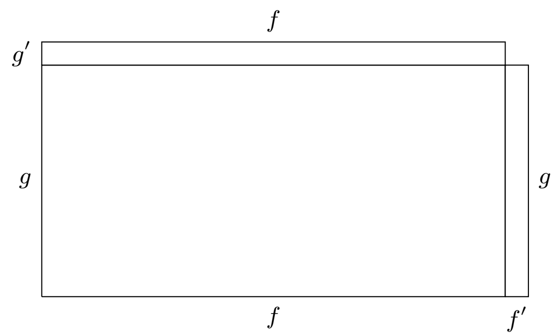
Rule.

Quotient Rule: $\left(\frac{f}{g}\right)' =$ ____________________

Figure 3.1: Intuitive, graphical, understanding of product rule



Picture the product fg as the area of a rectangle with sides f and g. Interpret $(fg)'$ as a small change in the area, pictured here in red.



In a similar way, we picture f' and g' as small changes in f and g. Then $f' \cdot g$ and $f \cdot g'$ are the areas of two skinny rectangles (again pictured in red)

Comparing the two previous pictures (and ignoring the very small white square in the corner that we've left out) we see that the red area in the first picture is the same as the red area in the second picture. In other words,

$$(fg)' = f' \cdot g + f \cdot g'$$

Here is the more verbose way to state this

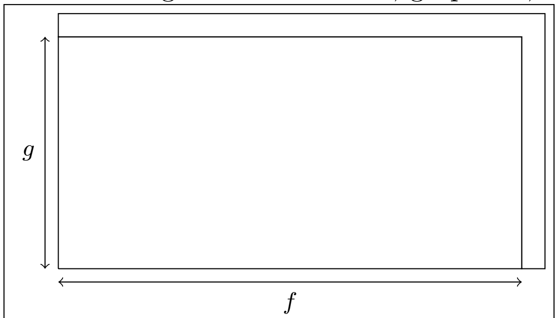
Rule.

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \underline{\hspace{6cm}}$$

Example 3. Find $\dfrac{d}{dx}\dfrac{x}{e^x}$.

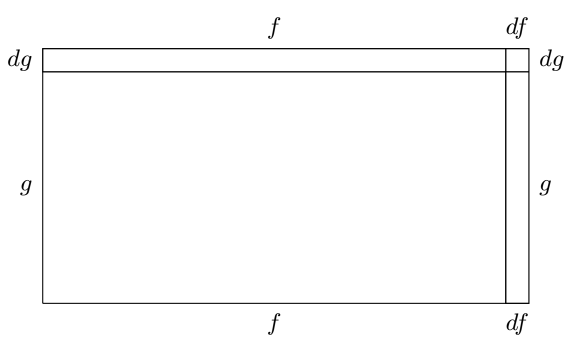
Example 4. Find the derivative of $\dfrac{2t + \frac{1}{t}}{2 + \sqrt{t}}$.

Figure 3.2: Intuitive, graphical, understanding of product rule



Picture fg as a rectangle with sides f and g, let $d(fg)$ represent an infinitely small change in fg, pictured in red (well, a *small* change is picture in red).

Now, the red part, the change in fg, not only represents $d(fg)$, but can also be connected to changes in sides f and g.



Using this, we can break the red part up into two rectangles and a square with areas $df \cdot g$, $g \cdot df$, and $df \cdot dg$.

So, the red part is $d(fg)$, and this equals the sum of two rectangles and a square

$$d(fg) = df{\cdot}g + f{\cdot}dg + df{\cdot}dg$$

Now, since df and dg are infinitely small, we have $df \cdot dg \approx 0$. This gives us the crucial idea of the whole picture/proof

$$\boxed{\begin{array}{c} d(fg) = df \cdot g + f \cdot dg \\ \text{Change area } = df \cdot g + f \cdot dg \end{array}}$$

Now, we "divide" by dx to get

$$\frac{d}{dx}fg = \frac{df}{dx} \cdot g + f\frac{dg}{dx}$$

Example 5. Suppose $h(x)$ is a function such that $h(5) = 3$ and $h'(5) = -2$. Find the derivative of $e^x h(x)$ at $x = 5$.

3.3 Trigonometric functions

Rule.

Sine Rule: $\frac{d}{dx}\sin(x) =$ __________

Comments. Note: this formula is only correct in **radians**.

□

Example 1. Find the equation of the tangent line at $x = \pi/6$ for $\sin(x)$.

Example 2. Find $\dfrac{d}{dx}(x\sin(x))$

Rule.

Cos Rule: $\dfrac{d}{dx}\cos(x) =$ ____________

Comments. The proof is left as homework.

Rule.

Tan Rule: $\dfrac{d}{dx}\tan(x) =$ ____________ or ____________

$\square$

Example 3. Find $\dfrac{d}{dx}\dfrac{1+\sin(x)}{x+\cos(x)}$, and simplify.

Rule.

$$\frac{d}{dx}\cot(x) = -\csc^2(x), \quad \frac{d}{dx}\sec(x) = \sec(x)\tan(x), \quad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

3.4 The Chain rule!!

Rule.

Chain rule: $\frac{d}{dx}f(g(x)) =$ ____________________

Rule (Chain Rule in words)**.**

$\frac{d}{dx}f(g(x)) =$ the derivative of the function (don't change the) times the derivative of the function.

Rule (Chain Rule in Leibniz notation)**.**

$$\frac{dy}{dx} = \_\_\_\_\_\_\_\_\_\_\_\_$$

where y is a function of u and u is a function of x

Rule (Chain Rule function by function).

ordinary rule		chain rule version
$\frac{d}{dx}x^n = nx^{n-1}$	$\longrightarrow$	______
$\frac{d}{dx}e^x = e^x$	$\longrightarrow$	______
$\frac{d}{dx}\sin(x) = \cos(x)$	$\longrightarrow$	______
$\frac{d}{dx}\cos(x) = -\sin(x)$	$\longrightarrow$	______
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\longrightarrow$	______

For each formula, put whatever you want in each , *as long as you put the same thing in each* .

Example 1. Find $\frac{d}{dx}\sin(x^2+1)$.

Example 2. Find $\dfrac{d}{dx}\sqrt{4x^2 + x}$.

Example 3. Find $\dfrac{d}{dt}\left(\dfrac{\sin(t)}{t^2 + t + 1}\right)^{15}$

Example 4. Find $\frac{d}{dx}(3x^2 - 5\sqrt{x})e^{\sin(\sqrt{\tan(x)})}$

Example 5. Find $F'(-11)$ where $F(x) = f(g(x))$ and

$$\begin{aligned} f(-3) &= 5 & g(-3) &= \frac{3}{2} \\ f'(-3) &= 2 & g'(-3) &= -\frac{1}{7} \\ f(-11) &= 7 & g(-11) &= -3 \\ f'(-11) &= 2 & g'(-11) &= 5 \end{aligned}$$

Rule.

General exponential rule: $\frac{d}{dx}a^x = a^x \ln(a), \text{for } a > 0$

□

3.5 Implicit derivatives

Example 1. Find the slope at $x = 1/2$ on the circle $x^2 + y^2 = 1$, by solving explicitly for y.

Example 2. Find the slope at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the circle $x^2 + y^2 = 1$, by working implicitly: remember, y is a function of x.

How to take implicit derivatives: You start with an equation involving x and y. Take the derivative, $\frac{d}{dx}$ of both sides. Anytime you see an x, you just take the derivative like normal. Anytime you see a y, you treat it like a hidden function and use the chain rule (and possibly the product and quotient rules). Here's a simpler way to say this: take the derivative of anything involving y just the same as you would if it were x, but then multiply by y' or $\frac{dy}{dx}$. [1] Finally, you solve for y'.

[1] It is often confusing to students why we treat y this way, I'll try to offer a little explanation here. The crucial difference is between a function and an equation. When we write $f(x) = x^2 + y^2$ we are defining a function $f(x)$. We can plug in any number we want for x, and get the output from the formula $x^2 + y^2$. For example, $f(1) = 1 + y^2$. Now, in this case, y has no relation to x. We could have that $y = 3$ or $y = 5$, or just treat it as an unknown constant, or as an unrelated variable. In this case, if we take the derivative $\frac{d}{dx}$ of $f(x)$ we get that $\frac{d}{dx}y = 0$, and so $\frac{d}{dx}f(x) = 2x$. This

Example 3. Find the derivative at $x = 2/3$, $y = 4/3$ for the shape defined by $x^3 + y^3 = 3xy$. Remember: y is a function of x.

---

makes sense because the graph of $f(x)$ is a parabola, shifted either up or down due to the value of y, and such a parabola should have slope given by $2x$, just like $f(x) = x^2$.

Now, suppose $x^2 + y^2 = 1$. Here, the fact that we have an equation means that y cannot be a number unrelated to x. If I plug in $x = 1$, then I must have $y = 0$. There is no choice about y, it is determined by x. Thus, y is a function of x. Looking at it the other way, we are not free to plug in different values of y and still have a function $f(x)$. We cannot have $y = 3$ and then graph $f(x)$. There is no $f(x)$ here and there is no value of x that corresponds to $y = 3$. Again, y is not unrelated to x, it is an implicit function of x.

Now, when we take the derivative of this equation, we have to do it in a way that (1) is correct given that y is a function of x, and (2) is correct given that we could in fact replace y with $\sqrt{1-x^2}$. So, if we have $y = g(x)$, then $y^2 = (g(x))^2$ and the derivative becomes $2g(x) \cdot g'(x)$. This is the same as we found (but in different notation), $2y \cdot y'$. This would be true no matter what function y equals. We will have other examples where y is a different function of x, but we will still have that the derivative of y^2 is $2y \cdot y'$. Finally, given the fact that $y = \sqrt{1-x^2}$ we should get the correct thing when we take the derivative of $(\sqrt{1-x^2})^2$. On the one hand, we can simplify this first and take the derivative of $1 - x^2$ and get $-2x$. On the other hand, we should be able to apply the chain rule and get

$$2(\sqrt{1-x^2}) \cdot \boxed{\sqrt{1-x^2}}' = 2\sqrt{1-x^2} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = -2x.$$

The point is, all of these different ways of looking at $x^2 + y^2 = 1$ have to give the same result, the one we got in Example 1, and it can't be the same as taking the derivative of $f(x) = x^2 + y^2$.

Example 4. Find y' for $e^y \cos(x) = 1 + \sin(xy)$

Example 5. [3.5#15] Find y' where $e^{x/y} = x - y$

Inverse trig functions

Recall that if we know one trig function, then we know them all[2].

Example 6. Suppose $\sin(\theta) = \dfrac{5}{7}$ (and $0 < \theta < \pi/2$). Find $\cos(\theta)$ and $\tan(\theta)$.

[2] As I like to say, in an evil tone of voice, "one trig function to rule them all, one trig function to find them, one trig function to bring them all and in the darkness bind them."

Example 7. Suppose $\sin(y) = x$, find $\cos(y)$ and $\tan(y)$.

Example 8. Find $\dfrac{d}{dx}\sin^{-1}(x)$.

Rule.

Sine Inverse: $\dfrac{d}{dx}\sin^{-1}(x) =$ ____________

Rule. **Cos inverse:** $\dfrac{d}{dx}\cos^{-1}(x) =$ ____________

Rule. **Tan inverse:** $\dfrac{d}{dx}\tan^{-1}(x) =$ __________

Example 9. Find the derivative of $\sqrt{1-x^2}\arctan(\pi x + 7)$

3.6 Logarithmic differentiation

Example 1. Find $\dfrac{d}{dx}\ln(x)$

ln: $\dfrac{d}{dx}\ln(x) = \underline{\qquad}$

Example 2. Find the derivative of $y = \sqrt[3]{\dfrac{x^4+1}{x^4-1}}$

Procedure for logarithmic differentiation: Start with y =something.

Take ln( ) of both sides, and use the properties of ln( ) to simplify the right hand side.

Take implicit derivatives, solve for y' and replace y with the formula you started with.

Example 3. Find the derivative of $y = x^x$.

Example 4. Find a general formula for the derivative of $f(x)^{g(x)}$.

Example 5. Prove the rule $\frac{d}{dx}x^n = nx^{n-1}$ for any real number n.

We can summarize all of our basic rules for derivatives, as shown on the next page.

Derivatives

Here all the derivative rules and techniques that we have learned this semester. With them, you can take the derivative of any function that you have ever seen! (Note: there are other functions in the world, that you don't know how to take the derivative of, but these involve definitions and formulas that you have probably never seen.)

Formulas marked with a "*" should definitely be memorized.

Basic functions

$$* \frac{d}{dx}x^n = nx^{n-1} \text{ for all real numbers } n$$

The derivatives of x^2, x^3, $\sqrt{x} = x^{1/2}$ were found in 2.8, using the definition of derivative. The derivatives of $1/x = x^{-1}$, $1/x^2 = x^{-2}$ were found in 3.1, again using the definition. The derivative of x^n, for all real numbers n, was found in 3.6 by using logarithmic differentiation.

$$* \frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$
$$* \frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$
$$* \frac{d}{dx}\tan(x) = \sec^2(x) \qquad \frac{d}{dx}\cot(x) = -\csc^2(x)$$

The derivatives of $\sin(x)$ and $\cos(x)$ were found in 3.3, using the definition, trig identities, and the special limits $\lim_{h\to 0}\frac{\sin(h)}{h} = 1$ and $\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0$. The rest of these derivatives were found in 3.3 using the quotient rule.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\csc^{-1}(x) = \frac{-1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}(x) = \frac{-1}{1+x^2}$$

All of these derivatives were found in 3.5, by using implicit derivatives to obtain a relationship between each of these functions and the noninverse function.

$$* \frac{d}{dx}e^x = e^x \text{ and } * \frac{d}{dx}\ln|x| = \frac{1}{x}$$

The derivative of e^x was found in 3.1 using the special limit $\lim_{h\to 0}\frac{e^h-1}{h} = 1$. The derivative of $\ln(x)$ was found in 3.6 using implicit derivatives.

Combinations

$$*(f+g)' = f' + g' \qquad *(f\cdot g)' = f'\cdot g + f\cdot g'$$
$$*(f-g)' = f' - g' \qquad *(f/g)' = \frac{f'\cdot g - f\cdot g'}{(g)^2}$$

$$*(f(g(x)))' = f'(g(x))\cdot g'(x)$$

(the combination $f(x)^{g(x)}$ is described below)

Here is how some of our basic functions look when combined with the chain rule:

$$* \frac{d}{dx}\Box^n = n\Box^{n-1}\cdot\Box' \qquad * \frac{d}{dx}\sin(\Box) = \cos(\Box)\cdot\Box'$$
$$* \frac{d}{dx}e^{\Box} = e^{\Box}\cdot\Box' \qquad * \frac{d}{dx}\cos(\Box) = -\sin(\Box)\cdot\Box'$$
$$* \frac{d}{dx}\ln(\Box) = \frac{1}{\Box}\cdot\Box' \qquad * \frac{d}{dx}\tan(\Box) = \sec^2(\Box)\cdot\Box'$$

You should imagine putting other functions inside the boxes. Of course, the exact same stuff should go inside each box in one of these equations.

The derivatives of $f \pm g$ were proven in 3.1 using basic definitions. The product and quotient rules were proven in 3.2 using the basic definition and some tricks. The chain rule was proven in 3.4, also using the defintiion and some tricks.

Techniques

Implicit derivatives. (1) You start with an equation involving x's, y's, and/or numbers. (2) You take the derivative of both sides of the equation. When you do this you view y as a function of x and use the chain rule, product rule, etc. as needed. (For example $\frac{d}{dx}y^2 = 2yy'$, $\frac{d}{dx}\sin(y) = \cos(y)y'$, $\frac{d}{dx}xy = y + xy'$, etc..) (3) You solve the new equation for y'.

Logarithmic differentiation. (1) You start with a function, often of the form $y = f(x)^{g(x)}$. (2) Take ln of both sides, bring the exponent down in front. (3) Take the implicit derivative. (4) Solve for y'.

3.7 Rates of change in the natural sciences

Recall that derivative = rate of change. The main difficulty in this section is just seeing different ways of interpreting the rate of change, and extracting from each problem just what the function is, and what the variables are. Since I do not know all the applications that are in this book, nor an equal number of applications that are not in the book, I will follow the examples in the book more closely than I usually do.

Example 1. Identify the two functions in Example 1 from the book, and what their derivatives mean.

Example 2. Identify what the function is in example 2 from the book, and what its derivative means.

Example 3. Identify what the function is in example 3 from the book, and what its derivative means.

Example 4. Identify what the function is in example 4 in the book, and what its derivative means.

Example 5. Identify what the function is in example 5 from the book, and what its derivative means.

Example 6. Identify what the function is in example 6 from the book, and what its derivative means.

Example 7. Identify what the function is in example 7 from the book, and what its derivative means.

Example 8. Identify what the function is in example 8 from the book, and what its derivative means.

Example 9. The graphs in #5 from the book, show the *velocity* of two different particles. When is each particle speeding up? When is it slowing down?

Example 10. In #11 in the book, a company is making square computer chips. We analyze the function there, and its derivative.

Example 11. In #13(b),(c) in the book, we calculate the change in area of a circle as a function of radius.

Example 12. In #17 in the book, we analyze the change in mass of a rod, i.e. linear density.

Homework comments

For the homework, you may want to have a skeleton key guide to what the functions are in the problems. Here it is

- #7, you are given position. Take the derivative to get velocity. Take the derivative again to get acceleration.
- #8, you are given position. Take the derivative to get velocity.
- #12, start with $V(x) = x^3$. Calculate $\frac{dV}{dx}$ at $x = 3$ (and interpret).
- #15, you are given surface area S. Calculate $\frac{dS}{dr}$ (and interpret).
- #16, you are given volume V. Calculate $\frac{dV}{dr}$ (not for part (a)) (and interpret).
- #17, you are given mass $M = 3x^2$. Calculate $\frac{dM}{dx}$.
- #20, you are given force F, where G, m and M are constants. Calculate $\frac{dF}{dr}$ (and interpret).
- #23, you need to start by finding a formula for the population P, as in example 6 from the book. Then take $\frac{dP}{dt}$.
- #24, you are given a formula for the population f, where a, b are constants. Find $\frac{df}{dt}$ and use this (with f) to solve for the constants a and b.
- #29, you are given cost C. Take $\frac{dC}{dx}$ to find the marginal cost.
- #30, you are given cost C. Take $\frac{dC}{dx}$ to find the marginal cost.

3.9 Related rates

The basic idea here is that we have an equation, and every variable in it is changing (or could be changing) with time. If the variable is changing with time, that means that it is a *function* of time, although we do not usually know what formula it has. Thus, if we take the time derivative, $\frac{d}{dt}$, of the equation, we treat each variable as an *implicit* function of t, and so we take the derivative in the same way as we did with implicit derivatives. The only difference now is that this approach applies to every variable, not just y, and we're taking the derivative with respect to t, not x.

There are two parts of every related rates problem. The pre-calculus part involves finding the formulas to use, possibly more than one. The calculus part involves, taking the derivative $\frac{d}{dt}$, plugging in numbers and solving for the remaining quantity. Part of what students find confusing is *combining* these steps, so I want to de-confuse things and *separate* them. We'll start with three examples that involve only the pre-calculus side, then do one example that involves only taking derivatives, and finally do some examples that combine the steps.

Example 1. Imagine that a 10 m long ladder is leaning against a wall and starts to slip, so that the top of the ladder is going down, and the bottom of the ladder is moving out away from the wall.

(a) Find an equation that relates the positions x and y of each end the ladder.
(b) Discuss what happens to x and y as the ladder slides.
(c) What do can you say, without actually taking derivatives, about $\frac{dx}{dt}$ and $\frac{dy}{dt}$?

Example 2. Imagine that a balloon is being blown up.

(a) Find an equation that relates the volume of the balloon V, to the radius r.
(b) Discuss what happens to V and r as the balloon is blown up.
(c) What can you say, without actually taking derivatives, about $\frac{dV}{dt}$ and $\frac{dr}{dt}$?

Example 3. Imagine that gravel is moving along a conveyor belt, up a ramp, and falling into a pile that makes a cone.

(a) Find an equation that relates the volume of the pile, V, to the radius of the pile, r, and the height of the pile, h.
(b) Discuss what happens to V, r and h as the gravel is added.
(c) What can you say, without actually taking derivatives, about $\frac{dV}{dt}$, $\frac{dh}{dt}$ and $\frac{dr}{dt}$?

Example 4. In each of the following equations, treat each variable as a function of time, and take $\frac{d}{dt}$ of both sides of the equation.

(a) Volume of a cube: $V = x^3$.
(b) Area of a circle: $A = \pi r^2$.
(c) Area of a square: $A = x^2$.
(d) Area of a rectangle: $A = l \cdot w$.
(e) Volume of a cylinder: $V = \pi r^2 h$.
(f) Volume of a sphere: $V = \frac{4}{3}\pi r^3$.
(g) Lengths in a right triangle: $z^2 = x^2 + y^2$.
(h) Surface area of sphere: $S = 4\pi r^2$.
(i) Volume of a cone: $V = \frac{1}{3}\pi r^2 h$.
(j) Angle made by tracking an object: $\tan(\theta) = \frac{y}{x}$.

Example 5. A ladder that is 10 m long is leaning against a wall. The bottom end is 2 m out from the wall and starts to slide farther out at a speed of 1 m/s. How fast is the top sliding down?

Example 6. A balloon is being blown up with air at a rate of $9.3\,\text{in}^3/\text{s}$. Find how fast the radius is increasing when the volume is $1000\,\text{in}^3$.

Example 7. (c.f. Stewart, 3.9#4) A lecturer is positioning a projector in a large room. They start with the projector close to the screen and move it back. As they move it back, the rectangle of light that it makes grows larger. Suppose at a certain point in time, the length of the rectangle is 72 inches, the height is 48 inches, the length is increasing by $2\,\text{in}/\text{s}$ and the height is increasing by $1.3333\,\text{in}/\text{s}$. How fast is the area increasing at this moment?

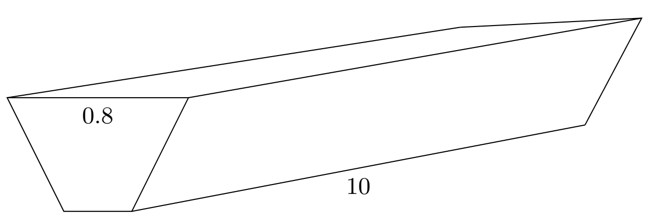
Example 8. (c.f. Stewart 3.9#20) A boat is being pulled into a dock by a rope connected to the front of the boat. The point where the rope connects to the boat is $2\,\text{m}$ higher than the point where the rope connects to the dock. The rope is being pulled in at a rate of $0.5\,\text{m}/\text{s}$. How fast is the boat approaching the dock, when the distance between the boat and the dock is $10\,\text{m}$?

Example 9. (Stewart 3.9#16)

A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m away from the building?

Here are some hints specific to the homework assignment.

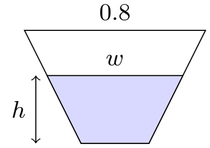
3.9 #25: We have a trough shaped as follows



The hard part is finding a formula for the volume V of the water as it fills the trough and has height h. The volume is given conceptually by

$$V = \text{area of water in cross section} \times 10$$

so now we need to figure out the area of water in a cross section, as pictured

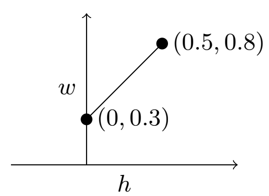


To do this we need to know the formula for the area of a trapezoid (look it up), and we need a formula for w in terms of h (actually, we don't *need* a formula for w, we can use related rates with both w and h there, but eventually we need to know how to find w when $h = 0.3$, so we might as well bite the bullet now).

To solve for w, note that we have two data points:

$$h = 0 \Rightarrow w = 0.3$$
$$h = 0.5 \Rightarrow w = 0.8$$

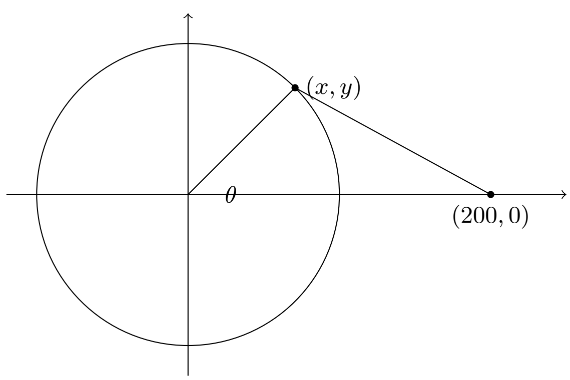
Thus, the formula for w gives a straight line through two points $(0, 0.3)$ and $(0.5, 0.8)$



Since this is straight line, it's easy to see that the formula for w is $w = h + 0.3$.

So, we now we have a formula for w in terms of h, then we have a formula for area in terms of h, then we have a formula for the volume of the water in terms of h, then we do related rates.

#43 You can set this problem up on an (x, y)-coordinate system, mark θ as shown



It's easy to see that $x = 100\cos(\theta)$ and $y = \sin(\theta)$, and therefore the distance D between the two people is given by

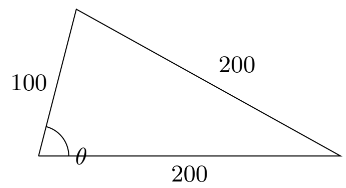
$$D = \sqrt{(100\cos(\theta) - 200)^2 + (100\sin(\theta))^2}$$

Actually, it might be nicer to use D^2 in the following equation

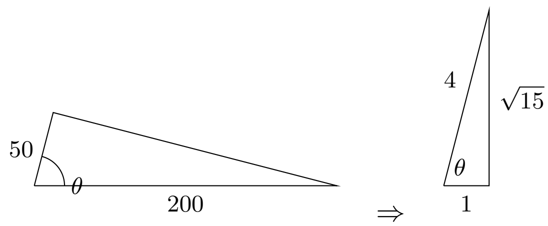
$$D^2 = (100\cos(\theta) - 200)^2 + (100\sin(\theta))^2$$

Now, we can take the time derivative of this, the result will involve D, $\dfrac{dD}{dt}$, θ and $\dfrac{d\theta}{dt}$. We know to plug in $D = 200$ and need to find $\dfrac{dD}{dt}$. We will also need to plug in θ and $\dfrac{d\theta}{dt}$.

Actually, we don't need to find θ, we need to find $\cos(\theta)$ and $\sin(\theta)$. Look at the triangle



If we cut this triangle in half to make a right triangle, and rescale it we see that $\cos(\theta) = \frac{1}{4}$ and $\sin(\theta) = \sqrt{15}/4$

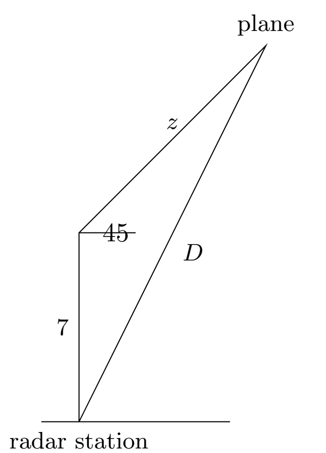


Finally, we need to know $\frac{d\theta}{dt}$. The key to this is the runner's speed. The speed is $7\,\mathrm{m/s}$. This translates into a speed along the circumference of the track. Recall the definition of θ in radians:

$$\theta = \frac{s}{r}$$

where s is arc-length and r is radius. In this case $r = 100$, so $\theta = \frac{1}{100}s$. Thus, $\frac{d\theta}{dt} = \frac{1}{100}\frac{ds}{dt}$, where $\frac{ds}{dt}$ is the speed of the runner.

WeBWorK #2. In one version of the problem, we have a plane flying over a radar station at an altitude of 7 km and at an angle of 45°. If z is the distance the plane has travelled from this position over the radar station, and D is the distance between the plane and the radar station, then we have the following picture:



Now, the Law of Cosines says that

$$D^2 = z^2 + 7^2 - 2 \cdot z \cdot 7 \cos(135)$$

Now you can take the time derivative of this. You need to plug in something for $\frac{dz}{dt}$, which is given, and for z (which you find by using $\frac{dz}{dt}$ and the amount of time that has gone by). Finally, you can find D by using the above equation.

Here's how the problem finishes, assuming a speed of 14 km/min and a travel time of 3 min

$$2D\frac{dD}{dt} = 2z\frac{dz}{dt} - 2 \cdot 7 \cdot \cos(135)\frac{dz}{dt}$$

$$\frac{dD}{dt} = \frac{(z - 7 \cdot \cos(135))\frac{dz}{dt}}{D}$$

After 3 minutes we have $z = 14 \cdot 3$ and $D = \sqrt{(14 \cdot 3)^2 + 7^2 - 2 \cdot 14 \cdot 3 \cdot 7 \cdot (135)}$ so

$$\frac{dD}{dt} = \frac{(14 \cdot 3 - 7 \cdot \cos(135))14}{\sqrt{(14 \cdot 3)^2 + 7^2 - 2 \cdot 14 \cdot 3 \cdot 7 \cdot \cos(135)}} \approx 13.922\,\text{km/min}$$

Chapter 4

Applications of Derivatives

4.1 Maximums and minimums

Definition. Let $x = c$ be in the domain of $f(x)$.

$x = c$ is an	absolute maximum	if ________________	for all x	
$x = c$ is a	local maximum	if ________________	for all x	.
		(c cannot be an endpoint)		
$x = c$ is an	absolute minimum	if ________________	for all x	.
$x = c$ is a	local minimum	if ________________	for all x	.
		(c cannot be an endpoint)		

We first learn to understand this definition by looking at graphs.

Example 1. Identify the local/absolute max/mins in the graph in textbook 4.1 figure 1 (p. 271). (Assume that the domain of the function is everything that appears in the graph, and nothing else.)

Theorem 1 (Fermat)**.** If $x = c$ is a local min/max then $f'(c) = 0$ or

.

□

Definition. If $f'(c) = 0$ or $f'(c)$ is undefined we call c a critical point.

Fermat's Theorem justifies our approach to finding mins and max's, which always starts with finding the critical points.

Theorem 2 (Absolute max/min test (aka "closed interval method"))**.** To find the absolute max/min of a function $f(x)$ on an interval $[a, b]$, do the following.

1. Find the critical points of $f(x)$ in the interval $[a, b]$ (i.e. find $f'(x)$, solve $f'(x) = 0$ and identify where $f'(x)$ is undefined).
2. List the $f(x)$-values (i.e. y-values) at a, b and at each critical point. The absolute max is the biggest y-value in the list. The absolute min is the smallest y-value in the list.

4.2 L'Hospital's Rule

Throughout this section, let "lim" be one of the following:

$$\lim = \lim_{x\to a}, \quad \lim_{x\to a^+}, \quad \lim_{x\to a^-}, \quad \lim_{x\to\infty}, \quad \lim_{x\to-\infty}$$

Then we have seen the following limits before

$\lim \frac{f(x)}{g(x)} = \underline{\hspace{4cm}}$ if we have $\frac{\#}{\#(\neq 0)}$ (i.e. both limits exist and $\lim g(x) \neq 0$)

$\lim \frac{f(x)}{g(x)} = \underline{\hspace{1cm}}$ if we have $\frac{\#}{\pm\infty}$ (i.e. $\lim f(x) = \# \neq \pm\infty$ and $\lim g(x) = \pm\infty$)

$\lim \frac{f(x)}{g(x)} = \underline{\hspace{2cm}}$ if we have $\frac{\#(\neq 0)}{0}$ (i.e. $\lim f(x)$ exists and $\neq 0$ and $\lim g(x) = 0$)

$\lim \frac{f(x)}{g(x)} = \underline{\hspace{2cm}}$ if we have $\frac{\pm\infty}{\#}$ (i.e. $\lim f(x) = \pm\infty$ and $\lim g(x) = \# \neq \pm\infty$)

Note: we are starting to do arithmetic with "extended real numbers" here. This means we work with the usual real numbers, $\mathbb{R}$, together with something we call infinity, ∞ (as well as $-\infty$). Here, ∞ is not a number, but we can still use some of our usual operations on it. So if r is a positive real number we have $r + \infty = \infty$, $r \cdot \infty = \infty$, $\infty - r = \infty$, $\frac{r}{\infty} = 0$, $\frac{r}{0} = \pm\infty$, etc. (See the Wikipedia article on the extended real number line for more information about this.) However, other operations we cannot do, at least not without some other method of calculation. For example, $\infty - \infty = ?$, $\frac{\infty}{\infty} = ?$, etc. These question marks can sometimes be filled in, but not always. We will learn the technique for filling them in now.

Theorem 1 (L'Hospital's Rule)**.** Suppose that $\lim \frac{f'(x)}{g'(x)}$ exists or equals $\pm\infty$.

If $\lim \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$ or $\lim \frac{f(x)}{g(x)} = \frac{0}{0}$ then

$$\lim \frac{f(x)}{g(x)} = \underline{\hspace{4cm}} .$$

Comments. Some things to keep in mind:

- It is crucial that you check the condition $\lim \frac{f(x)}{g(x)}$ *before* you use L'Hospital's Rule, and that this appear on your paper. It is crucial that if $\lim \frac{f'(x)}{g'(x)}$ does not exist, then you do not draw any conclusions about $\lim \frac{f(x)}{g(x)}$.
- You can use L'Hospital's Rule more than once, so that $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = \lim \frac{f''(x)}{g''(x)}$, etc.
- We often need to know a limit like $\lim_{x\to\infty} f(x)$. We will say things like "plug ∞ into $f(x)$" or "$f(\infty)$". There are two ways to justify this: you can work with the extended reals, as described above, or you should translate the phrase "plug in ∞" into "take the limit as x goes to ∞".

  In this manner, we can write $e^{\infty} = \infty$, $e^{-\infty} = 0$, $\ln(\infty) = \infty$, $\ln(0) = -\infty$, $\frac{1}{\pm\infty} = 0$, $\frac{1}{0} = \pm\infty$.

Example 1. Find $\lim_{x\to 0} \frac{\sin(x)}{x}$.

Example 2. Find $\lim_{x \to (\pi/2)^+} \dfrac{\cos(x)}{1 - \sin(x)}$

Example 3. Find $\lim_{x \to \infty} \dfrac{\ln(x)}{x}$.

Example 4. Find $\lim_{x \to 0} \dfrac{\cos(x)}{x}$.

Example 5. Find $\lim_{x\to\infty} \frac{x^2}{e^{0.01x}}$

Example 6. Find $\lim_{x\to 0} \frac{1-\cos(x^{10})}{x^{20}}$. Hint: Do LH once, simplify x's, and then do LH again. Compare your answer to the graph of $\frac{1-\cos x^{10}}{x^{20}}$.

Variations on L'Hospital's Rule.

- $\lim f(x)g(x) =$ ________ : rewrite as a fraction, and use LH.
  (Also we could have $0 \cdot (-\infty)$, and $\pm\infty \cdot 0$, etc. To rewrite $f(x)g(x)$ as a fraction, note that $f(x)g(x) = \dfrac{f(x)}{1/g(x)} = \dfrac{g(x)}{1/f(x)}$.)
- $\lim f(x) - g(x) =$ ___________ : rewrite as a fraction, and use LH.
  (To rewrite $f(x)-g(x)$ as a fraction may require creativity: maybe get a common denominator, maybe write it as $\dfrac{f(x)}{1} - \dfrac{g(x)}{1}$ and multiply by conjugate square roots, etc.).
- $\lim(f(x))^{g(x)} =$ _____ , ______ , _______ : Let $y = f(x)^{g(x)}$, take natural log, rewrite as a product, rewrite as a fraction, apply L'Hospital, get back to original limit using e: $\lim y = e^{\lim g(x)\ln(f(x))}$.

Example 7. Find $\lim_{x \to 0^+} x\ln(x)$

Example 8. Find $\lim_{x \to 1^+} \ln(x) \tan(\pi x/2)$.

Example 9. Find $\lim_{x \to \pi/2^-} (\sec(x) - \tan(x))$

Example 10. Find $\lim_{x \to 0^+} (\csc(x) - \cot(x))$.

Example 11. Find $\lim_{x \to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln(x)} \right)$ (Hint: get a common denominator, use LH, then simplify.)

Example 12. If you have a debt of P (in dollars), compounded n times per year, at an annual interest rate of r, then the amount you will owe after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

(a) Suppose you have \$100,000 of debt, at 7% interest, for 5 years. How much will you owe if it's compounded monthly? (b) Daily? (c) Compounded every instant?

Example 13. Find $\lim_{x\to 0^+} (1+\sin(4x))^{\cot(x)}$

Some Homework problems solutions

Example 14. [4.4#39] $\lim_{x\to\infty} x\sin(\pi/x)$

$\lim_{x\to\infty} x\sin(\pi/x) = \infty\sin(\pi/\infty) = \infty\cdot\sin(0) = \infty\cdot 0$✓

$\lim_{x\to\infty} \frac{\sin(\pi/x)}{1/x} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{\cos(\pi/x)(\frac{-\pi}{x^2})}{\frac{-1}{x^2}}$

(Simplify!)

$\lim_{x\to\infty} \frac{\cos(\pi/x)\pi}{1}$

$\cos(\pi/\infty)\pi = \cos(0) = \pi$

Note: you could have tried to turn this product into a fraction the other way, using the fact that $\frac{1}{\sin} = \csc$ and gotten this

$$\lim_{x\to\infty} \frac{x}{\csc(\pi/x)}$$

but then this wouldn't have helped, because the next step would have been

$$\stackrel{\text{LH}}{=} \lim_{x\to\infty} \frac{1}{-\csc(\pi/x)\cot(\pi/x)(\frac{-\pi}{x^2})} = \frac{1}{-\csc(0)\cot(0)0}$$

But $\csc(0) = \infty$, and so this didn't work, at least not without additional steps.

Example 15. [4.4#40] $\lim_{x\to-\infty} x^2e^x = (-\infty)^2e^{-\infty} = \infty \cdot \frac{1}{\infty} = \infty \cdot 0\checkmark$

Rewrite as a fraction, put e^x on the bottom as $1/e^x = e^{-x}$.

$\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$

$\stackrel{\text{LH}}{=}= \lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \frac{\infty}{-e^{-(-\infty)}} = \frac{\infty}{\infty}\checkmark$

Do L'Hospital's Rule again:

$\stackrel{\text{LH}}{=} \lim_{x\to-\infty} \frac{2}{e^{-x}} = \frac{2}{e^{-(-\infty)}} = \frac{2}{\infty} = 0$

Note: you could have tried putting x^2 on the bottom of the fraction. You'd get $\frac{e^x}{x^{-2}}$, and after applying L'Hospital's Rule, you'd get $\frac{e^x}{-2x^{-3}}$. This is still indeterminate, but note that if you apply L'Hospital's Rule again, it doesn't help. You started with x^{-2}, and then get x^{-3}, and then x^{-4}, etc. It's better to leave x^2 on top, so you can get rid of it by taking the derivative.

Example 16. [4.4#47] $\lim_{x\to1}\left(\frac{x}{x-1} - \frac{1}{\ln(x)}\right) = \frac{1}{1-1} - \frac{1}{\ln(1)} = \frac{1}{0} - \frac{1}{0} = \infty - \infty\checkmark$

Rewrite as a fraction:

$\lim_{x\to1} \frac{x\ln(x)}{(x-1)\ln(x)} - \frac{(x-1)}{(x-1)\ln(x)} = \lim_{x\to1} \frac{x\ln(x)-(x-1)}{(x-1)\ln(x)}$

$\stackrel{\text{LH}}{=} \lim_{x\to1} \frac{1\ln(x) + x\cdot\frac{1}{x} - 1}{1\ln(x) + (x-1)\cdot\frac{1}{x}}$

$= \lim_{x\to1} \frac{\ln(x)}{\ln(x)+1-\frac{1}{x}} = \frac{\ln(1)}{\ln(1)+1-\frac{1}{1}} = \frac{0}{0}\checkmark$

$\stackrel{\text{LH}}{=} \lim_{x\to1} \frac{\frac{1}{x}}{\frac{1}{x}+0+\frac{1}{x^2}} = \frac{1}{1+1} = \frac{1}{2}$

Example 17. [4.4 #49] $\lim_{x\to\infty}(\sqrt{x^2+x} - x) = \sqrt{\infty} - \infty = \infty - \infty\checkmark$

$= \lim_{x\to\infty}(\sqrt{x^2+x}-x)\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty}\frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x} = \lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}$

Divide the top and bottom by the largest simplified power of x from the bottom. This is x since we have $x = \sqrt{x^2}$.

$\lim_{x\to\infty}\frac{\frac{1}{x}\cdot x}{\frac{1}{x}(\sqrt{x^2+x}+x)} = \lim_{x\to\infty}\frac{1}{\sqrt{\frac{1}{x^2}(x^2+x)}+\frac{1}{x}\cdot x}$

$\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$

Note: We didn't do L'Hospital's rule here for two reasons. First of all, we know how to do this problem without L'Hospital's rule: this is straight from section 2.6. Secondly, if we did L'Hospital's rule, it doesn't work. When you take the derivative of the bottom, part of what you will get is the derivative of the square root, which is $\frac{x}{\sqrt{x^2+x}}$. This part alone is of the form $\frac{\infty}{\infty}$, and so you would need to do L'Hospital's rule again. Then you will get $\frac{1}{\frac{x}{\sqrt{x^2+x}}} = \frac{\sqrt{x^2+x}}{x}$. This is again of the form $\frac{\infty}{\infty}$, and so you would need to do L'Hospital's rule again. But doing it again just starts the cycle over again with the square root.

Example 18. [4.4 #55] $\lim_{x\to 0}(1-2x)^{1/x} = (1-0)^{1/0} = 1^{\infty}\checkmark$.

$y = (1-2x)^{1/x}$

$\ln(y) = \frac{1}{x}\ln(1-2x)$

$\lim_{x\to 0}\frac{1}{x}\ln(1-2x) = \lim_{x\to 0}\frac{\ln(1-2x)}{x}$

$\overset{\text{LH}}{=} \lim_{x\to 0}\frac{\frac{1}{1-2x}(-2)}{1} = \frac{\frac{-2}{1-0}}{1} = -2$

e^{-2}

Example 19. [4.4, #64] $\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1} = \left(\frac{2}{2}\right)^{\infty} = 1^{\infty}\checkmark$

$y = \left(\frac{2x-3}{2x+5}\right)^{2x+1}$

$\ln(y) = (2x+1)\ln\left(\frac{2x-3}{2x+5}\right) = (2x+1)(\ln(2x-3) - \ln(2x+5))$

$\lim_{x\to\infty}(2x+1)(\ln(2x-3) - \ln(2x+5)) = \lim_{x\to\infty}\frac{\ln(2x-3)-\ln(2x+5)}{\frac{1}{2x+1}}$

$\overset{\text{LH}}{=} \lim_{x\to\infty}\frac{\frac{1}{2x-3}(2) - \frac{1}{2x+5}(2)}{-(2x+1)^{-2}(2)}$

Cancel the three (2)'s, and rewrite the big fraction, so that instead of dividing by $-(2x+1)^{-2}$, multiply by its reciprocal.

$= \lim_{x\to\infty}\left(\frac{1}{2x-3} - \frac{1}{2x+5}\right)(-(2x+1)^2)$

$= \lim_{x\to\infty}\left(\frac{(2x+5)-(2x-3)}{(2x-3)(2x+5)}\right)\frac{-(4x^2+4x+1)}{1} = \lim_{x\to\infty}\frac{8}{4x^2+4x-15}\cdot\frac{-(4x^2+4x+1)}{1}$

$= \lim_{x\to\infty} -8\cdot\frac{4x^2+4x+1}{4x^2+4x-15} = -8\cdot\frac{4}{4} = -8$

e^{-8}

Note: I used twice the short-cut rule for horizontal asymptotes of rational functions (see page 31). Namely: given the same powers of x on the top and bottom of the fraction, the limit as x goes to infinity, i.e. the horizontal asymptote, is the ratio of the leading coefficients.

4.3 Function analysis

In a sense, this section has nothing new in it, it's a combination of our previous work. The point is to analyze a function using calculus, and then to make a sketch of its graph.

To sketch the graph of a function $f(x)$, find (some of) the following:

$f(x)$ **parts**

- Symmetry:
  - ∗ if ____________________ then $f(x)$ is *even*. This means that the graph to the left of the y-axis is the the right side. Note: $x = 0$ will be a local max or min. Examples of even functions: or any power of x.
  - ∗ if __________________ then $f(x)$ is *odd*. This means that the graph to the left of the y-axis is the the right side. Note: the graph will go through the origin (unless $x = 0$ is a vertical asymptote). Examples of odd function: or any power of x.
  - ∗ Note: even×even=even, even×odd=odd, odd×odd = even (so multiplying functions is like adding even/odd numbers). Thus, $x^2 \sin(x)$ will be odd, and $x^4 \cos(x)$ will be even. Also, if g = even is even then $f(g(x))$ = even for any function f.
- if $f(x)$ is periodic (like , , etc.), then we only need to analyze it for one full (like $[0, 2\pi]$).
- x and y intercepts.
- Vertical and horizontal asymptotes.
- intervals of positivity/negativity.

$f'(x)$ **parts** Find the points, (including y-values), intervals of .

$f''(x)$ **parts** Find when $f''(x)$ equals ___ or is , find the intervals of and points.

Note: Not all of the above will be needed in every problem. For instance, finding symmetry is not absolutely necessary, but can save time.

Example 1. Analyze the function xe^{-x^2} and graph the result by hand.

Example 2. Analyze $f(x) = \dfrac{1}{x^8} - \dfrac{2 \times 10^8}{x^4}$, use your calculator where necessary, but graph the result by hand.

Example 3. [Extra example: not covered in class] Analyze the function $\dfrac{x}{x^3 - 1}$.

Example 4. [4.5,#11] $f(x) = \dfrac{1}{x^2-9}$.

$f(x)$-stuff.

x-int: $0 = \dfrac{1}{x^2-9}$, no solution.

y-int: $f(0) = \dfrac{1}{0-9} = -\dfrac{1}{9}$

Symmetry: even.

HA: $y = 0$ since x^2 is on the bottom and there are no x's on top.

VA: $x = \pm 3$ since this gives division by 0.

f pos/neg:

-3		3
$f(-4) = +$	$f(0) = -$	$f(4) = +$

$f'(x)$-stuff.

$f'(x) = -\dfrac{2x}{(x^2-9)^2}$

$f'(x) = 0$ at $x = 0$.

$f'(x)$ DNE at $x = \pm 3$.

f' pos/neg

-3	0	3	
$f'(-4) = +$	$f'(-1) = +$	$f'(1) = -$	$f'(4) = -$

$x = 0$ is l.max, $y = -\dfrac{1}{9}$.

$f\uparrow$: $(-\infty, -3) \cup (3, 0)$

$f\downarrow$: $(0, 3) \cup (3, \infty)$

$f''(x)$-stuff

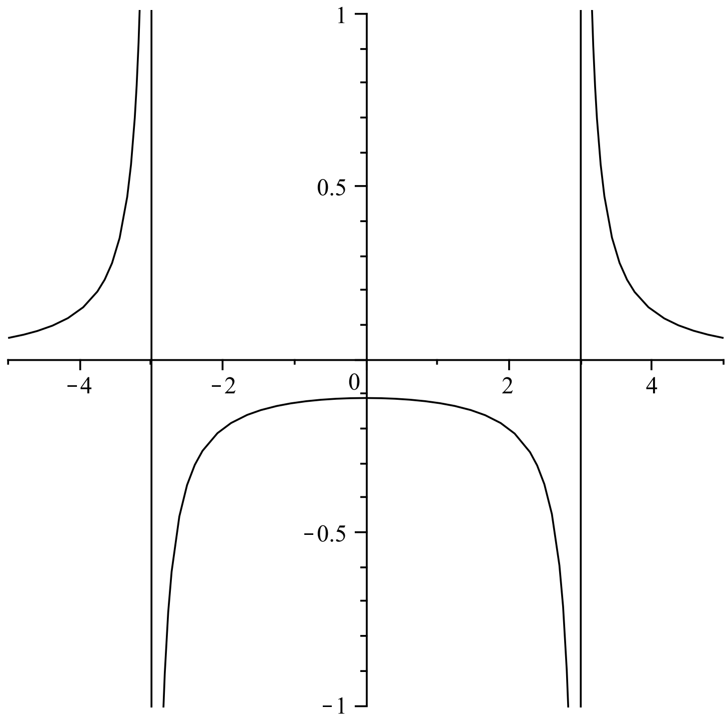
$f''(x) = \dfrac{6(x^2+3)}{(x^2-9)^3}$

$f''(x) = 0$: no solution.

$f''(x)$ DNE at $x = \pm 3$.
$f''(x)$ pos/neg

-3	3	
$f''(-4) = +$	$f''(0) = -$	$f''(4) = +$

f is C.U. $(-\infty, 3) \cup (3, \infty)$
f is C.D. $(-3, 3)$
Putting it all together:



Example 5. [4.5 #31] $f(x) = 3\sin(x) - \sin^3(x)$
y-int: $f(0) = 0$.
x-int: $f(x) = 0$, $0 = \sin(x)\big(3 - \sin^2(x)\big)$, $\sin(x) = 0$, $x = 0, \pi, 2\pi, 3\pi$, etc.
Symmetry: $f(x)$ is odd.
Periodic: yes. So, if we want, we can do only work from 0 to 2π, and then extend our results.
VA: none.
HA: none.
f pos/neg (between 0 and 2π)

0	π	2π
$f(1) = +$	$f(4) = -$	

$f'(x)$ stuff.
$f'(x) = 3\cos(x) - 3\sin^2(x)\cos(x)$
$0 = 3\cos(x) - 3\sin^2(x)\cos(x)$
$0 = 3\cos(x)(1 - \sin^2(x))$
$\cos(x) = 0$ OR $1 - \sin^2(x) = 0$

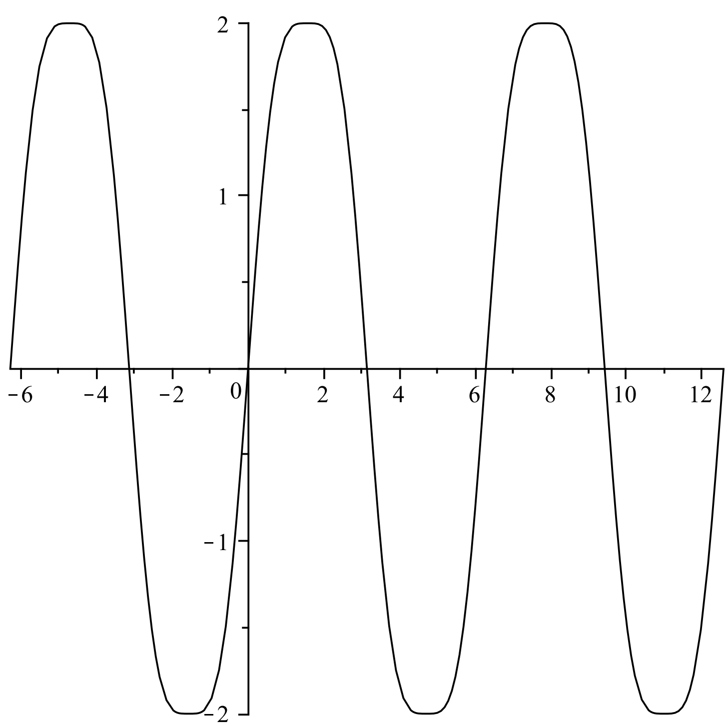
$x = \pi/2,\ 3\pi/2,\ 5\pi/2,\ 7\pi/2, \ldots$
$f'(x)$ pos/neg (between 0 and 2π)

0	$\pi/2$	$3\pi/2$	2π
$f'(0.5) = +$	$f'(3) = -$	$f'(5.5) = +$	

$x = \pi/2$ is l. max, $y = f(\pi/2) = 2$
$x = 3\pi/2$ is l.min, $y = f(3\pi/2) = -2$
$f \uparrow$: $(0, \pi/2) \cup (3\pi/2, 2\pi)$
$f \downarrow$: $(\pi/2, 3\pi/2)$
$f''(x)$-stuff
$f''(x) = -3\sin(x) - 6\sin(x)\cos^2(x) + 3\sin^3(x)$
$f''(x) = 3\sin(x)\big(1 - 2\cos^2(x) + \sin^2(x)\big)$
$f''(x) = -9\sin(x)\cos^2(x)$ (using the identity $\sin^2(x) + \cos^2(x) = 1$)
$0 = -9\sin(x)\cos^2(x)$ means $\sin(x) = 0$ OR $\cos^2(x) = 0$.
$x = 0, \pi/2, \pi, 3\pi/2, 2\pi, 5\pi/2, \ldots$

0	$\pi/2$	π	$3\pi/2$	2π
$f''(0.5) = -$	$f''(2.5) = -$	$f''(3.5) = +$	$f''(5.5) = +$	

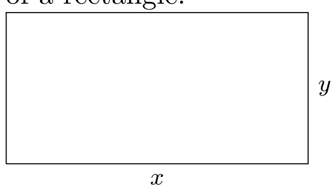
Inflection points: $x = \pi$
f C.U. $(\pi, 3\pi/2) \cup (3\pi/2, 2\pi)$
f C.D. $(0, \pi/2) \cup (\pi/2, \pi)$
Putting it all together:



4.4 Optimization

The basic idea of this section is to find absolute max/mins of real-world problems. Sometimes you are given a single function to optimize, but most often you are asked to combine two functions.

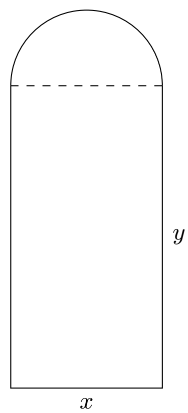
Example 1. (a). Find formulas for the following the area A and the perimeter P of a rectangle.



(b). Set $A = 100$ and solve for y. Plug this into the formula for P and simplify.
(c). Find the minimum of P (include x and y values).

Example 2. (a). Find formulas for the area A and the cost C of a rectangular fence which costs \$30 per meter on the north and south sides, and \$50 per meter on the east and west sides.
(b). Set $C = \$5000$ and solve for y. Plug this into the formula for A and simplify.
(c). Find the maximum of A (include x and y values).

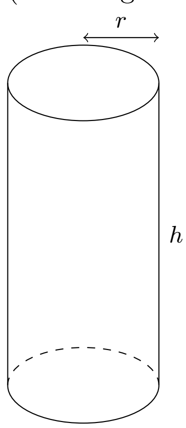
Example 3. (a). Find formulas for the area A and the perimeter P of the shape below (which is a rectangle with a half-circle on top. The perimeter does not include the dashed line between the circle and the rectangle).



(b). Set $P = 10$ and solve for y. Plug this into the formula for A and simplify.

(c). Maximize A (include x and y values)

Example 4. (a). Find formulas for the area A and the volume V of the cylinder (including the top and bottom) below.



(b). Set $A = 1$ and solve for h. Plug this into the formula for V and simplify.
(c). Maximize V (include h and r values).

4.9 Anti-derivatives

Definition. An **anti-derivative** of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. As an abbreviation for "the anti-derivative of $f(x)$" we write $\int f(x)\,dx$. Note that both "$\int$" and "dx" are part of this notation, they are somewhat like parentheses (and), the important stuff goes between them. Also, dx plays the role of telling us what variable we are using for the anti-derivative, just like $\frac{d}{dx}$ does for the derivative.

Example 1. Find an anti-derivative of $3x^2$ and verify this.

Theorem 1. If $F(x)$ is an anti-derivative of $f(x)$, then $F(x) + C$ is also an anti-derivative for each constant C.

□

Example 2. Find three anti-derivatives of $2\sin(x)$.

Theorem 2. If $F(x)$ and $G(x)$ are both anti-derivatives of $f(x)$, then $G(x) = F(x) + C$ for some constant C.

□

As a result of the previous theorem, we always write anti-derivatives using $+C$ at the end of the formula.

Example 3. Turn each of the following derivative formulas around, and make a formula for the indicated anti-derivative:

$$\frac{d}{dx}x^n = nx^{n-1} \implies \int nx^{n-1}\,dx = \rule{3cm}{0.4pt}$$
$$\frac{d}{dx}e^x = e^x \implies \int e^x\,dx = \rule{3cm}{0.4pt}$$
$$\frac{d}{dx}\sin(x) = \cos(x) \implies \int \cos(x)\,dx = \rule{4cm}{0.4pt}$$
$$\frac{d}{dx}\cos(x) = -\sin(x) \implies \int -\sin(x)\,dx = \rule{4cm}{0.4pt}$$
$$\frac{d}{dx}\tan(x) = \sec^2(x) \implies \int \sec^2(x)\,dx = \rule{4cm}{0.4pt}$$
$$\frac{d}{dx}\ln(x) = \frac{1}{x} \implies \int \frac{1}{x}\,dx = \rule{4cm}{0.4pt}$$

Sometimes we want to write the anti-derivative formula differently. For instance, we want to know the anti-derivative of $\sin(x)$, not $-\sin(x)$, and similarly we want to know the anti-derivative of x^n, not nx^{n-1}. This is easy to do once we use the constant multiple rule:

$$\frac{d}{dx}Cf(x) = Cf'(x) \Rightarrow \int Cf(x)\,dx = \rule{5cm}{0.4pt}$$

Now it's easy to rewrite the formulas for $\sin(x)$ and x^n:

$$\int \sin(x)\,dx = \rule{3.5cm}{0.4pt}$$

$$\int x^n\,dx = \rule{2.5cm}{0.4pt} \qquad \text{if } \rule{3cm}{0.4pt}$$

Example 4. Find the anti-derivative $F(x)$ of $f(x) = x^7 - 5\sin(x)$ such that $F(0) = 7$.

One of the most important types of derivatives and anti-derivatives involve position, velocity and acceleration. Since we know

$$\text{velocity} = \frac{d}{dt}\,\text{position} \qquad \text{and} \qquad \text{acceleration} = \frac{d}{dt}\,\text{velocity}$$

this means, by definition, that

$$\text{velocity} = \int \text{acceleration}\,dt \qquad \text{and} \qquad \text{position} = \int \text{velocity}\,dt$$

Example 5. On the moon, acceleration due to gravity is given by $-1.6\,\mathrm{m/s^2}$. A ball is thrown straight up (at time $t = 0$) with a velocity of $0.5\,\mathrm{m/s}$ from a height of $2\,\mathrm{m}$. (a) Find the velocity function $v(t)$. (b) Find the height function $h(t)$.

Chapter 5

The Definite Integral

5.1 The Definite Integral

Definition. Let $f(x)$ be a function defined on the interval $[a, b]$. We define the following

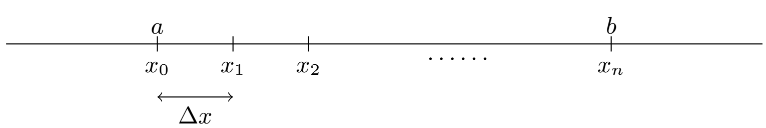
$$\underbrace{\qquad\qquad}_{\text{the symbol}} = \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{the name}} = \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{the formula}}$$

$$= \underbrace{\qquad\qquad\qquad\qquad}_{\text{the formula in } \Sigma \text{ notation}}$$

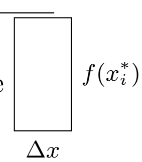
with the details about Δx and each x_i^* given below.

Details:

- We divide the interval $[a, b]$ into n equal pieces, each of width $\Delta x =$ __________



  where $x_0 = a$, $x_1 =$ ____________ , $x_2 =$ ______________ , $\ldots$, $x_i =$ ______________ , $\ldots$, $x_n =$ ____________________

- $f(x_i^*)\Delta x$ is the (signed) area of the i^{th} rectangle 

- x_i^* is any x-value chosen in the i^{th} interval. In the limit above, as n goes to infinity, it does not matter how you choose each x_i^*.

Comments.

- We often use a finite value of n to approximate the infinite infinite limit in the preceding definition. For a finite value of n, we usually use one of the following procedures to pick x_i^*

LHS (Left Hand Sum):	$x_i^* = x_{i-1}$, the left edge of the i^{th} rectangle.
RHS (Right Hand Sum):	$x_i^* = x_i$, the right edge of the i^{th} rectangle.
MP (Midpoint):	$x_i^* = \dfrac{x_{i-1}+x_i}{2}$, the x-value half way between the left edge and the right edge of the i^{th} rectangle.

- Interpretations: If $f(x) \geq 0$, then $\int_a^b f(x)\,dx$ is the area between $f(x)$, the x-axis and between $x = a$ and $x = b$. If $f(t)$ is velocity then $\int_a^b f(t)\,dt$ is the net distance traveled from $t = a$ to $t = b$. In general, $\int_a^b f(x)\,dx$ is the signed area.
- We will soon find $\int_a^b f(x)\,dx$ by taking the anti-derivative of $f(x)$, but this is not the definition of $\int_a^b f(x)\,dx$.
- We call the sum $f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x$ a Riemann sum with n steps.
- Many applied problems have an integral as their answer. Often, these problems are figured out first using a sum of terms, this sum is then turned into $\sum$ notation, and this is then turned into an integral. This thought process is shown in Figure 5.1.

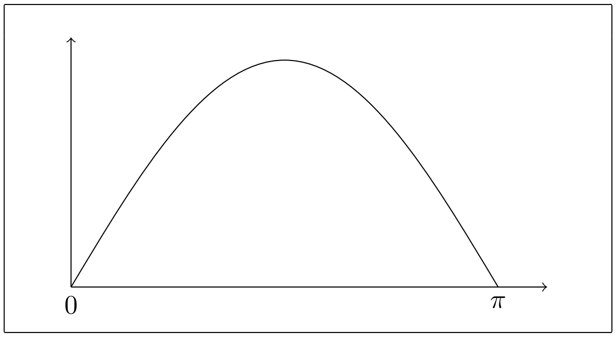
Example 1. Approximate the area between the x-axis and the curve $y = \sin(x)$ from $x = 0$ to $x = \pi$ using a Midpoint Riemann sum with $n = 6$, write your answer using integral notation, and draw a picture illustrating the Riemann sum.

Figure 5.1: Turning an applied problem into an integral

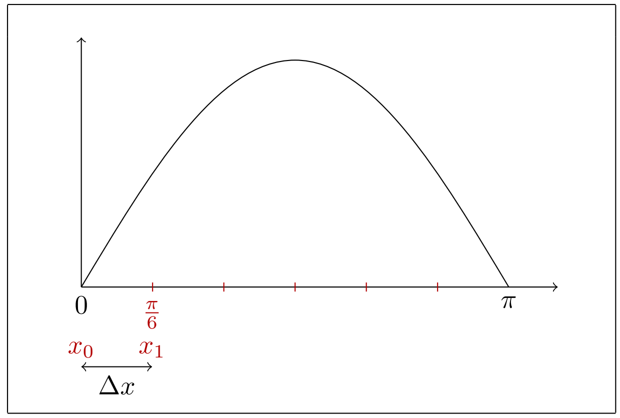
Initial work	A formula involving a sum of things, each of which involves a function $f(x)$ multiplied by something else of the form Δx
	$\rightsquigarrow$
Introduce $\sum$ notation	$\displaystyle\sum_{i=0}^{n} f(x_i^*)\Delta x$
	$\rightsquigarrow$
Take limit	$\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} \quad f(x_i^*) \quad \Delta x$
	$\rightsquigarrow \quad \rightsquigarrow \quad \rightsquigarrow$
This equals integral	$\displaystyle\int_a^b \quad f(x) \quad dx$

The next page shows a complete walk through of many steps in a Riemann Sum.

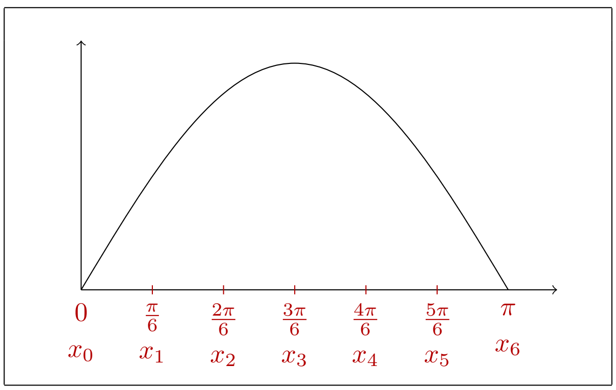
Walking through the 6 step Midpoint Rule Riemann Sum for $\int_0^\pi \sin(x)\,dx$



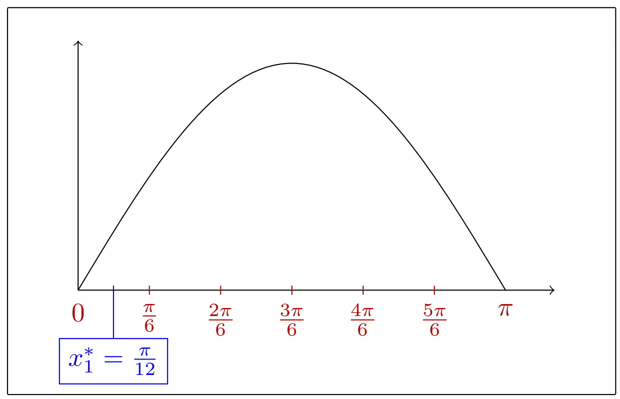
We start by drawing the curve $\sin(x)$ from 0 to π. The picture doesn't have to be accurate, but it helps see how the interval on the x-axis will be broken up.



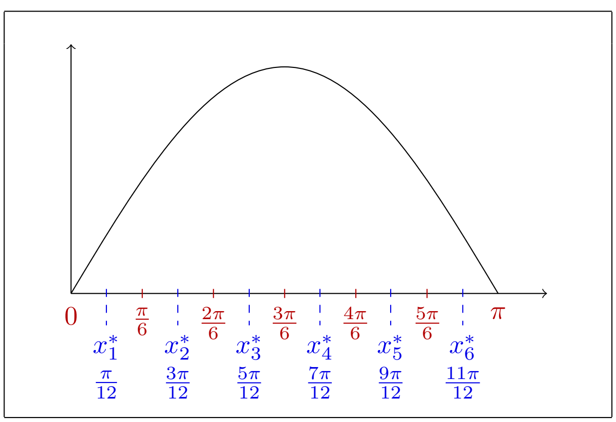
Now, if we break the whole interval into 6 equal pieces, it's easy to see that each piece should have length $\Delta x = \frac{\pi}{6}$. Thus, $x_0 = 0$ and $x_1 = \frac{\pi}{6}$



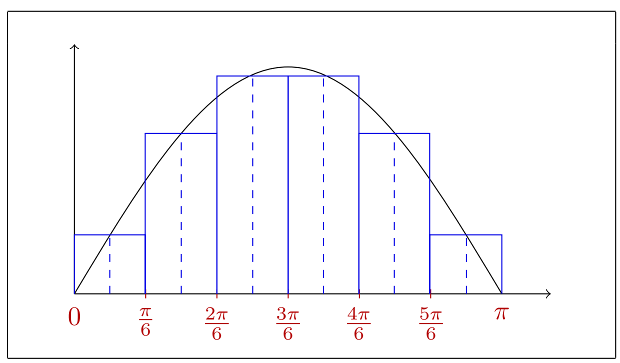
Now, we figure out the next x-value by adding Δx to x_1. Thus, $x_2 = \frac{2\pi}{6}$, $x_3 = \frac{3\pi}{6}$, etc.



Now we need to figure out the first midpoint. This is halfway between 0 and $\frac{\pi}{6}$. With a little thought, you should see that the halfway point is $\frac{\pi}{12}$. In other words, you have $x_1^* = \frac{\pi}{12}$.



The second midpoint is $\frac{3\pi}{12}$. There are two ways to calculate this: take the average of $\frac{\pi}{6}$ and $\frac{2\pi}{6}$. Or, add Δx to the first midpoint, $\frac{\pi}{12}$. Note that in the second way, we have $\Delta x = \frac{\pi}{6} = \frac{2\pi}{12}$, and so the same pattern applies to all the midpoints: just keep adding 2π to the numerator to get the next midpoint.

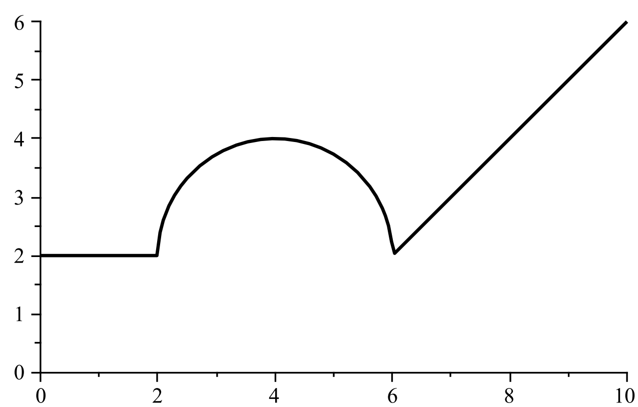


Now we draw rectangles, where the left and right edge come from x_0, $x_1, \ldots,$ x_6, but the top edge has height of $\sin(x_1^*)$, $\sin(x_2^*), \ldots,$ $\sin(x_6^*)$.

The sum of the areas of these rectangles is given by
$\sin(\pi/12) \cdot \frac{\pi}{6} + \sin(3\pi/12) \cdot \frac{\pi}{6} + \sin(5\pi/12) \cdot \frac{\pi}{6} + \sin(7\pi/12) \cdot \frac{\pi}{6} + \sin(9\pi/12) \cdot \frac{\pi}{6} + \sin(11\pi/12) \cdot \frac{\pi}{6}$

Example 2. Approximate the area between the x-axis and the curve $y = \sqrt{1-x^2}$ from $x = -1/2$ to $x = 1/2$ using a Left Hand Riemann, Midpoint, and Right hand Riemann sums with $n = 5$ and write your answer using integral notation.

Example 3. L $\int^{10}$ $f(x)\,dx$ exactly.



5.2 The Fundamental Theorem of Calculus

Example 1. Let $F(x) = \int_0^x 2t + 1\,dt$.

Find $F(0)$, $F(1)$ and $F(2)$ (you may want to know the area of a trapezoid is $\frac{1}{2}b(h_1 + h_2)$). Find $F(2) - F(1)$. What is the geometric description of $F(2) - F(1)$?

Theorem 1 (Fundamental Theorem of Calculus I)**.** Let $f(x)$ be a continuous function defined on $[a, b]$ and define

$$F(x) = \int_a^x f(t)\,dt$$

Then $F(x)$ is an anti-derivative of ________ . In other words, $\frac{d}{dx}F(x) =$ ________ .

One thing that students often find confusing in the previous theorem is the use of two variables: x at the top of the integral sign $\int_a^x$, and t in $f(t)\,dt$. In some sense, this is a technical point that you don't need to worry about too much, but here's the issue. The variable x states how far to the right we are taking the area. The variable t is used to define the curve. They are playing different roles, as shown by the previous example where x.

On the other hand, this technical point is often something students don't understand, and ignore, and it ends up causing no problem! So my advice is to think about, try to understand it, and then move on without worries.

$\square$

Comments. The above proof justifies the correctness of the Fundamental Theorem of Calculus I, but it doesn't explain it much. Here's an intuitive explanation. Suppose you define a function $F(x)$, that measures the amount of area, up to some value x, under a curve defined by $f(t)$. The rate of change of $F(x)$ is how much the area is changing at the point x; this is given by the height of the function $f(t)$ at $t = x$, i.e. $f(x)$. The bigger $f(x)$ is, the more the area is changing, and the smaller $f(x)$ is, the less the area is changing. Finally, this makes it clear why the value a doesn't appear in the theorem: whatever the height $f(a)$ is, it doesn't tell you how much the area is changing at x.

Example 2. (a) Let $F(x) = \int_1^x \sin(t^2)\,dt$. Find the derivative of $F(x)$.

(b) Let $F(x) = \int_1^{7x+1} \sin(t^2)\,dt$. Find the derivative of $F(x)$.

Example 3. Find $F'(x)$ where $F(x) = \int_2^{x^2} \frac{\ln(t)}{t}\,dt$.

Theorem 2 (Fundamental Theorem of Calculus II)**.** Let $f(x)$ be a continuous function defined on $[a, b]$. Then

$$\int_a^b f(x)\,dx = \underline{\hspace{4cm}}$$

where $F(x)$ is any anti-derivative of $f(x)$.

Comments. Notation: we usually write $F(x)\Big|_a^b$ as an abbreviation for $F(b) - F(a)$.

□

Example 4. Find $\int_0^{\pi} \sin(x)\,dx$.

Example 5. Find $\int_1^2 \frac{2}{x} - 7\sqrt[5]{x^3}\,dx$.

5.3 The Net Change Theorem

Recall that FTCII had two functions $f(x)$ and $F(x)$ where $F(x)$ is an anti-derivative of $f(x)$. This means that $f(x) = F'(x)$. If we write the theorem this way, it tells us about the net change of $F(x)$.

Theorem 1. The net change of a function $F(x)$ on an interval $[a, b]$ is defined as $F(b) - F(a)$. If we know the derivative $F'(x)$, then the net change can be calculated as

$$F(b) - F(a) = \int_a^b \underline{\hspace{3cm}} \quad dx$$

Example 1. 5.4#61. Recall our linear density example from section 3.7. We have a rod with mass, and $m(x)$ is the amount of mass from position 0 to position x in the rod. Then $\rho(x)$ is the derivative of $m(x)$, and we call $\rho(x)$ the linear density.

Suppose that $\rho(x) = 9 + 2\sqrt{x}$ kg/m for a rod of length 4 m. Find the total mass of the rod.

Example 2. Let $v(t) = \dfrac{\sqrt{t}}{16}$ be a velocity function. Find the total distance travelled from $t = 0$ to $t = 60$.

5.4 U-substitution

Example 1. (a) Find the derivative of $\sin(x^2)$.

(b) Find the anti-derivative $\int x\cos(x^2)\,dx$.

Comments. In the previous example, part (a) was the chain rule, and then part (b) was essentially doing the chain rule backwards. The main idea of the technique we learn next is to make a system for doing the chain rule backwards.

Here is a brief outline of the technique of U-substitution.

0. You are given an integral $\int h(x)\,dx$ where $h(x)$ is some complicated function of x.
1. Fill in the following

$$u = \qquad\qquad\qquad\qquad \text{(you get to pick this)}$$

$$du = (\qquad\qquad\qquad\qquad\qquad) \cdot dx \qquad \text{(you don't get to pick this)}$$

Most often you pick u to equal

.

2. Fill in the following

$$\int h(x)\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \int f(u)\,du.$$

Make sure that all the x's (including dx) cancel by the last step; whatever you're left with, call it $f(u)$.

3. Find the anti-derivative

$$\int f(u)\,du = F(u)$$

4. Inside of $F(u)$, replace u with the same "something" involving x, that you picked in step 1.

Example 2. (a) Find the anti-derivative $\int \sin(3x+7)\,dx$.

(b) Find the anti-derivative $\int x^7 \sin(x^8)\,dx$.

(c) Find the anti-derivative $\int e^x \sin(e^x)\,dx$.

Example 3. (a) Find the anti-derivative $\int \sin(3x+7)\,dx$.

(b) Find the anti-derivative $\int x^7 \sin(x^8)\,dx$.

(c) Find the anti-derivative $\int e^x \sin(e^x)\,dx$.

Example 4. Find $\int \tan(x)\,dx$. (Note: this gives us the anti-derivative of one of our last basic functions.)

Example 5. Find $\int \frac{\cos(\ln(x))}{x}\,dx$

Example 6. Find $\int \frac{7\sin(x)\cos(x)}{1+\cos^2(x)}\,dx$

Appendix A

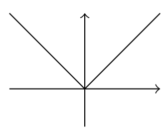
Absolute values

Here is a list of the basic properties of absolute values.

1. Absolute value of x is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

2. The easy way to think of $|x|$: just make x positive!
3. The graph of the absolute value is



4. Absolute value of $x - y$ equals

$$|x - y| = \begin{cases} x - y & \text{if } x \geq y \\ y - x & \text{if } x < y \end{cases}$$

5. The easy way to think of $|x - y|$: the distance between x and y.
6. Algebraic properties of absolute value:

$$a|x - y| = |ax - ay| \quad \text{if } a > 0$$

7. Solving $|x - y| < a$:

$$\begin{aligned} |x - y| < a \text{ is equivalent to } & -a < x - y < a \\ \text{is equivalent to } & y - a < x < y + a \end{aligned}$$