Geometry:

Euclidean

and

Noneuclidean

by

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2020
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Chapter 1

Constructions

1.1 Worksheet on Elementary Constructions
Worksheet on Elementary Geometric Constructions

MA 431 Geometry

January 15, 2020

Goals

After going through this whole worksheet, and the discussions that accompanied it, and getting feedback on some of the work you submitted for it, you should be able to answer the following questions.

1. What are the rules of elementary geometric constructions?
2. What tools are examples of tools we are allowed to use for constructions?
3. Where did Geometry originate?
4. What are some of the basic Geometric facts that we need to know to show that our constructions are correct?
5. What is a list of five, or ten, basic constructions that we know how to do?
6. Can we use string constructions to make more complicated geometric shapes?
1 Geometry on the Ancient Nile

Task 1. Form into groups, two people per group (one group can have three people).

The year is sometime around around 4000 B.C. Kanofer lives on the banks of the river Nile with his family, farming the same piece of land that he was born on, that his father worked before him, and that his children and grandchildren will someday work. One week ago the fields emerged from under the water. Everyone has been working hard to reclaim their land and prepare for the next planting. Erased boundaries are recreated, puddles are drained, flotsam and errant stones removed, swells leveled, boundary markers placed, and furrows dug, in endless hours pushing through hot muddy air. Towards the end of the reclaiming period, after most of the furrows have been dug, Kanofer becomes convinced that his neighbor Ahmes has taken some land that had been in Kanofer’s field before the inundation. He thinks that Ahmes has moved the line that separates the two farms.

This is a serious matter, the family’s food and welfare depend on their fields, and so Kanofer approaches Ahmes, and with barely contained anger accuses him of moving the line. Ahmes rejects the accusation and takes Kanofer to the point where the river meets the boundary between their fields. He points to a tree, about a fifth of a mile from the river: “Since the time of our fathers, a person can stand here, one foot in each field, and raise his eyes straight along the boundary, until he was looking at that tree over there.”

But Kanofer claims that the line Ahmes has made is not equally disposed between their fields: it hits the river at such an angle as to enlarge Ahmes’s field at the expense of Kanofer’s. Kanofer walks into Ahmes’s field, crushing furrows as he goes, and says “This is where the boundary should be!”

Ahmes replies, “Kanofer, surely it is not fair to curve your field into mine?”

Kanofer says with scorn “Ahmes, the boundary should be a straight line from the tree to this point.”

The two men argue further about where the point should be placed along the river that will define the boundary between their fields. What they want is to place this point so that the angle made by the boundary between the fields and the river, is the same angle for each farmer. Eventually, they consult the wise woman Khrehduonkh for a fair way to find this point. She gives them a solution, that is simple, easy to understand, and requires only a lot of rope: about a quarter mile of it.

Can you find her solution? (See the picture on the following page. You can use string instead of rope.)

How many geometric concepts are involved in this problem?

Be prepared to write your answers on the board and discuss them with the rest of the class.

---

1This story is set shortly after the development of agriculture, the widespread specialization of labor, the invention of the wheel and of writing, and about 1000 years before the time Pharaohs and the first monumental pyramids. This is probably the time that math as an independent subject came into existence. Certainly, by the time of the pyramids, mathematics was developed enough to include the four operations of arithmetic, formulas for area and volumes of various shapes, as well as some range of simple geometric constructions.

Historically, Kanofer and Khrehduonkh are the father and mother of Imhotep. Imhotep lived around 2950 B.C. and is sometimes called the first person in history, for he is the first person who is both nonfictional and remembered for his accomplishments rather than being a king. He was the head chancellor for the pharaoh Djoser (aka Zojer), architect for the first great pyramid and high priest. In later centuries he was rumored to be a doctor, and basically master of all knowledge.
Figure 1: Kanofer and Ahmes’ fields
The tree in the middle of the image is the one they are basing their discussion upon.
(Modern image of a tree on the banks of the Nile, at location 26.061071, 32.767309)
Problem 1. For tomorrow:
(a) Make a list of all the geometric terms we used in this task (these are the terms we will have to define later more carefully).
(b) Give a complete solution of the main problem, and solve it, using purely geometric language (no rivers, no trees).
   “Complete solution” means the following: State carefully and completely what you are given, what the goal of the construction is, what the steps of the solution are, and why the solution is correct.
Compare and contrast two different versions of the solution of the tree and fields problem below.

**Narrative**

The rope is long enough to stretch from the tree to the river. Using the same length of rope, with one end fixed at the tree, Kanofer and Ahmes mark two positions where the other end of the rope touches the river bank.

Mark the rope to the length between the two positions on the river bank. Fold that portion of rope in half, and use this new length of rope to mark a position $x$ halfway between the first two. The angle between the river bank and the line from the tree to $x$ is the same on both sides of the line. To see this, note that the two smaller triangles are mirror images of each other.

**Geometric notation**

**Proposition.** Given a line and a point not on the line, drop a perpendicular from the point to the line, i.e. construct a second line that goes through the point and is perpendicular to the first line.

**Proof.** Let $A$ be the point. Pick any point $B$ on the line, and set the length of string equal to the length $AB$. Using the string mark another point $C$ such that $AB$ and $AC$ have the same length.

Set the length of string equal to the length of $BC$. Fold this length in half and mark the midpoint $P$ of $BC$.

Claim: $AP$ is perpendicular to $BC$.

Note that $\triangle ABP$ is the mirror image of triangle $\triangle AC P$. Therefore $\angle APB = \angle APC$, and so each is $90^\circ$.  

\[\Box\]
In my opinion there are two things, that are related, that could make this proof better. First, basically what makes a mathematician a mathematician is we would like to assume as few mathematical things as possible. As a result we make a big deal about every one we do assume, put them altogether, hope there are not many, and label them officially so it’s clear what we are assuming. In geometry we usually call these assumptions axioms. We didn’t do that in the previous proof, or anywhere before it. What did we assume? In other words, what are the ground rules?

Secondly, we want our axioms, and really all of our work, to be described as mathematically as possible, as opposed to ways that depend upon the physical tools we have. This is related to the first, because we want few, clear, unambiguous assumptions, and so we don’t want our work to depend on what sort of string we have, or how long it is, or how stretchy it is, etc. In fact, we’d like to write down the mathematical properties that we are assuming, that the string approximates, and describe them rather than the string. For instance, instead of explicitly mentioning that we fold the string in half, we could assume “Axiom: Given any line segment we can construct the midpoint”. That might sound like a bigger assumption than “just figure out what you can do with a piece of string” but it’s unambiguous. Besides, it makes clear also what we are not assuming: for instance, this axiom makes it clear we are not assuming we can fold a piece of string into thirds. Now maybe you can, and maybe you can’t fold it into thirds, with a real piece of string it’s ambiguous.

In this solution we used the following concepts from geometry:

- **points**  The tree marks a point, we constructed two additional points on the river, and our final solution depends on a third point along the river.
- **lines**  The final solution really is to construct the line from the tree to the river.
- **triangles**  The argument for the two angles on either side of the dividing line being equal describes triangles.
- **angles and distances**  We talked about the angle made by the line with the river, and the distances between points.
- **circles**  These were *implicit* not *explicit*: the first two points we marked on the river were defined by being the same distance from the tree. In other words, they are on the same circle with center at the tree. (Perhaps it seems like a bit of stretch to say that this implies circles, but in fact, when I drew the solution of this problem on the board, I made little circular arcs to mark these points, so circles are more present than the may appear.)

**Combinations of points**  Two points determine a line, that’s why we needed the point along the river to combine with the tree to mark the boundary line.

**Combinations of circles and lines**  Circles intersect lines: that’s how we determined the first two points on the river.
2 Rounding up basic constructions

Before we do the next problem, let’s make sure we agree on the rules. For string, at the simplest level, it seems to me what we can do is this:

1. If you have two known points, then you can use the string to draw the straight line through them (by pulling the string tight).
2. If you have two known points, then you can use the string to draw a circle with center at one point, and circumference through the other point.
3. Given two points you can find the midpoint (by folding the string in half).
4. You can intersect lines and circles.

You cannot construct, or know, any other points aside from ones produced in this fashion. E.g. if you are given a line segment, you know two points: the ones at the end. You can draw a circle and know one or two more points. You can draw lines through the new points, and maybe now you know 4 points, etc. You cannot move the string so that it becomes tangent to curve, or rotate a line, etc. You cannot “measure” lengths or angles.

**Proposition.** On a given line segment, construct an equilateral triangle (that has one side equal to the given line segment).

**Proof.** Let $AB$ be the given line segment

\[
\begin{array}{c}
A \\
B
\end{array}
\]

Draw a circle with center at $A$ and circumference through $B$. Also draw a circle with center at $B$ and circumference through $A$.

\[
\begin{array}{c}
A \\
B
\end{array}
\]

Let $C$ be one of the points in the intersection of the two circles.

Claim: the triangle $\triangle ABC$ is equilateral.
Since $\overline{AC}$ and $\overline{AB}$ are both in the circle with center $A$, we have $\overline{AB} = \overline{AC}$. Since $\overline{AB}$ and $\overline{BC}$ are both in the circle with center $B$, we have $\overline{AB} = \overline{BC}$. This shows that all three sides are equal.

\qed
Task 2. Form into groups, two people per group (one group can have three people). Write down as many basic geometric constructions as you can think of. You do not need to solve all of them, just the statement of the problem: what’s given, and what you would be asked to find/do/construct. Oh, think of a pithy name too.

For instance, in the previous task and problem, you might write: Drop a perpendicular (that’s the pithy name): Given a line, and a point not on the line, we construct a new line that goes through the point and is perpendicular to the first line.

Be prepared to write your answers on the board and discuss them with the rest of the class.
Task 3. Form into groups, two people per group (one group can have three people).
Find string constructions for the following problems
1. Erect a perpendicular
2. Bisect an angle
3. Construct a square (on a given length)
4. Construct a regular hexagon (on a given length)
Task 4. Get out your compass. See if you can make a triangular grid kind of like this, but bigger (in the sense of more triangles).
Task 5. Get out your compass. See if you can make this
Task 6. Get out your compass. See if you can construct an equilateral triangle with an inscribed circle:
Task 7. Form into groups, two people per group (one group can have three people).

Above I asked you to think of many basic constructions, but here’s one I’m guessing you didn’t think of. Using the two points shown below, draw several ellipses that have the two points as foci. Here’s how:

Pick two points on the string, a little farther apart than the two points on the paper. One person hold the two string-points on the two paper-points. The other person pulls the string taught with the pencil. Keeping the string taught, move the pencil from one end to the other and draw on the paper.

Do this several times, producing several different ellipses (at least the following: one that’s as fat as possible, one that’s as narrow as possible, and one that’s in the middle).
**Task 8.** Form into groups, two people per group (*one* group can have three people). Now we’ll draw a parabola with focus and directrix given by the point and line below (we have three different foci, so we’ll draw three different parabolas. Here’s how:

We’ll need a piece of string, a ruler, a right angle L/T/triangle and a piece of tape. Tape one end of the string near the top of the right angle L/T/triangle.

Pull the string taught and mark the point where it touches the very bottom of the right angle L/T/triangle.

One person put your straight edge on the line below, put the L/T/triangle flush against that straight edge so that the long edge is vertical on the page.

The other person put the marked end of string on the bottom focus and hold it there.

Catch your pencil in the string, pull the string tight and keep your pencil flush against the vertical edge. This is one point on the parabola. Now do two things simultaneously (you’ll need all four hands!): move the vertical edge left and right, while you do this keep the bottom of the right angle flush with the straight edge below, and keep the pencil flush against the vertical edge.
1.2 Turning Ruler and Compass Constructions into Algebra

1.2.1 Basic Operations

Definition 1.2.1. The following are the allowable steps for a straightedge and compass construction (“SC” stands for “straight edge and compass”):

SC1 If we are given, or have already constructed, two distinct points, then we can draw the unique line connecting them.

SC2 If we are given, or have already constructed, two distinct points, then we can draw the unique circle with center at one point, and circumference through the other point.

SC3 If we are given, or have constructed, two nonparallel lines, then we can draw their point of intersection.

SC4 If we are given, or have constructed, a circle and a line through the circle, then we can draw the point or points of intersection.

SC5 If we are given, or have constructed, two circles that overlap or touch, then we can draw the point or points of their intersection.

Challenge. Later you will be assigned the following problem, so you may want to think about it now:

Let points $A$ and $B$ be given, and let $AB$ be the distance between them. Let $C$ be a circle with center $A$ and radius $r$. Let $D$ be a circle with center $B$ and radius $s$. Classify when circles $C$ and $D$ intersect, 0, 1, or 2 times, in terms of $AB$, $r$ and $s$.

1.2.2 Background Geometry

Comments. We assume now all the basics of Euclidean geometry. In particular we assume that the following statements are true and have been proven:

- Pythagorean Theorem and Converse: If $a$, $b$ and $c$ are the sides of a triangle $\triangle ABC$, with $c$ the longest side, then

$$\triangle ABC \text{ is a right triangle } \iff a^2 + b^2 = c^2.$$

- Let $\triangle ABC$ and $\triangle DEF$ have sides of length $a, b, \ldots, f$, and angles of measure $\alpha, \beta, \gamma, \delta, \varepsilon, \eta$ as shown:

1. SSS: If $a = d, b = e$ and $c = f$, then $\triangle ABC \cong \triangle DEF$.
2. SAS: If $a = d, \beta = \varepsilon$ and $c = f$, then $\triangle ABC \cong \triangle DEF$.
3. ASA: If $\alpha = \delta, b = e$ and $\gamma = \eta$, then $\triangle ABC \cong \triangle DEF$.
4. AAS: If $\alpha = \delta, \beta = \varepsilon$ and $c = f$, then $\triangle ABC \cong \triangle DEF$.

Courant and Robbins proposed this set of axioms. They are often called “ruler and compass” constructions, but “ruler” is a bit misleading: the constructions do not use (are not allowed to use) any measurements or marks on the straight edge.
5. AAA: If $\alpha = \delta$, $\beta = \varepsilon$ and $\gamma = \varphi$, then $\triangle ABC \sim \triangle DEF$.

- Thales Theorem: Given a semicircle with diameter $AB$, if $C$ is any point on the circumference then $\triangle ABC$ is a right triangle:

![Triangulo](image)

- We assume our basic constructions: bisecting angles and line segments, erecting and dropping perpendiculars.
- We assume that we can “copy” line segments and angles (later we’ll see how Euclid proved that you can “copy” them).

Comments. Our goal is to be able to answer the following question:

**Question:** If we start with a given line segment, and call it length 1, what other lengths can we construct, using S&C alone?

Although constructions are classically Euclidean, we will frame the above question from a modern, non-Euclidean perspective. The following definition allows us to do this.

**Definition 1.2.2.** A real number $r$ is **constructable** if we can start with a line segment of length 1 and construct a line segment of length $r$ using only the SC axioms.
Activity 1. Shown below are two line segments with lengths \( a \) and \( b \).
(a) Using only S&C steps, construct a line segment with the same length as \( a + b \).
(b) Using only S&C steps, construct a line segment with the same length as \( a - b \).
(c) Cheat: using your ruler, measure \( a \) and \( b \) (metric measures are easier), calculate \( a + b \) and \( a - b \) with your calculator, and compare it to what you got using S&C steps.
Activity 2. Shown below are two line segments with lengths \( a \) and \( b \).
(a) Using only S&C steps, construct a line segment with the same length as \( ab \).

---

(b) Cheat: using your ruler, measure \( a \) and \( b \) (metric measures are easier), calculate \( ab \) with your calculator, and compare it to what you got using S&C steps.
Activity 3. Shown below are two line segments with lengths $a$ and $b$.
(a) Using only S&C steps, construct a line segment with the same length as $a/b$.

(b) Cheat: using your ruler, measure $a$ and $b$ (metric measures are easier), calculate $a/b$ with your calculator, and compare it to what you got using S&C steps.
**Activity 4.** Using Geogebra, with the perspective set on “geometry” (not “basic geometry”), do the following constructions:

(a) Start with a line segment of length 1, construct a line segment of length 5, using only the SC axioms, and the following guidelines.

Under the “geometry” perspective, there is a button for creating a circle by choosing the center, and then entering the radius. But, no cheating: the radius needs to be a number that we have constructed!

“Already constructed” is a recursive definition: given 1 as constructible, you might construct 2 (i.e. a line segment of length 2). If you've already constructed 1 and 2, now you can use them as radii and line segments anywhere you want to construct other numbers.

For instance, maybe you start with 1, and then construct, somehow, 11. Maybe you then construct 6. Now you could make line segments and radii of length 6 or 11 anytime you want. Thus, you could construct 5 by making a line segment of length 11, marking off a circle of radius 6 from one end and “cutting” this off to leave length 5.

(b) Start with a line segment of length 1, construct a line segment of length $2\frac{3}{8}$, using only the SC axioms.

Use the circle setting as described above, but again, you can’t enter 1/8 for the radius, we have to construct 1/2, then 1/4, then 1/8. Since we have already done bisectors from scratch, we can use the short-cut for them, i.e. you can use the bisector button.

**Comments.** Based on the first of these activities, we can see that 1, 2, 3, . . . , are all constructible. If we adopt a sense of direction on the number line, then we can view also the negative numbers, −1, −2, −3, . . . , as being constructible. In other words, the first activity shows this:

\[ \mathbb{Z} \subseteq \{ \text{all constructible numbers} \}. \]

Based on the second activity, we know that there are constructible numbers that are not integers, for instance, $\frac{1}{8}$.

**Theorem 1.2.3.** Given two constructible numbers, $a$ and $b$, it is possible to construct $a + b$, $a - b$, $ab$ and $\frac{a}{b}$ (assuming for the last result that $b \neq 0$).

**Proof.** We leave the the statements for $a + b$, $a - b$ and $\frac{a}{b}$ as exercises (the first two are very easy, and the last follows the same proof we’ll give for $ab$).

Start with $BC$ of length 1.

Using Prop I.11, erect a perpendicular through $C$.

Using SC3, draw a circle with center $C$ and radius $a$. 
Using SC4, let $A$ be the intersection of this circle and the perpendicular.

Using SC1, draw $\overline{AB}$.

Now we let $\overline{EF}$ be a line segment with length $b$.

Using Prop I.11, erect a perpendicular through $F$.

Using Prop I.23, construct an angle with vertex $E$ and one side given by $\overline{EF}$, such that the angle is congruent to $\angle ABC$. 
Using SC2, let $D$ be the intersection of the other side of the angle with the perpendicular through $F$.

Let $x$ be the length of the line segment $DF$.

Now the two triangles $\triangle ABC$ and $\triangle DEF$ have all the same angles, and are therefore similar. Thus, the ratios of corresponding sides are equal. So we have

$$\frac{b}{1} = \frac{x}{a}$$

which implies $x = ab$.

Thus, we have constructed a segment $DF$ whose length equals the product $ab$.

**Activity 5.** Using Geogebra, with the perspective set on “geometry” (not “basic geometry”), do the following constructions:

(a) Start with a line segment of length 1.5, and a line segment of length 2.5 and construct a line segment of length $(1.5) \times (2.5)$, using the technique described in the proof of the previous theorem.

(The main new button to use is one for copying an angle. Again, we could do this the long way, using all the steps in the SC axioms, or we can use the shortcut. I’ll allow the shortcut. The “angle button, with an $\alpha$ next to it, actually means “measure an angle”. Then one of the buttons under this allows you to construct an angle with a given measurement.)

(b) Start with a line segment of length 1 and construct a line segment of length $\frac{3}{2}$. 
Corollary 1.2.4. The set of all constructible numbers forms a subfield of $\mathbb{R}$.

Comments. There is one more algebraic property that the field described in the previous theorem satisfies: it is closed under taking square roots.
Activity 6 (Pythagorean Spiral, or Theodorus’ Spiral). Shown below is a line segment of unit length 1 (it’s actually exactly 2 cm, I’ve scaled the picture by a factor of 2 to make it easier for you to draw).

(a) Add a line segment at the bottom of the given one to make the given line segment part of a right triangle. How long is the hypotenuse? Cheat: measure your constructed line segment (if you measure in metric, don’t forget to divide by 2 since the scale is $2 : 1$).

(b) Erect a perpendicular of length 1 from one end of your last hypotenuse. This starts a new right triangle. How long is its hypotenuse? Measure to verify.

(c) Repeat the last step, starting from your last triangle.

(d) Repeat the last step, starting from your last triangle. How long is the last hypotenuse? Measure to verify.

Solution:
Theorem 1.2.5. If $r$ is a constructible number, then so is $\sqrt{r}$.

Proof. Let $AB$ have length $r$. Extend $AB$ past $B$ to a point $C$ with $BC = 1$. Construct a semi-circle with $AC$ as the diameter. Erect a perpendicular at $B$, and let $D$ be the intersection of the perpendicular with the semi-circle. From Thales’ Theorem the triangle $\triangle ACD$ is a right triangle. Let $x = BD$. We claim that $x = \sqrt{r}$.

By construction $\triangle ABD$, $\triangle BCD$, and $\triangle ACD$ are all right triangles. Since $\triangle ACD$ and $\triangle ABD$ share an angle, namely at $A$, they are similar. Similarly, $\triangle ACD$, $\triangle BCD$ share an angle and are similar. Therefore, by transitivity, $\triangle ACD$ and $\triangle BCD$ are similar. Now we set up ratios of corresponding sides

\[
\frac{x}{r} = \frac{1}{x} \quad \Rightarrow \quad x^2 = r \quad \Rightarrow \quad x = \sqrt{r}
\]

\[\square\]
Activity 7. Given below is a line segment of length $r$. Apply the proof of the constructibility of $\sqrt{r}$ to construct a line segment with length equal to the $\sqrt{r}$. Measure to verify.
Theorem 1.2.6 (SC Constructibility Critereon). A real number \( r \) is SC-constructible if and only if we may obtain \( r \) by applying the operations \(+, -, \times, \div\) and \( \sqrt{\cdot} \) a finite number of times to 1 and to the numbers produced starting from 1. (In other words, the set of constructible numbers is the smallest field that consists of real numbers and that contains all square roots of elements inside it.)

Proof. Define two sets:

\[
X = \{ r \in \mathbb{R} \mid r \text{ can be constructed by starting with 1 and applying the S&C operations} \}
\]

\[
Y = \{ r \in \mathbb{R} \mid r \text{ can be constructed by starting with 1 and applying the operations } +, -, \times, \div, \sqrt{\cdot} \text{ operations} \}
\]

The last 4 tasks show the following: for all \( r \in \mathbb{R} \) such that \( r \) can be calculated using \( 1, +, -, \times, \div \) and \( \sqrt{\cdot} \), we also have that \( r \) can be constructed using 1 and SC1–SC5. Thus, \( Y \subseteq X \).

To finish the proof, we need to show that \( X \subseteq Y \). We will do this in a similar fashion to the opposite direction: we show that at every step, each application of SC1–SC5, if it produces a new length \( r \), then that length will be something we can calculate using 1, \(+, -, \times, \div\) and \( \sqrt{\cdot} \).

Note that SC1 and SC 2 do not produce any new lengths. Thus, for these, there is no need to prove anything.

Let’s convert to Cartesian geometry: lines and circles will be defined by equations. Constructible points will be ones with constructible \( x \) and \( y \) values. Constructible lines will be ones that go through two constructible points. Constructible circles will be ones with constructible centers and constructible radii.

Let’s see what new length SC3 can give us. Let \( \ell \) and \( m \) be two constructible lines that intersect. In Cartesian geometry we can describe \( \ell \) and \( m \) with two equations

\[
ax + by = e
\]

\[
 cx + dy = f.
\]

If \( \ell \) and \( m \) are constructible, then \( a, b, c, d, e \) and \( f \) are constructible. Then the intersection of \( \ell \) and \( m \) is given in terms of some rational functions of \( a, \ldots, f \). If we want to be more explicit, we can say that

\[
x = \frac{1}{ad - bc}(de - bf)
\]

\[
y = \frac{1}{ad - bc}(af - ce)
\]

but the point here is not the exact formula, but that the formula only uses \( +, -, \times \) and \( \div \) and \( a, \ldots, f \). Therefore, SC3 creates the new points in this intersection, but these new points will still be 5-op-constructible.

Let’s see what new lengths SC4 can give us. Let \( \ell \) be a constructible line and \( C \) be a constructible circle. Then we can describe \( \ell \) and \( C \) with two equations

\[
ax + by = c
\]

\[
(x - d)^2 + (y - e)^2 = f^2,
\]

where \( a, b, c, d, e \) and \( f \) are constructible. Then the intersection of these two equations can be calculated using only \( +, -, \times, \div \) and \( \sqrt{\cdot} \) and the numbers \( a, \ldots, f \). Therefore, SC4 creates the new points in this intersection, but these new points will still be 5-op-constructible.
Let’s see what new lengths SC5 can give us. Let $\mathcal{C}$ and $\mathcal{D}$ be constructible circles. Then we can describe $\ell$ and $\mathcal{C}$ with two equations
\[
(x - a)^2 + (y - b)^2 = c^2 \\
(x - d)^2 + (y - e)^2 = f^2,
\]
where $a, b, c, d, e$, and $f$ are constructible. Then the intersection of these two equations can be calculated using only $+, -, \times, \div$ and $\sqrt{\cdot}$ and the numbers $a, \ldots, f$. Therefore, SC5 creates the new points in this intersection, but these new points will still be 5-op-constructible.

The preceding arguments show that all new points produced by SC2, SC4 and SC5 will also be points that are 5-op-constructible. This finishes the if and only if proof. \square

Example 1. How many different kinds of numbers can you make up that are constructible? Can you make up some that are (probably) not constructible?

Solution: We know that all the natural numbers, 1, 2, 3, 4, . . . are constructible.

We know that fractions are constructible, $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{2}{5}, \ldots$.

We know that square roots are constructible: $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{1/2}, \sqrt{1/3}, \sqrt{2/3}, \sqrt{2/5}, \ldots$.

Are there others? What about cube roots? What about more complicated combinations of numbers and roots?

Definitely yes on the latter: $\sqrt{1 + \sqrt{2}}, \sqrt{2/3 + \sqrt{5/7}}$, and even
\[
r = -\frac{1}{16} + \frac{\sqrt{17}}{16} + \frac{1}{16} \sqrt{34 - 2\sqrt{17}} + \frac{1}{8} \sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17} - 2\sqrt{34 + 2\sqrt{17}}}}.
\]

What about $\sqrt[3]{2}$? If you think about it, it seems unlikely that we can make a cube root equal to some combination of square roots. On the other hand, $\sqrt[3]{2}$ is approximately equal to $\sqrt[3]{395}/250$, so maybe it’s possible? We will say more about this in a later section.

1.2.3 Constructible angles

Example 2. Make up as many examples of constructible angles as you can.

Solution: We know that $90^\circ$ is constructible (since we can erect a perpendicular).

We also know that we can bisect an angle, so $45^\circ$, $22.5^\circ$, $11.25^\circ$, $45/8^\circ = 5.625$, $45/16^\circ = 2.8125$, $45/32^\circ = 1.40625$, . . . , are all constructible.

We can add angles too (by copying one angle onto another). So, we can start with $1.40625$ and add it to itself repeatedly.

Let’s shift to radians: We can start with $\pi/128$ (i.e. 1.40625°) and add it to itself repeatedly: $\pi/128, 2\pi/128, 3\pi/128, 4\pi/128, 5\pi/128, 6\pi/128, 7\pi/128, 8\pi/128, \ldots$. This will include $\pi/4, \pi/2$, etc.

Are there others?

Yes: since we can construct a regular equilateral triangle, we can make $\pi/3$. We can cut this one in half repeatedly, and then also add all of those together $\pi/6, \pi/12, \pi/24, \pi/48, \pi/96$, . . . . We can also take any of these, for instance, the smallest one, and add it to itself repeatedly: $\pi/96, 2\pi/96, 3\pi/96, 4\pi/96, 5\pi/96, 6\pi/96, 7\pi/96, 8\pi/96$.

We can repeat the above patterns, bisecting and adding, to get things like: $\pi/6 + \pi/16$. Can you provide a set description of all the fractions we’ve described so far?

So far we have described how to make any angle of the form
\[
\theta = a \frac{\pi}{2^n} + b \frac{\pi}{3 \cdot 2^m}, \quad a, b, n, m \in \mathbb{N}
\]
Aside from repeating the above patterns (i.e. bisecting, and adding), are there any others? What about $\pi/5$? Well, in theory, we constructed $2\pi/5$ when we made the regular pentagon. And then we can construct $\pi/5$ through bisection.

What about $2\pi/n$ for any $n$? It was a lot of work to construct $2\pi/5$, it’s not clear if it should be possible for $2\pi/7, 2\pi/9,$ etc.

**Comments.** The previous section laid out the basic properties of constructible lengths, constructible real numbers (the same thing), gave some powerful results about what numbers/lengths are constructible, and made some assertions about what numbers/lengths are not constructible. In this section we start to do the same thing for angles. In this section we assume the basics of trigonometry.

**Lemma 1.2.7.** The angle $\alpha$ is constructible if and only if $\cos(\alpha)$ is a constructible length/number.

**Proof.** The angle $\alpha$ is constructible if and only if the points $A, B$ and $C$ in the following are constructible

if and only if the points $C$ and $D$ are constructible

if and only if the length $\overline{AD}$ is constructible, if and only if $\cos(\alpha)$ is constructible. □

**Corollary 1.2.8.** The angle $\alpha$ is constructible if and only if the length $\cos(\alpha)$ is constructible, i.e. if and only if $\cos(\alpha)$ can calculated using natural numbers and $+, -, \times, \div, \sqrt{}$.

**Corollary 1.2.9.** The angle $\pi/12$ is constructible.

**Proof.**

$$\cos(\pi/12) = \frac{2 + \sqrt{3}}{2}$$ □
1.3 Some constructibility classifications

In the previous sections we have done a few things: We classified which real numbers are constructible; We have given a representative list of constructible real numbers; We have shown how the question of which angles are constructible can be turned into a question of which lengths are constructible; We have given a criterion that is sufficient to show some things are not constructible; We have used that criterion to confirm that $3\sqrt{2}$ is not constructible, to show that $20^\circ$ is not constructible, and therefore that it is not possible to construct the trisection of all angles.

So what remains to be done? We are essentially done with answering the constructability question for lengths: we know what they should look like, and we know in theory how to prove that things that don’t look like this are not in fact constructible. But for angles the situation is not as clear, because it’s not obvious, for example, what we can do with $\alpha = 1^\circ$, or $\alpha = \pi/9$, since it’s not obvious what exact value $\cos(1^\circ)$ or $\cos(\pi/9)$ have.

So what we do here is give a complete statement of exactly which angles of the form $2\pi/n$, $n \in \mathbb{N}$, are constructible, and as a result give a complete statement of exactly which regular $n$-gons are constructible.

**Definition 1.3.1.** An natural number $p$ is prime if it is impossible to write $p = nm$ for any natural numbers $n$ and $m$ satisfying $1 < n, m < p$.

**Definition 1.3.2.** A prime $p$ is a Fermat prime if $p = 2^{2^n} + 1$ for some $n \geq 0$.

**Example 3.** Find the first few Fermat primes.

**Solution:**

$n = 0$: $p = 2^0 + 1 = 3 = F_0$.

$n = 1$: $p = 2^2 + 1 = 5 = F_1$.

$n = 2$: $p = 2^4 + 1 = 17 = F_2$.

$n = 3$: $p = 2^8 + 1 = 257 = F_3$ (yes, it’s prime: it’s not divisibly by 2, 3, 5, 7, 11, 13).

The next number is probably too large to do without a calculator: $F_4 = 2^{2^4} + 1 = 65537$ which is also prime.

It is not known for sure, one way or the other, if there are any other Fermat primes! There may be none, or there may be infinitely more.

For an up to date list of what is known about Fermat numbers, go [here](#).}

**Theorem 1.3.3** (Gauss’s Constructibility Criterion). The angle $2\pi/n$ can be constructed with straightedge and compass if and only if $n = 2^k$ or

$$n = 2^kp_1p_2 \ldots$$

where each $p_i$ is a Fermat prime and distinct from the other $p_j$.

**Corollary 1.3.4.** The regular $n$-gon can be constructed with straightedge and compass if and only if $n = 2^k$ or

$$n = 2^kp_1p_2 \ldots$$

where each $p_i$ is a Fermat prime and distinct from the other $p_j$.

**Comments.** It would be a bit too long of a diversion for us to prove this here. But, in the next section we will sketch a result about when a number is non-constructible, and in the section after that show that some angles are not constructible. These results give a pretty good idea of the flavor of how to prove part of Gauss’s criterion.

**Example 4.** Compare Gauss’s Criterion to the integers $n = 3, \ldots, 10$.

**Solution:**
• $n = 3$. Odd primes distinct.✓ (i.e. we didn’t use the same odd prime twice to factor 3.)
\[ 3 = 2^{k+1} \text{ for } k = 0.✓ \]

• $n = 4$. Odd prime factors distinct.✓ (this is true because there are no odd prime factors, so, out of all the odd prime factors that we have, they are all distinct.)
All odd prime factors have the form $2^{2k} + 1$.✓ (this is true, again, because there are no odd prime factors.)

• $n = 5$. Odd prime factors distinct.✓
\[ 5 = 2^{k+1} \text{ for } k = 1.✓ \]

• $n = 6 = 2 \cdot 3$. Odd prime factors distinct.✓ (There is only one odd prime factor.)
\[ 3 = 2^{k+1} \text{ for } k = 0.✓ \]

• $n = 7$. Odd prime factors distinct.✓
\[ 7 = 2^{k+1} \text{ for } k =?_. You can verify that } k = 1 \text{ doesn’t work, the result is less than } 7. \text{ And } k = 2 \text{ doesn’t work, the result is bigger than } 7. \text{ Therefore only some decimal value of } k \text{ could make this equal } 7, \text{ and we don’t allow decimal values of } k. \]

The regular 7-gon is not constructible.

item $n = 8$. Odd prime factors distinct.✓ (There are no odd prime factors, so “all” of them are distinct.)
All the odd primes have the form $2^{2k} + 1$.✓

• $n = 9 = 3 \cdot 3$. So the odd prime factors are not distinct.

The regular 9-gon is not constructible.

• $n = 10 = 2 \cdot 5$. This is constructible since 2 is ok, and 5 is a Fermat prime, and there are no repeat Fermat primes.

1.4 Impossibility of constructions

Activity 8. Translate the problem posed in the following story into a mathematical statement:

Around 400 B.C. the citizens of the city Delos were struck by a plague that would not leave them. For months people were sickened, and they came to believe that the plague was sent by the god Apollo, and so the citizens sought help by going to Apollo’s oracle at Delphi.

The oracle responded that the city could lift the plague by doubling the size of the altar dedicated to Apollo. The altar was in the shape of a cube and made of gold. It was easy to figure out how to make another gold cube that was twice as big in all dimensions. But, when this was done, the plague did not lift.

The city then asked Plato for advice. Plato said that the oracle’s statement had been misinterpreted (as they usually were). The altar needed to be doubled in volume but not in each length. However it appeared to be hard to make another cube with twice the volume of the original one. Plato interpreted this as Apollo wanting the citizens to devote themselves to studying geometry, and that this would eventually lead to an answer.
**Solution:** We are given a physical cube. Let $1$ represent the length of its side. Then the cube has volume $V = 1$. We wish to construct a cube of volume $V = 2$. Thus, we need to construct, using straight edge and compass, the length $x$ of the side of the cube with volume 2. Thus, we need to solve

$$x^3 = 2.$$ 

In other words, we need to construct the length $\sqrt[3]{2}$.

To bad for the citizens of Delos that this problem had no solution!

**Definition 1.4.1.** An polynomial $p(x)$ with rational coefficients is irreducible over $\mathbb{Q}$ if it impossible to write $p(x) = q(x)f(x)$ for any polynomials $q(x)$ and $f(x)$ with rational coefficients and satisfying $0 < \deg p(x), \deg f(x) < \deg p(x)$.

**Theorem 1.4.2.** If the real number $r$ is constructible, then it is not the root of an irreducible cubic polynomial over $\mathbb{Q}$.

**Comments.** We can’t prove this theorem here, but it is a fairly easy result in an Abstract Algebra course. We’ll give two sketches, the first is super brief, and the second a little longer.

**Very brief proof sketch.** For contradiction, suppose $r$ is constructible, and the root of an irreducible cubic. Since $r$ is the root of an irreducible cubic, we can calculate $r$ using some sort of $\sqrt[3]{\cdot}$. But, since $r$ is constructible, we can calculate $r$ using some combination of $\sqrt{s}$. But in general, $\sqrt[3]{\cdot}$ is not equal to any combination of $\sqrt{s}$.

**Sketch/proof by made up examples.** For contradiction, suppose $r$ is constructible, and the root of an irreducible cubic. Since $r$ is the root of an irreducible cubic, suppose, for example (this assumption is why this proof is not a real proof!!!) that $r$ is a root of

$$p(x) = x^3 + 2x^2 + x + 1.$$  

(This polynomial is actually irreducible). Then

$$r^3 + 2r^2 + r + 1 = 0$$

$$r^3 = -2r^2 - r - 1$$

It turns out that the irreducibility of $p(x)$ means that the coefficients used on the right hand side here are unique: i.e. this is the only way to write $r^3$ as a combination of lower powers of $r$. (Actually this is pure linear algebra: the span of the set $1, r, r^2, r^3, r^4, \ldots$, is finite dimensional and the basis is $1, r, r^2$.) So $r^3$ equals the unique linear combination of $r^2, r$ and 1 just shown.

On the other hand, since $r$ is constructible, it can be calculated using square roots. Suppose for example that $r = 5 + \sqrt{13}$. Then $r$ is a root of the polynomial

$$q(x) = x^2 - 10x + 12$$

(It’s true that $r$ is a root of this polynomial.) Then

$$r^2 - 10r + 12 = 0$$

$$r^2 = 10r - 12$$

But then we can replace $r^2$ in the above expression for $r^3$ and get

$$r^3 = -2r^2 - r - 1$$

$$= -2(10r - 12) - r - 1$$

$$= -21r + 23$$

But this contradicts the uniqueness of $r^3$ given above.
Corollary 1.4.3. The number $\sqrt[3]{2}$ is not constructible.

Proof. The number $\sqrt[3]{2}$ is clearly the root of $x^3 - 2$. It remains only to prove that $x^3 - 2$ is irreducible. But $x^3 - 2$ factors as shown:

$$x^3 - 2 = (x - \sqrt[3]{2})(x^2 + \sqrt[3]{2}x + 2^{2/3}).$$

Since neither of the factors we’ve used here is defined over $\mathbb{Q}$, we see that $x^3 - 2$ is irreducible (actually, we haven’t quite proven that it’s impossible to factor $x^3 - 2$ over $\mathbb{Q}$, but for any proof of this we will have to assume some uniqueness of factorization properties, which can then be applied to finish the proof).

Corollary 1.4.4. It is not possible to “duplicate the cube”: the ancient Greek city of Delos was doomed!

Proposition 1.4.5. The angle $20^\circ$ is not constructible.

Proof. Suppose, for contradiction, that $20^\circ$ is constructible. Then $\cos(20)$ is constructible. But, we claim that $\cos(20)$ is a root of the following cubic, and that the cubic is irreducible

$$8x^3 - 6x - 1.$$

To see that the $\cos(20)$ is a root of this cubic, “recall” the trig identity

$$\cos(3\alpha) = 4\cos^3(\alpha) - 3\cos(\alpha).$$

Plugging in $\alpha = 20^\circ$ gives

$$\cos(60) = 4\cos^3(20) - 3\cos(20)$$

and

$$\frac{1}{2} = 4\cos^3(20) - 3\cos(20)$$

and

$$0 = 8\cos^3(20) - 6\cos(20) - 1$$

and

$$0 = 8x^3 - x - 1, \quad x = \cos(20).$$

Now, to see that this polynomial is irreducible over $\mathbb{Q}$, note that if it factored, then one of the factors would be linear. In other words, if the polynomial is not irreducible we would have

$$8x^3 - x - 1 = (x + a)(bx^2 + cx + d)$$

for some values of $a, b, c, d \in \mathbb{Q}$. Then the linear factor $x + a$ would imply that $x = -a$ is a root of the polynomial. But the rational root test shows that the only possible rational roots of this polynomial are $\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}$ and $\pm \frac{1}{8}$. We can check directly, by plugging these in, that none are roots. Therefore, the polynomial does not factor, and therefore it is irreducible.

Activity 9. You know how to bisect any angle using ruler and compass: See if you can find a construction using ruler and compass to trisect any angle.

Corollary 1.4.6. It is not possible to trisect every angle using straightedge and compass constructions.

Proof. If it was possible to trisect every angle, then it would be possible to trisect a $60^\circ$ angle. This would mean we could construct a $20^\circ$ angle, which we have just seen is impossible.
Appendix

Theorem 1.4.7 (Unique Factorization of Natural Numbers). The natural numbers have the following two properties:

1. Every natural number \( n \) can be written as a product of powers of prime numbers, like this
   \[
   n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}
   \]
   where each \( p_i \) is prime and distinct from the other \( p_j \).

2. The primes and the powers used in part (1) are unique, up to the order they are written in.

Example 5. The natural number \( n = 1234567890 \) has prime factorization of

\[
   n = 2 \cdot 3^2 \cdot 5 \cdot 3607 \cdot 3803
\]

This is the only way to factor \( n \), ignoring differences of the order we write the primes in.

Theorem 1.4.8 (Unique Factorization of Polynomials). Polynomials with coefficients in \( \mathbb{Q} \) have the following two properties

1. Every \( p(x) \) with coefficients in \( \mathbb{Q} \) can be written as a product of powers of irreducible polynomials with coefficients in \( \mathbb{Q} \), like this
   \[
   p(x) = (p_1(x))^{a_1} (p_2(x))^{a_2} \cdots (p_r(x))^{a_r}
   \]
   where each \( p_i(x) \) is an irreducible polynomial with coefficients in \( \mathbb{Q} \) and distinct from the other \( p_j(x) \).

2. The irreducible polynomials, and the powers, used in part (1) are unique, up to the order they are written in, and ignoring any differences of constant multiples.

Example 6. The polynomial

\[
   p(x) = x^6 + 11x^5 + 51x^4 + 184x^3 + 373x^2 - 195x - 425
\]

has in irreducible factorization of

\[
   p(x) = (x - 1)(x + 1)(x^2 + x + 17)(x + 5)^2
\]

We can factor \( p(x) \) in other ways, but only two other ways: by putting the factors in a different order, or by multiplying and/or dividing some factors by a constant:

\[
   p(x) = \frac{1}{9}x^2 + \frac{17}{9} + \frac{1}{2}x \cdot \frac{6x}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2}
\]

1.5 Theoretical Origami

1.5.1 Basic operations

Comments. This section is devoted to a different kind of geometric construction: paper folding, formalized in the name origami. We start by exploring what can be done in the simplest situations.

Activity 10 (Folding). Start with a full sheet of paper (it doesn’t need to be origami paper or even square).

(a) Draw two “random” dots and find the unique fold going through both of them.
(b) Draw two “random” dots and find the unique fold that takes the first dot onto the second.

(c) Draw two “random” intersecting lines and fold the paper so that one line lands completely on top of the other. Discuss the geometric relationship between the two original lines and the folded line.

(d) Draw a line and a point not on the line. Fold the paper so that the fold goes through the given point and is perpendicular to the given line.

(e) Draw a line and two points $A$ and $B$ with $AB$ greater than or equal to the distance between $B$ and $\ell$. Fold the paper so that the fold goes through $B$, and $A$ lands on top of $\ell$.

Comments. Now we formalize the kind of folding steps we did above in the activity. The goal is to produce a list of operations that explicitly show what we are assuming can be done, just like the list of SC operations explicitly showed what we assumed could be done with a straight edge and compass.

The operations below were first made explicit in the work of Humiaki Huzita (OO1–OO5, 1989) and Koshiro Hatori (OO6 and OO7, 2002).

**Definition 1.5.1.** The following are the allowable steps for an origami construction (“OO” stands for “Origami Operation”):

**OO1** Given two distinct points, we can find the unique fold going through both points. Given two folds that intersect, we can find the point of intersection.

**OO2** Given two distinct points, we can find the unique fold so that one point lands exactly on top of the other.

**OO3** Given two lines, we can find a fold so that one line lands completely on top of the other.

**OO4** Given a point and a line, we can find a fold that goes through the point and is perpendicular to the line.

**OO5** Given two points $A$ and $B$ and a line $\ell$, if $AB \geq \text{distance}(B, \ell)$ then we can find a fold that goes through $B$ and that makes $A$ land on top of $\ell$.

We assume the following about these folds: we can talk about where a point “goes” when we apply the fold, and then we have two points, the original, in its original location, and the new one, where the original “moved” to. Also, we assume that folding preserves angles and distances. In other words, if $A$, $B$ and $C$ are three points, and we apply a fold and those three points “move” to $A'$, $B'$ and $C'$, then $AB = A'B'$ and $\angle ABC = \angle A'B'C'$.

Comments. The Origami Operations are not identical to the S&C operations, but many of them are similar and one might expect that similar constructions will be possible using Origami. This turns out to be the case. Next we explore some of these similarities.

(a) Draw a “random” angle. Construct its bisection.
(b) Draw a “random” line segment. Construct its bisection.
(c) Draw a line segment $AB$, a line $\ell$ not containing $A$ or $B$, and a point $C$ on $\ell$. Construct a point $D$ on $\ell$ such that $CD = AB$. (This is analogous to intersecting $\ell$ with the circle centered at $C$ and with radius $AB$.)
(d) Let $A$, $B$, $C$ and $D$ be four distinct points with $AB = a$ and $CD = b$ and $a > b$. Show that we can construct line segments of length $a + b$ and $a - b$ (you don't actually have to do the construction, but give an argument that it is possible.)
(e) Draw a line $\ell$ and a point $P$ not on this line. Construct a line through $P$ parallel to $\ell$.

1.5.2 Constructible numbers and lengths

Lemma 1.5.2. Given lengths $a$ and $b$, we can construct $a + b$, $a - b$, $a^2$, $ab$, and $a/b$ (provided $b \neq 0$).

Proof. The assertions about $a + b$ and $a - b$ are contained in the above activities. Now we show how to construct $a^2$. Start with a line segment of length 1. Erect a perpendicular (one of the previous activities) at one end. Mark off a length $a$ (one of the previous activities) on this perpendicular. Now extend (or contract) the base to a length of $a$.

Again erect the perpendicular. Extend (or contract) the hypotenuse to make a right triangle with base $a$ and height $x$. Then the right triangles are similar and so the height will be $x = a^2$.

The assertion about $a/b$ is left as an exercise.

Theorem 1.5.3. Anything that is constructible with straight edge and compass is constructible with origami.

Proof. We show first that any length that is SC-constructible is also OO-constructible. We do this by examining each SC axiom in turn, and showing that any length that it produces can be also produced by using OO operations.

What does it mean for an SC axiom to produce a new length? It means that we start with a given figure (a set of points, line segments and circles), apply the SC axiom, and construct a new point, such that the new point and an old point make a line segment with a new length. A careful reading of the SC axioms show that SC1 and SC3 do not, directly, produce any new length: they do not by themselves produce a new point.

Thus, we need only show the following: given any figure, given any point constructed using this figure and SC2, SC4 and SC5, we may construct the same point using OO1–OO5.

SC2: this axiom can produce a new point by intersecting two lines. OO1 creates the same point by intersecting two folds.
CHAPTER 1. CONSTRUCTIONS

SC4: Suppose a point has been constructed using SC4. We are given a circle \( C \), and a line \( \ell \) that appears to go through the circle. We need to show that the intersection of the line and the circle is constructible using using OO1–OO5.

For the purpose of origami, we can’t take as “given” the whole circle, i.e. all the points on the circumference. We will assume that we are given: \( A \), the center of the circle, and \( B \), one point on the circumference of the circle, and the line that “appears” to go through the circle.

Since \( \ell \) appears to go through the circle, this means that \( AB \) is greater than the distance from \( A \) to \( \ell \). Thus, we can apply OO5 to find a fold that goes through \( A \) and that takes \( B \) onto \( \ell \). Let \( B' \) be the point that \( B \) is taken to by this fold. By definition, \( B' \) is on \( \ell \). Since folding preserves distances, we have \( AB' = AB \), so \( B' \) is on the circle. Therefore \( B' \in \ell \cap C \), and so \( B' \) is one of the intersections of \( \ell \) and \( C \).

We leave as homework how to construct the second intersection.

For SC5, suppose we have two circles with centers \( O \) and \( P \), and radii \( r \) and \( s \) respectively, so that \( OP < r + s \).

Then the circles intersect, and we need to construct an intersection using origami operations.

To be clear: we assume that \( O, P, r \) and \( s \) are all given, that they are OO-constructed.

Let \( z = OP \), so that \( z \) is constructible. Then, by Lemma 2, so is \( z^2 \), \( r^2 \) and \( s^2 \), and \( z^2 + r^2 - s^2 \) and \( \frac{z^2+r^2-s^2}{2z} \). Let \( x = \frac{z^2+r^2-s^2}{2z} \). Note that

\[
\begin{align*}
& z < r + s \\
& z - r < s \\
& (z - r)^2 < s^2 \\
& z^2 - 2rz + r^2 < s^2 \\
& z^2 + r^2 - s^2 < 2rz \\
& \frac{z^2 + r^2 - s^2}{2z} < r \\
& x < r.
\end{align*}
\]

Since \( x < r < z \) we have there exists a point \( C \) on \( OP \) with \( OC = x \). Furthermore, \( C \) is in the interior of the circle centered at \( O \). Therefore, the perpendicular through \( C \) will intersect the circle centered at \( O \). Let \( A \) be this intersection and let \( y = OA \). Now we need to prove that \( A \) is also contained in the circle centered at \( P \). In other words, we need to prove that \( AP = s \).

Then \( \triangle OAC \) is a right triangle and satisfies

\[
x^2 + y^2 = r^2.
\]
Also, \( \triangle ACP \) is a right triangle. Then
\[
\overline{AP}^2 = (z - x)^2 + y^2 \quad \text{(Pythagorean Theorem)}
\]
\[
= z^2 - 2zx + x^2 + y^2 \quad \text{(FOIL-ing)}
\]
\[
= z^2 - (z^2 + r^2 - s^2) + r^2 \quad \text{(Replacing } x \text{ and } x^2 + y^2)
\]
\[
= s^2.
\]
Thus, \( \overline{AP} = s \) and so \( A \) is also in the circle centered at \( P \).

Activity 12 (Folding Polygons).

• Take a piece of notebook paper, and tear off every edge, leaving a somewhat irregular shape. Now, fold a perfect square within your irregular shape.

• Start with a square piece of paper. Fold an equilateral triangle such that one of the edges of the paper forms one of the sides of the triangle.

1.6 Origami beyond straightedge and compass

Comments. In this section we add two more operations that can be done with paper folding, that are not possible with straightedge and compass.

Activity 13. Draw a “random” line \( \ell \) near the bottom of your page and a “random” point \( A \) about an inch or two above \( \ell \).

(a) Find a fold that takes \( A \) onto the line. Draw the fold with a solid line.

(b) Repeat part (a) until you have moved \( A \) to 10 or 20 points along the line \( \ell \), and turn each fold into a solid line.

(c) All the lines that you have drawn (aside from \( \ell \)) make the outline of a shape: what shape is it?

Solution: The folds form the envelope of a parabola. What this means is that there exists a unique parabola such that every fold is tangent to this parabola; if we draw enough folds then the parabola is outlined by folds.
This can be proven by using the following property: given a point and a line, the set of points that are equidistant from the given point and line form a parabola.

As above, let \( \ell \) be the given line and \( A \) the given point. Let \( X \) be equidistant from \( A \) and \( \ell \). Then the tangent line through \( X \) will reflect \( A \) onto \( \ell \).

**Activity 14.** Draw a pair of “random”, intersecting lines \( \ell_1 \) and \( \ell_2 \). Draw a pair of “random” points \( A \) and \( B \) that are a couple of inches away from each other and a couple of inches away from \( \ell_1 \) and \( \ell_2 \).

Find a fold that takes \( A \) onto \( \ell_1 \) and \( B \) onto \( \ell_2 \).

**Definition 1.6.1.** The extra origami operations are as follows:

**OO6** Given two points \( A \) and \( B \), and two lines \( \ell_1 \) and \( \ell_2 \), we can fold the paper so that \( A \) is taken to \( \ell_1 \) and \( B \) is taken to \( \ell_2 \).

**007** Given a point \( A \) and two lines \( \ell_1 \) and \( \ell_2 \), we can fold the paper so that \( A \) is taken to \( \ell_1 \) and the fold is perpendicular to \( \ell_2 \).
1.6.1 Trisections

Comments. The additional origami constructions described above allow us to trisect an angle.

Theorem 1.6.2. Given any angle, we can construct its trisection using origami operations.

Proof. We start by describing the construction, and then prove that it works.

We assume that we are given an angle $\angle ABC$. Furthermore, it suffices to prove that trisections are possible when $\angle ABC < 90^\circ$ (otherwise we can take just part of $\angle ABC$, the part that is greater than $90^\circ$, trisect this and then add $30^\circ$).

Using origami operations we can make a square having $BC$ as one of the sides. Then $BA$ intersects another edge of the square, and we assume that $A$ is actually on the edge.

We start with a square piece of origami paper containing angle $\angle ABC$.

Fold the bottom edge to the top, and then unfold, creating line $DE$. Fold the bottom edge to the middle line, and then unfold, creating line $FG$.

Apply Origami Operation 6 to find a fold so that $D$ is taken onto line $AB$ and $B$ is taken onto line $FG$. Let $D'$, $F'$ and $B'$ be the points that $D$, $F$ and $B$ were taken to by this fold.

Unfold the paper again. We claim that $\angle ABC$ is trisected by the segments $BF'$ and $BB'$.

Let $H$ be the intersection of $BF'$ and $FG$. Drop a perpendicular from $B'$ to the line $BC$, intersecting at the point $J$. 
We will prove two triangle congruences: $\triangle BD'F' \cong \triangle BB'F'$ and $\triangle BB'F' \cong \triangle BB'J$, in homework.

Exercise. True or False, with proof: If we make the above construction by making a different line $DE$, parallel to $BC'$, followed by applying all the same steps as above, then the result will still be a trisection of the angle.

### 1.6.2 Origami Constructible numbers and lengths

**Theorem 1.6.3** (OO Constructible Length Criterion). A real number $r$ is origami constructible if and only if we may obtain $r$ by starting with the number 1, and apply a finite number of operations of the form $+, -, \times, \div, \sqrt{\_}$ and $\sqrt[3]{\_}$.

### 1.6.3 Origami Constructible polygons

**Theorem 1.6.4** (OO Constructible Polygon Criterion). A regular $n$-gon is origami constructible if and only if all the prime factors of $n$ are: 2, 3, and distinct primes of the form $2^k 3^\ell + 1$.

For a succinct proof of this result, see this paper by Antonio Marcén or this Masters of Arts thesis by Hwa Young Lee.

### 1.7 Surveying

**Activity 15.** From 1730 to 1738, colonists in the territories of Pennsylvania and Maryland were at odds over a boundary: specifically, they could not agree where the boundary was between the two territories. Disagreements included one group of people seizing land belonging to another, homeowners being charged taxes by both territories, people committing crimes in one territory but then fleeing to the other, etc. In the end, hostilities took a dozen or so lives, and created a huge rift between the governors of Pennsylvania and Maryland.

The colonists appealed to the King of England for a resolution and the resulting Royal decree defined what the border would be. It was not defined by a natural boundary such as a river, nor was it given an explicit set of coordinates, or with a marked line on a highly accurate map. After all, they didn’t know the exact coordinates of almost any part of the American colonies or have a highly accurate map.

Part of the boundary was defined geometrically. For this part, two things were known: “The Middle Point,” a point that was exactly halfway across what is now known as the Delmar Peninsula, and a circle with a radius of 12 miles with center at the top of the courthouse tower in the town of New Castle, now in Delaware.

The relevant part of the eastern border of Maryland was defined as follows: it would be the line through the Middle Point and tangent to the west side of the given circle.
Let’s call this the Tangent Line. In 1751 a survey team consisting of two surveyors from Maryland and two from Pennsylvania tried to find the Tangent Line. They worked on the ground, with chains and sextants and levels and telescopes, but failed to find an accurate solution. Their line was neither terribly straight, nor tangent to the circle. Eventually, in 1763, a team headed by Charles Mason and Jeremiah Dixon found the Tangent Line. The same duo surveyed another line too: the famous line, now known as the Mason-Dixon line, that runs due east and west. The east-west Mason-Dixon line formed the north border of Maryland, and in the Civil War it separated the slave holding states from the free states.

The picture below represents the problem you will be given next class. You will have a large sheet of paper with two known points: MP, the Middle Point, and NC, the point at the courthouse in New Castle. You will be given the line extending due north from MP. You know a few points on the circle with center at NC. Your task will the same as Mason and Dixon: Find the tangent line and the point of tangency.

Hint: you may not be able to construct this exactly: you cannot run one single straight edge from MP to the circle: in real life the line was about 80 miles long so they had to work in steps; the circle did not have the entire circumference marked, but “only” a handful of points. You cannot fold the paper, you cannot use a ruler longer than 12 inches. You cannot use a compass larger than 6 inches.

You can approximate the solution using your ruler and measurements, trigonometry, your calculator, your protractor, trial and error (followed by an iterated trial, etc.), and almost anything else you can think of provided it follows the above rules.

We will decide afterwards who has the most accurate answer. To decide this we can “cheat”: we can use longer straight edges, measuring tapes, large compasses, fold the paper, etc. We may need to come to an agreement on how exactly to measure accuracy, including both the Tangent Line and the Tangent Point.

Whichever team has the most accurate answer will win $5.
1.8 Paradoxes, and fallacies

Comments. One very important part of our study of geometry, is the formal deductive reasoning. This means proving things carefully, and not taking anything for granted. Below, we include two fallacies that should remind us things are not always what they appear to be.

Example 7. The Curry Paradox
(a) The following picture shows two 5 × 13 rectangles, that have been subdivided into triangles and a rectangle. Things appear to be equal, but something is wrong. What is the most obvious thing that is wrong?
(The black grid is only placed here for your convenience and is not intended to be part of the picture.)

(b) Solve the paradox by figuring out what things are not really what they appear to be.

Solution:
(a) In making these pictures I have just rearranged red and green triangles, without manipulating them at all. Obviously the area of the whole rectangle (blue triangle + red triangle + green triangle + yellow rectangle) does not change under this rearrangement. Obviously this means that the two yellow rectangles have the same area. Obviously this means that $15 = 16$. Obviously this causes some problems in mathematics!

(b) To resolve this paradox it suffices to identify some visual fallacies, which I’ll do below. But the instructive thing about this example is not the specific mistake that was made this time, but rather (1) that pictures can be deceiving, and (2) that if we are going to rely on a picture in a proof, we should double-check that it’s correct.

Now, what are the mistakes in this picture? Well, as with any mistake, there’s lots of things that you can find that went wrong. Each picture is wrong all by itself: the blue “triangle” is not lined up with the red and green triangles. You can convince yourself of this by noting that the slopes of the hypotenuses all differ: the blue slope is $\frac{5}{13} \approx 0.3846$, the red slope is $\frac{3}{8} \approx 0.3750$ and the green slope is $\frac{2}{5} = 0.4$ (no, it’s not a coincidence that these numbers, 2, 3, 5, 8, 13, all come from the Fibonacci sequence).

Example 8. See if you can find the mistake in the following proof. You may assume that the following are all correct:

C1 Given an angle we can construct the angle bisector,

C2 Given a line segment we can construct a perpendicular bisector,

C3 Given a certain line and a certain point, we can construct through the point a new line that is perpendicular to the given line (“drop perpendiculars”).

P1 any point on a perpendicular bisector of a line segment has equal distances to the two endpoints of the line segment,

P2 The angles in a triangle add to 180°,

P3 ASA implies congruent triangles,

P4 SSS implies congruent triangles,

P5 The Pythagorean Theorem.

We use the following notation: $AB$ for the (infinite) line through $A$ and $B$, $\overline{AB}$ for a line segment (and also for the length of the line segment), $\triangle ABC$ for a triangle with vertices $A$, $B$ and $C$, and $\angle C$ for the angle in $\triangle ABC$ at vertex $C$.

Proposition 1.8.1 (False Proposition). Every triangle is an isosceles triangle.

Proof.
Let \( \triangle ABC \) be any triangle. Construct the line that bisects the angle \( \angle ACB \), and construct the perpendicular bisector of \( AB \), and let the point \( D \) be their intersection.

From \( D \) drop perpendiculars to \( AC \) and \( BC \), letting the points of intersection be \( E \) and \( F \). Now \( \triangle DCE \) and \( \triangle DCF \) are both right triangles (the right angles are at \( E \) and \( F \) respectively), and \( \angle DCE = \angle DCF \), since the line between them is the bisector of \( \angle C \). Therefore all the angles in \( \triangle DCE \) and \( \triangle DCF \) are equal (since the interior angles of each triangle add to \( 180^\circ \)). Since \( \triangle DCE \) and \( \triangle DCF \) have a side in common, \( DC \), we have ASA congruence: \( \triangle DCE \cong \triangle DCF \). Therefore \( CE = CF \) and \( DE = DF \).

Also, since line \( GD \) is the perpendicular bisector of \( AB \), we have that \( AD = BD \). Since \( \triangle ADE \) and \( \triangle BDF \) are right triangles, we apply the Pythagorean theorem and conclude that their third sides are also equal. Therefore, by SSS we have congruence \( \triangle ADE \cong \triangle BDF \).

Therefore \( EA = FB \). Since we already had \( CE = CF \), this means
\[
CE + EA = CF + FB \\
CA = CB
\]

Solution: To find this mistake, it might help to break down the prose proof into a series of assertions, as small as possible, and then see if we can justify every assertion. In effect, it might help to turn it into a two column proof.

Here are the statements.

1. Let \( \triangle ABC \) be any triangle.
2. Construct the line that bisects the angle \( \angle ACB \).
3. Construct the perpendicular bisector of \( AB \).
4. Let the point \( D \) be the intersection of the lines constructed in the two previous steps.
5. Drop a perpendicular from \( D \) to \( AC \).
6. Let \( E \) be the intersection of the line just constructed with \( AC \).
7. Drop a perpendicular from \( D \) to \( BC \).
8. Let \( F \) be the intersection of the line just constructed with \( BC \).
9. \( \triangle DCE \) is a right triangle.
10. \( \triangle DCF \) is a right triangle.
11. \( \angle DCE = \angle DCF \).
12. \( \angle DCE + \angle CED + \angle EDC = 180^\circ \).
13. \( \angle DCF + \angle CFD + \angle FDC = 180^\circ \).
14. \( \angle EDC = \angle FDC \).
15. \( \angle DCE = \angle DCF \) and \( DC = DC \) and \( \angle CDE = \angle CDF \).
16. \( \triangle DCE \cong \triangle DCF \).
17. \( CE = CF \) and \( DE = DF \).
18. \( AD = BD \).
19. \( \triangle ADE \) is a right triangle.
20. \( \triangle BDF \) is a right triangle.
21. \( AE^2 + DE^2 = AD^2 \).
22. \( BF^2 + DF^2 = BD^2 \).
23. \( AE = BF \).
24. \( \triangle ADE \cong \triangle BDF \).
25. \( EA = FB \).
26. \( CE + EA = CF + FB \).
27. \( CA = CB \).

Statement 1 does not need justification, no logical assertion is being made, just the introduction of a label. Statement 2 is justified by construction C1. Statement 3 is justified by construction C2. Statement 4 breaks down in two ways: we don’t know for sure that the two lines intersect, maybe they could be parallel. In reality, if we make our
picture accurate, the lines do intersect, but the point $D$ does not lie within the triangle. Most of the rest of the statements are correct, but not statement 27. It turns out that one of the points $E$ or $F$ lies on the side of the triangle, and one lies on an external extension. Thus, we can't add the line segments shown in statement 26 because, for example, $F$ might not between $B$ and $C$, Thus, we should have something like

$$\pm CE \pm EA = \pm CF \pm FB$$

where the $+$s and $-$s are chosen to make things correct. (See some nice explanation and demonstrations at [www.cut-the-knot.org](http://www.cut-the-knot.org) (All Triangles are Isosceles) [http://www.scribd.com/doc/19346327/4/THE-ISOSCELES-TRIANGLE-FALLACY-ANALYSED]())
Activity 16. Find and install a computer geometry application such as Geogebra, Geometer’s Sketchpad, Cabri, or something similar.

(a) Perform the construction used in Euclid’s Book I, Proposition 1, with your computer geometry application. Be sure to define each line segment, circle, and intersection, as geometric objects, not just pictures.

Make sure your construction passes the drag test (i.e. if you drag either endpoint of the original line segment, the whole construction should move with the drag).

(b) Perform the construction used in the fallacious proof that all triangles are isosceles. Make sure your points are labelled the same way as in the proof.

Again, make sure the construction passes the drag test.

(c) The crucial part of the fallacious proof came down to the following steps:

\[
\begin{align*}
CE &= CF \\
EA &= FB \\
CE + EA &= CF + FB \\
CA &= CB
\end{align*}
\]

Find the length-measuring tool on Geogebra, and use it to figure out which of these steps is not correct, and how you could change it to make it correct (your change should involve which lengths are added or subtracted, and this should depend on whether or not \(E\) or \(F\) are on the triangle’s sides, or on an extension of them).

1.9 Why should we study geometry?

Comments. Giving a definitive answer to any “why” question is impossible, but I’ll offer a few reasons and comments here as to why we should study geometry. I’ll equivocate a little more: the answer to this question needs to be relative to the audience, so I’ll answer it more than once.

If you’re below the age of a high school student, you need to study geometry from the qualitative perspective: you should know what straight lines are, circles, and angles. You should know a little about area and perimeter.

At higher ages, a critical divide exists between two reasons to study geometry, or, rather, any mathematics: for the thought process or for the applications. As I mentioned above, Dudley argues for the thought process. Another plank in this platform is put forth by Paul Lockhart, in a fairly well-known essay titled A Mathematician’s Lament. However, the main point of Lockhart’s essay is not just that the thought process is about deductive, rational thinking; to him it is about a lot more than that, including creativity and discovery; but still, it’s the thought process that matters, not applications.

I’ll note that both Dudley and Lockhart would probably agree than there is little value in trying to teach kids more math at the expense of watering down the thought process. It would be better to get one great year of high school math than be made to take two years of bad high school math. For instance, there is little value in increasing the number of students who take Calculus, unless there is on emphasis on the thought process. I think Lockhart would even go so far as to say it would be worth it to have one great idea in calculus, discovered, explored, proved by the students for the entire semester, rather than teach Calculus the usual way.
1.10 Appendix on Field Extensions


The main ideas for the impossibility of certain constructions involve polynomials and factoring them. In particular, I asserted something like “you can’t make a cube root out of combining square roots,” which seems pretty likely, but I hope some readers would still want to be a little more formal in justifying this. Let’s see if we can provide a (pretty) rigorous proof. What do I mean by pretty rigorous? Well, we’ll still need to assume some things, like linear algebra and polynomial division. But otherwise, we should be able to fill in the steps.

**Definition 1.10.1.** Suppose \( r \) is a real number and \( p(x) \) is a polynomial with coefficients in \( \mathbb{Q} \) such that three properties hold

1. \( p(r) = 0 \)
2. the leading coefficient of \( p(x) \) is \( 1 \)
3. the degree of \( p(x) \) is minimal among all polynomials for which \( r \) is a root

Under these conditions we say that \( p(x) \) is a **minimal polynomial** for \( r \).

**Example 9.** Let \( r = \sqrt{2} + \sqrt{2} \). Define three polynomials

\[
\begin{align*}
    f(x) &= 3x^4 - 12x^2 + 6 \\
    g(x) &= x^4 - 4x^2 + 2 \\
    h(x) &= x^8 - 8x^6 + 20x^4 - 16x^2 + 4
\end{align*}
\]

With work we can show that

\[
    f(r) = 0 \quad \text{and} \quad g(r) = 0 \quad \text{and} \quad h(r) = 0.
\]

The first one, \( f(x) \) cannot be the minimal polynomial, since it doesn’t have leading coefficient of \( 1 \).

The third one, \( h(x) \) cannot be the minimal polynomial, since it has a degree that is larger than it needs to be.

This suggests that \( g(x) \) is the minimal polynomial, and this is correct, but we don’t know how to prove it yet.

**Lemma 1.10.2.** Let \( r \) be a real number and let \( p(x) \) be an irreducible polynomial with rational coefficients, leading coefficient of \( 1 \), and such that \( p(r) = 0 \). Then \( p(x) \) is the minimal polynomial for \( r \).

**Proof.** Let \( f(x) \) be a minimal for \( r \). Apply polynomial division to write \( p(x) = f(x)q(x) + s(x) \) where \( \deg s(x) < \deg f(x) \), i.e. \( s(x) \) is the remainder after division. Then

\[
    0 = p(r) \quad \text{by assumption}
\]

\[
    = f(r)q(r) + s(r) \quad \text{by polynomial division}
\]

\[
    = s(r) \quad \text{since } f(r) = 0
\]

Then \( s(r) = 0 \) and \( \deg s(x) < \deg f(x) \), and \( f(x) \) was supposed to have minimal degree among all polynomials for which \( r \) is a root. Combined, this means that \( s(x) \) isn’t a non-trivial polynomial at all, so \( s(x) = 0 \), i.e. there is no remainder. This means that \( f(x) \) divides \( p(x) \). But \( p(x) \) is irreducible, so \( f(x) = p(x) \), (give or take a constant multiple, but since they both have leading coefficient of \( 1 \), the constant multiple would have to be \( 1 \)).
Lemma 1.10.3. such that $p(x)$ is irreducible over $\mathbb{Q}$ and $p(r) = 0$. Let $X$ be the set

$$X = \{1, r, r^2, r^3, \ldots \}.$$ 

and let $V$ be the $\mathbb{Q}$-span of $X$, i.e. the set of all linear combinations of $1, r, r^2, \ldots$, where the coefficients come from $\mathbb{Q}$.

Then

1. $V$ is a finite dimensional vector space over $\mathbb{Q}$ with dimension equal to the degree of $p(x)$.
2. If $n = \deg p(x)$ then $1, \ldots, r^n-1$ forms a basis for $V$.

Proof. Part (1) follows from part (2), which we will prove.

First we show that the indicated set spans $V$. Let $p(x) = a_n x^n + \ldots + a_1 x + a_0$ with $a_n \neq 0$. Since $p(r) = 0$ we have

$$a_n r^n + \cdots + a_1 r + a_0 = 0$$

This means that $r^n$ can be written as a linear combination of lower powers of $r$

$$r^n = \frac{1}{a_n} (-a_{n-1} r^{n-1} - \cdots - a_1 r - a_0)$$

But we can repeat this process for higher powers of $r$ as well:

$$r^{n+1} = r(r^n) = r \left( \frac{1}{a_n} (-a_{n-1} r^{n-1} - \cdots - a_1 r - a_0) \right)$$

where we can expand on the right and then reduce $r^n$ again. By induction, we can reduce any power of $r$ that is higher than $r^{n-1}$, which shows that the claimed basis spans.

Now we sketch the proof that $1, r, \ldots, r^n-1$ is linearly independent. By the previous lemma, $p(x)$ has the smallest degree among all polynomials for which $r$ is a root.

Suppose now that

$$b_{n-1} r^{n-1} + \ldots + b_1 r + b_0 = 0.$$ 

Then $r$ is a root of

$$b_{n-1} x^{n-1} + \ldots + b_1 x + b_0.$$ 

But since $p(x)$ has the smallest degree of all polynomials for which $r$ is a root, we conclude that $b_{n-1} = b_{n-2} = \cdots = b_1 = b_0 = 0$. This means that $r^{n-1}, \ldots, r, 1$ are linearly independent.

Corollary 1.10.4. If $r$ is the root of an irreducible cubic, then $r$ is not the root of a quadratic.

Proof. Since $r$ is the root of an irreducible cubic, we have that $1, r$ and $r^2$ are linearly independent. If $r$ was also the root of a quadratic, then we would have a non-trivial linear combination of $1, r$ and $r^2$ that added to 0, which is impossible.

Corollary 1.10.5. If $r$ is the root of an irreducible cubic, then $r$ is not the root of a polynomial with powers $0, x^2$ and $x^4$.

Proof. Since $r$ is the
Chapter 2

Euclid’s Elements

2.1 Finite Geometry

Let’s look at some examples that are relevant to arguments you might make in the clubs and students problem.

In each of the following pictures, a straight line or a circle with diameter ≈ 1 inch, represents a club and a small dot • represents a student. For each picture, decide whether each logical statement about the clubs and students is true or false.

Here is a reminder about the logical statements in question: \( S \) and clubs are assumed to be nonempty.

Axioms:

CA0 The set of students is nonempty, and every club is nonempty.

CA1 Every student in \( S \) is a member of at least one club.

Propositions:

1. Every student in \( S \) is a member of at least two clubs.
2. Every club contains at least two members.
3. The set \( S \) contains at least four students.
4. There are at least 6 different clubs.
The above 7 pictures have some connection to your homework problem. What is it? How direct and exact can you make the connection?

For more on this subject, look up “Finite Geometry” or “Finite Affine Planes.” For instance this article Affine Planes: An Example of Research on Geometric Structures, by G"nter Pickert in The Mathematical Gazette. There are many open questions in this field, for instance, it is unproven whether or not every finite affine plane has order given by a prime power (i.e. is it the case that the number of points on every line is given by $p^k$ for some prime $p$).

Also see Four Finite Geometries by H. F. Mac Neish, in The American Mathematical Monthly.
2.2 Reading Euclid

2.2.1 Definitions and Postulates

The next two pages show a translation of the first two pages of Euclid's *Elements*, the most famous textbook on geometry. Actually, it is the oldest extant textbook on geometry. When you get right down to it, it's the oldest extant textbook of any kind, written about 300BC (Euclid may have been born about the year that Aristotle died, give or take 10 years). It's the second most printed book in the world, second to the Bible. (More information about the *Elements* available on Wikipedia)

Take a minute to read some of the text, and then I'll have a few comments about it.
1. A point is that of which there is no part.
2. And a line is a length without breadth.
3. And the extremities of a line are points.
4. A straight-line is (any) one which lies evenly with points on itself.
5. And a surface is that which has length and breadth only.
6. And the extremities of a surface are lines.
7. A plane surface is (any) one which lies evenly with the straight-lines on itself.
8. And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight-line.
9. And when the lines containing the angle are straight then the angle is called rectilinear.
10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands.
11. An obtuse angle is one greater than a right-angle.
12. And an acute angle (is) one less than a right-angle.
13. A boundary is that which is the extremity of something.
14. A figure is that which is contained by some boundary or boundaries.
15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.
16. And the point is called the center of the circle.
17. And a diameter of the circle is any straight-line, being drawn through the center, and terminated in each direction by the circumference of the circle. (And) any such (straight-line) also cuts the circle in half.
18. And a semi-circle is the figure contained by the diameter and the circumference cuts off by it. And the center of the semi-circle is the same (point) as (the center of) the circle.
19. Rectilinear figures are those (figures) contained by straight-lines: trilateral figures being those contained by three straight-lines, quadrilateral by four, and multilateral by more than four.
20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having unequal sides.
21. And further of the trilateral figures: a right-angled triangle is that having a right-angle, an obtuse-angled (triangle) that having an obtuse angle, and an acute-angled (triangle) that having three acute angles.

22. And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one another which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.

23. Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions).

† This should really be counted as a postulate, rather than as part of a definition.

Postulates

1. Let it have been postulated† to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).‡

† The Greek present perfect tense indicates a past action with present significance. Hence, the 3rd-person present perfect imperative ἔτεκτοι could be translated as “let it be postulated”, in the sense “let it stand as postulated”, but not “let the postulate be now brought forward”. The literal translation “let it have been postulated” sounds awkward in English, but more accurately captures the meaning of the Greek.

‡ This postulate effectively specifies that we are dealing with the geometry of flat, rather than curved, space.

Common Notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.†
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

† As an obvious extension of C.N.s 2 & 3—if equal things are added or subtracted from the two sides of an inequality then the inequality remains
Here are some of the things I notice as I read these. This textbook is not like modern textbooks: he offers no introduction or explanation, he starts with his first definition, the definition of a point. In a modern math book we wouldn’t try to define points. From a mathematical perspective, we know that there have to be some undefined terms. Each new thing that we want to define has to be defined in terms of simpler concepts, and these have to be defined in terms of still simpler concepts, and so on, until you get to the beginning. There has to be a beginning, otherwise you’d have an infinite chain of definitions, or a circular chain of definitions. So, something has to be given at the beginning that is not defined in terms of simpler concepts, and points are right there.

But Euclid tries. Granted, his definition isn’t really very good: “A point is that of which there is no part.” But on the other hand, it gives you some idea of what’s going on: points have no parts. We know they should be imagined as infinitely small as well, but Euclid doesn’t say this. His next definition is of a line, and kind of makes sense: it has length but not width, so it just has a single dimension. But, watch out: when Euclid says “line” he means what we would call a curve. You can tell this by definition 4, of a straight line. This definition is quite confusing to me, I don’t really have any idea what “lies evenly with points on itself” means. Probably it sounded better in the original Greek. I would guess it means something like it doesn’t go to one side or the other.

I’ll point out that he defines circle, and it’s quite a good definition. I’m surprised how much space it takes to define all the different concepts involving circles: a little over 100 words! He goes on to define equilateral triangles and squares, etc. Also the other thing I notice is that he does all of this without notation. Sometimes I find this elegant, and sometimes a little awkward, sometimes both within the same definition!

For instance, in “circle” he doesn’t need to name the center, but on the other hand, the word “center” doesn’t even show up until the next definition!

So a modern geometry book probably would have skipped the definitions of points and lines, but probably would have included the definitions of circles, triangles, etc. Next come the postulates. Aristotle has some sort of discussion about how Postulates are a little different than axioms because they state an action, or what can be done, not just an assumption. That may be true on some philosophical level, but in every modern math book these statements would probably be called “axioms”.

The reason we practiced with the student and club axioms is because this is how geometry is presented, especially in Euclid, starting with a very small set of axioms, and then proving everything from that. So, in theory, every proposition in Euclid would be proven using just these five postulates.

The final thing I’ll point out that Euclid does a little differently than we would now is with his “common notions”. Here he states things like if two things are equal to a third then they are equal to each other: in symbols, if \( a = b \) and \( b = c \) then \( a = c \), i.e. the transitive property of “equals”. Certainly in modern books we would not usually bother stating this. I conclude that in Euclid’s time, a typical reader of his book wouldn’t have the same basic logic, and semi-algebraic viewpoint that similar students would today.

**Exercise.** One of the more unsatisfying definitions in The Elements is of a straight-line: “A straight-line is (any) one which lies evenly with points on itself.” What does that mean? Did Euclid drop the ball on this? Or is it a decent attempt to help us understand what’s going on? Take a stand and answer this question.

You might want to read some of the following:

- Heath’s commentary on this definition (see pages 143–150 and 158–159).
- This discussion at Math Stack Exchange
- This blog posting at Intellectual Mathematics
2.2.2 Euclid I.1

The Elements are such a standard text, that even the numbering Euclid used for his results is universally preserved. So “Euclid I.1” means “Euclid, Book I, Proposition 1.” The Elements are divided into 12 “books” with each book being what we might call a pretty long chapter. Book I consists of 48 propositions. Euclid doesn’t name anything a “theorem” or a “corollary” or “lemma”: everything is a proposition.

Euclid I.1 states this: “To construct an equilateral triangle on a given finite straight-line.” In the next few pages I show a few different proofs of this result, and some discussion of them.
Euclid’s Proof

This copy of Euclid I.1 is from the D’Orville manuscript. It is the oldest intact copy, made in 888 AD in Constantinople. It is held by the Bodleian Library, Oxford University, and was made publically available by the Clay Mathematics Institute.
Euclid’s Proof

To construct an equilateral triangle on a given finite straight-line.

Let $AB$ be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line $AB$.

Let the circle $BCD$ with center $A$ and radius $AB$ have been drawn [Post. 3], and again let the circle $ACE$ with center $B$ and radius $BA$ have been drawn [Post. 3]. And let the straight-lines $CA$ and $CB$ have been joined from the point $C$, where the circles cut one another, to the points $A$ and $B$ (respectively) [Post. 1].

And since the point $A$ is the center of the circle $CDB$, $AC$ is equal to $AB$ [Def. 1.15]. Again, since the point $B$ is the center of the circle $CAE$, $BC$ is equal to $BA$ [Def. 1.15]. But $CA$ was also shown (to be) equal to $AB$. Thus, $CA$ and $CB$ are each equal to $AB$. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, $CA$ is also equal to $CB$. Thus, the three (straight-lines) $CA$, $AB$, and $BC$ are equal to one another.

Thus, the triangle $ABC$ is equilateral, and has been constructed on the given finite straight-line $AB$. (Which is) the very thing it was required to do.

This is a pretty literal translation of Euclid’s text into English, produced by Richard Fitzpatrick in 2007.

The comments in square brackets, such as “[Post. 3]” were not included in Euclid, but have been standard additions by most authors after him. The comments in parentheses such as “(respectively)” were added by Fitzpatrick where they seemed necessary for good English grammar.

Part of me notices that it’s a little hard to follow Euclid at times, but mostly I’m in awe that a proof written 2000 years ago is still so readable and accurate! Note that he is a bit wordier than we would be today, going out of his way, as mentioned in the introduction, to point out that two things are equal to a third, and so they are equal to each other, etc.
Modern Proof Style

Proof. Let $\overline{AB}$ be the given segment. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be circles with centers $A$ and $B$ respectively, and both circles having radius $\overline{AB}$. Let $C \in \mathcal{C}_1 \cap \mathcal{C}_2$. Since $B, C \in \mathcal{C}_1$ we have $\overline{AB} = \overline{AC}$ and similar reasoning shows $\overline{AB} = \overline{BC}$. Therefore

$$\overline{AC} = \overline{AB} = \overline{BC},$$

which means that $\triangle ABC$ is an equilateral triangle, as desired.

This is a more or less how a modern author would write this proof. Note that it’s a bit more concise than Euclid, especially in the step where all three line segments are asserted to be equal.

While we are here, can you see the logical gap that this proof has, and that Euclid had? Double check each logical statement, either on this page or the preceding one and ask if it is justified.
**Byrne’s Proof**

On a given finite straight line (___) to describe an equilateral triangle.

Describe __ and ___ (postulate 3); draw __ and __ (post. 1).

Then will △ be equilateral.

For ___ = ___ (def. 15);
For __ = __ (def. 15);
∴∴∴ = (axiom. 1);

And therefore △ is the equilateral triangle required.

Q.E.D.

This version of Proposition I was edited and written in 1847 (actually, I’ve recreated it here using a computer). It was produced by Oliver Byrne who had the great idea that Euclid would be easier to follow if, instead of all those annoying letters, we used colors to keep track of lines and circles.

The full Byrne edition is available online.
This proof is my version of Byrne's approach, but taken to a greater extreme. It's basically a proof without words. No modern mathematician writes a proof without words in general, but there is sometimes an ongoing column, or article in a magazine that shows an example. It's one of those things that's more amusing than useful. But, notice that you can pretty much follow the proof.
Two Column Proof

<table>
<thead>
<tr>
<th>Statement</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $\overline{AB}$ be a segment</td>
<td>Given</td>
</tr>
<tr>
<td>Let $\mathcal{C}_1 =$ circle with center $A$, radius $\overline{AB}$</td>
<td>Postulate 3</td>
</tr>
<tr>
<td>Let $\mathcal{C}_2 =$ circle with center $B$, radius $\overline{AB}$</td>
<td>Postulate 3</td>
</tr>
<tr>
<td>$C \in \mathcal{C}_1 \cap \mathcal{C}_2$</td>
<td>??</td>
</tr>
<tr>
<td>Let $\overline{AC}$ be a segment</td>
<td>Postulate 1</td>
</tr>
<tr>
<td>Let $\overline{BC}$ be a segment</td>
<td>Postulate 1</td>
</tr>
<tr>
<td>$\overline{AB} = \overline{AC}$</td>
<td>Definition 15</td>
</tr>
<tr>
<td>$\overline{AB} = \overline{BC}$</td>
<td>Definition 15</td>
</tr>
<tr>
<td>$\overline{AC} = \overline{AB} = \overline{BC}$</td>
<td>Common Notion 1</td>
</tr>
</tbody>
</table>

 Modern mathematicians don’t write two column proofs like this, but they are still a good way to practice, especially for students first learning proofs. The second column enforces tremendous discipline: every statement must have a justification! Even when you write proofs in paragraph mode, you should mentally ask yourself at every step for the justification. However, a modern mathematician avoids these proofs not because they want to avoid filling in the second column, but because the proof itself becomes very long and tedious to read. A well written proof should flow more than the one shown here, and repetitive steps should be made more obvious, and sequences of calculations should be combined, etc.

Before moving on to the next stage of rigor, ask yourself this: are there still hidden assumptions, aside from the one about $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$? I think there are, but they are ones that are mostly fillable. To use Postulate 1, what should we assume? What about Postulate 3? What about Definition 15, Common Notion 1, and Definition 20?

I think we need to assume that points are distinct when we make a line segment and when we draw a circle. If we are given a line segment, then we are given the assumption that the endpoints are distinct. But if we are going to assert that we can draw a line segment, then we should prove that the points we are considering are distinct.

Similarly, to assert that we have a triangle, we need to prove first that we have three distinct, noncollinear points.
Lamport Style Proof

1. Let \( \ell = \overline{AB} \) for some points \( A \) and \( B \).
   (a) Let \( \ell \) be a given line segment.
   (b) \( \ell \) has endpoints.
   (c) Let \( A \) and \( B \) be the endpoints of \( \ell \).
   (d) Then \( A \) and \( B \) are distinct.
   (e) Construct \( \overline{AB} \)
   (f) Then \( \ell = \overline{AB} \)

2. Apply Postulate 3 and let \( \mathcal{C}_1 \) be the circle with center \( A \) and radius \( \overline{AB} \).
3. Apply Postulate 3 and let \( \mathcal{C}_2 \) be the circle with center \( B \) and radius \( \overline{AB} \).
4. Let \( C \) be a point in \( \mathcal{C}_1 \cap \mathcal{C}_2 \)
   (a) Apply the Circle Intersection Lemma to see that \( \mathcal{C}_1 \) intersects \( \mathcal{C}_2 \).
5. Construct line segments \( \overline{AC} \) and \( \overline{BC} \).
   (a) \( A \neq C \).
      i. Suppose otherwise that \( A = C \) and get a contradiction.
         A. \( C \in \mathcal{C}_1 \)
         B. \( A \in \mathcal{C}_1 \) since \( A = C \).
         C. \( A \) is the center of \( \mathcal{C}_1 \).
         D. No point can be both the center of a circle and on the circle, a contradiction.
   (b) Construct \( \overline{AC} \)
   (c) \( B \neq C \) follows the same proof as 5(a)
   (d) Construct \( \overline{BC} \)
6. \( \overline{AC} = \overline{AB} \)
   (a) \( A \) is the center of \( \mathcal{C}_1 \).
   (b) \( B \) and \( C \) are on the circle \( \mathcal{C}_1 \).
   (c) All points on a circle have the same distance from the center
7. \( \overline{AB} = \overline{BC} \).
   (a) This follows the same proof as 6.
8. \( \overline{AC} = \overline{AB} = \overline{BC} \)
   (a) Combine steps 6 and 7.
9. Therefore, \( \triangle ABC \) is an equilateral triangle, as desired.

This proof style is inspired by Leslie Lamport: the idea is more or less to take every possible logical assertion down to the smallest possible steps and then justify those steps. Lamport introduces some levels/lists/hierarchy to make it easier to navigate than just presenting every assertion one after the other like in a two column proof. See this article for more detail.
Psuedo-Formal Proof

given
A, B
Point(A) and Point(B)
A not equal B
end given
Segment(A,B)
  by
  A not equal B
Postulate_1;
Circle(A,Segment(A,B))
  by
  A not equal B
Postulate_3;
let BCD = Circle(A,Segment(A,B))
Circle(B,Segment(A,B))
  by
  A not equal B
Postulate_3;
let ACE = Circle(B,Segment(A,B))
not empty(BCD intersect ACE)
  by
  Lemma_Circle_Intersection(BCD,ACE);
let C in (ACE intersect BCD)
  by
  not empty(BCD intersect ACE);
A not equal C
  by
  contradiction A = C
    A in BCD
    Length(A,A) = Length(A,B)
    by
    Definition_15;
    Length(A,B) = 0
    A = B
    false ;
Segment(A,C)
  by
  A not equal C;
B not equal C
  by
  contradiction B = C
    B in ACE
    Length(B,B) = Length(A,B)
    by

Definition_15;
Length(A,B) = 0
A = B
false ;
Segment(B,C)
  by
  B not equal C;
Length(Segment(A,C)) = Length(Segment(A,B))
  by
  C in BCD
  B in BCD
  A = Center(BCD)
Definition_15;
Length(Segment(B,C)) = Length(Segment(A,B))
  by
  A in ACE
  C in ACE
  B = Center(ACE)
Definition_15;
Length(Segment(A,C)) = Length(Segment(B,C))
  by
  Length(Segment(A,C)) = Length(Segment(A,B))
  Length(Segment(B,C)) = Length(Segment(A,B))
Transitivity_Equality (
    Length(Segment(A,C)),
    Length(Segment(A,B)),
    Length(Segment(A,B))
    );
Non_Collinear(A,B,C)
  by
  Lemma_Lengths_Imply_Non_Collinear(
    A,B,C,
    Length(Segment(A,B)),
    Length(Segment(B,C)),
    Length(Segment(A,C))
    );
EquilateralTriangle(A,B,C)
  by
  Non_Collinear(A,B,C)
  Length(Segment(A,C))
    = Length(Segment(A,B))
    = Length(Segment(A,B))
Definition_20;

This is my interpretation of what's known as a formal proof, at least in modern usage. The idea is this: similar to the Lamport style, break down the assertions into the smallest possible steps. But then, have a computer that can read the proof like a program. Each and every line is evaluated, and if you haven't skipped a step or made a mistake, then the computer runs the program/proof and returns the result “correct.”

Thousands of mathematical proofs have been formally verified in this sense, but many of the formal computer verified proofs that have been created are actually much
less readable than this (in other words, what I have here is considerably more human-friendly than the real formal proof verifiers). But the point is this: not to create proofs that are nice to read, but to have computers help double check our work so that we can be more confident that mathematics does not have significant errors lurking in the heart of the subject. See the Wikipedia article, or this article by Thomas Hales in the Notices of the American Mathematical Society. This snippet of an article shows an actual formal proof of the fact that $\sqrt{2}$ is irrational. There's also a more up to date online discussion about what formal proof should mean available here.

Finally, you can see actual results of formalization of all the axioms of geometry, in a 2018 Ph.D. thesis. This compares different systems and axioms, and also some proofs of results (mostly about independence of different axioms) On the Formalization of Fundations of Geometry by Pierre Boutry.

There are some examples as well of Bill Richter's work in implementing some arguments in HOL Light. here.
Interestingly, I think all of the proofs I’ve just shown for Euclid Proposition I.1 are the same proof! What I mean by that is that the logical steps and assertions are essentially the same: the same circles are defined, the intersection is used, etc. But of course the styles are extremely different. What I try to teach students is to write in the modern style, and I try not to be too pushy about it, but I am a little pushy. There’s a lot to be said for the modern style, although it almost certainly isn’t perfect. In any case, it is the current social norm for what a proof should look like, and so it’s a reasonable target to aim for.

In any case, to fill the hole (literally) that Euclid left, we need to add another axiom/postulate.

**Postulate 1** (Circle Continuity Postulate). 1: A line segment with one endpoint outside a given circle, and the other endpoint inside the circle will intersect the circle exactly once. 2: Given any circle, if a second circle contains a point inside the first and a point outside the first, then the two circles intersect twice.

**Example 1.** Figure-out and state a lemma that gives necessary and sufficient conditions in terms of $a$, $b$, and $c$, for when $C_1 \cap C_2$ equals 0 points, 1 point and 2 points. In other words you want to say: The intersection $C_1$ and $C_2$ has 0 points if and only if ________________, it has exactly 1 point if and only if ________________, it has 2 points if and only if ________________.

**2.2.3 A map of Book I**

Next, I’d like to get a quick overview of Euclid’s *Elements*, Book I. On the next page I show a directed graph of all 48 propositions that appear in that Book I.
As you look at the graph, it might be nice to find locate of the highlights of Book I, and I'll point out which ones we're going to do (and to some degree, why):

- Prop. 1, at the bottom, is the construction of an equilateral triangle.
- Prop. 2 states that we can add line segments
- Prop. 3 states that we can subtract line segments (aka cut off).
- Prop. 4, at the bottom right, is SAS congruence. We are not going to prove this. In fact, from the modern perspective it is not possible to prove this: it is taken as an axiom. (I have often wondered if it's equally possible to take SSS as an axiom. I have not been able to prove that it's possible, nor show that it's impossible.)
- Prop. 5 is *Pons Asinorum*: Equal sides imply equal angles.
- Prop. 8 is SSS congruence.
- Prop. 9 is bisecting an angle.
- Prop. 10 is bisecting a line segment.
- Prop. 11 is erecting a perpendicular.
- Prop. 12 is dropping a perpendicular.
- Prop. 13 states that two adjacent angles that make a straight line add to $180^\circ$.
- Prop. 14 states that two adjacent angles that add to $180^\circ$ make a straight line.
- Prop. 15 states that vertical angles are equal.
- Prop. 16 is the Weak Exterior Angle Theorem.
- Prop. 18 states that a larger side implies a larger angle.
- Prop. 19 states that a larger angle implies a larger side.
- Prop. 20 is the Triangle Inequality.
- Prop. 23 is the Copy an Angle Theorem.
- Prop. 26 is ASA congruence.
- Prop. 27 and 28 are versions of Equal Angles in Transversal Imply Parallel.
- Prop. 31 is the construction of a parallel line.

- Prop. 29 is Parallel Implies Equal Angles in Transversal.
- Prop. 30 is Transitivity of Parallels.
- Prop. 32 is the Strong exterior angle theorem.
- Prop. 47 is the Pythagorean Theorem.
- Prop. 48 is the converse of the Pythagorean Theorem.

I think this directed graph is somewhat wonderful. It shows at a glance how mathematics is actually structured: on result depends on other results, and they depend on other ones, and so on, back to the beginning. For instance, the Pythagorean Theorem depends upon almost all of Book I, or, to put it differently, basically, all of Book I is needed to prove the Pythagorean Theorem.

We will see later that non-Euclidean geometry is basically invented by changing the parallel postulate. As a result, all the results below the red line are valid in most non-Euclidean geometry, and none of the results above the red line will be valid. So the red line is a good indication of just how important the parallel postulate is.
2.2.4 Axiom systems

I’ll proceed very ad hoc in the following sections: I’ll introduce Euclid’s postulates, and then add just two postulates (in other words ignoring more subtle gaps). For more complete treatments I recommend the following:


3. William Richter *A Minimal Version of Hilbert’s Axioms for Plane Geometry*. This document is basically an article, unpublished, that lays out in 24 pages a modern treatment of Hilbert’s Axioms. “Modern” in the sense that it uses set theory. It’s intended as a short guide other textbooks, such as the one by Moise’s.

4. Edwin Moise, *Elementary Geometry from an Advanced Standpoint*, Pearson, 1990. This is meant as a college textbook, and treats the whole subject in a foundational manner, using Hilbert’s Axioms and set theory.

2.3 Some Important Propositions

Comments. In this section, we will visit a few of the well-known results from Euclid’s Book I: Pons Asinorum, Weak Exterior Angle Theorem, Equal angles in a transversal implies parallel, Playfair Axiom, Pythagorean Theorem.

2.3.1 Propositions that do not depend on the parallel postulate

**Example 2.** [Pons Asinorum: Euclid I.5] Translate the statement and proof below into modern language. Include a picture. Clean it up as much as you can (shorten it if possible, use fewer letters if possible, use the picture to define the labels, but not to “show” something is true, avoid making a simple logical statement more complicated).

**Proposition.** In an isosceles triangle, the angles at the base are equal to each other. Furthermore, if we extend the equal sides below the base, then the angles under the base are equal.

Let $ABC$ be an isosceles triangle having the side $AB$ equal to the side $AC$, and let the straight-lines $BD$ and $CE$ have been produced in a straight-line with $AB$ and $AC$ (respectively) [Post. 2]. I say that the angle $ABC$ is equal to $ACB$, and (angle) $CBD$ to $BCE$.

For let the point $F$ have been taken at random on $BD$, and let $AG$ have been cut off from the greater $AE$, equal to the lesser $AF$ [Prop. 1.3]. Also, let the straight-lines $FC$ and $GB$ have been joined [Post. 1].

In fact, since $AF$ is equal to $AG$, and $AB$ to $AC$, the two (straight-lines) $FA, AC$ are equal to the two (straight-lines) $GA, AB$, respectively. They also encompass a common angle, $FAG$. Thus, the base $FC$ is equal to the base $GB$, and the triangle $AFC$ will be equal to the triangle $ABG$, and the remaining angles subtend by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) $ACF$ to $ABG$, and $AFC$ to $AGB$. And since the whole of $AF$ is equal to the whole of $AG$, within which $AB$ is equal to $AC$, the remainder $BF$ is thus equal to the remainder $CG$ [C.N. 3]. But $FC$ was also shown (to be) equal to $GB$. So the two (straight-lines) $BF, FC$ are equal to the two (straight-lines) $CG, GB$, respectively, and the angle $BFC$ (is) equal to the angle $CGB$, and the base $BC$ is common to them. Thus, the triangle $BFC$ will be equal to the triangle $CGB$, and the remaining angles subtended by the equal sides will
be equal to the corresponding remaining angles [Prop. 1.4]. Thus, $FBC$ is equal to $GCB$, and $BCF$ to $CBG$. Therefore, since the whole angle $ABG$ was shown (to be) equal to the whole angle $ACF$, within which $CBG$ is equal to $BCF$, the remainder $ABC$ is thus equal to the remainder $ACB$ [C.N. 3]. And they are at the base of triangle $ABC$. And $FBC$ was also shown (to be) equal to $GCB$. And they are under the base.

Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.
Proof.
Let \( \triangle ABC \) be isosceles with \( AB = AC \). By Postulate 2 we extend \( AB \) into a ray \( \overrightarrow{AB} \), and then we let \( F \) be any point on the ray past \( B \). Similarly, we extend \( AC \) to a ray \( \overrightarrow{AC} \), and then let \( G \) be a point on the ray with \( AF = AG \).

Claim: \( \triangle AFC \cong \triangle AGB \).

Proof of claim: Apply SAS using the following:

- \( AF = AG \) (choice of \( G \))
- \( \angle A = \angle A \)
- \( AC = AB \) (isosceles assumption).

Claim: \( \triangle BFC \cong \triangle CGB \).

Proof of claim: Apply SAS using the following:

- \( CF = BG \) (CPCTC, \( \triangle AFC \cong \triangle AGB \))
- \( \angle BFC = \angle CGB \) (same as previous)
- \( BF = CG \) (subtract \( AB = AC \) from \( AF = AG \))

Now we can conclude \( \angle CBF = \angle BCG \), i.e. the angles under the base are equal, half of what we wanted to prove.

Furthermore, the last triangle congruence shows that \( \angle CBG = \angle BCF \), and the first shows that \( \angle ABG = \angle ACF \), whence

\[
\angle ABC = \angle ABG - \angle CBG = \angle ACF - \angle BCF = \angle ACB.
\]

Example 3. Note that the previous result is exactly how Euclid stated it, but for most of us, we don’t care about the angles under the base. If we leave those out, does the proof get any simpler? Restate the result, and follow the same main idea of the proof, but instead of using \( F \) and \( G \) on the outside of \( \triangle ABC \), put them on the line segments \( AB \) and \( AC \) respectively.

Proposition. In \( \triangle ABC \), if \( AB = AC \), then \( \angle ABC = \angle ACB \).

Proof.
Let \( F \) be any point, aside from the endpoints, in the segment \( \overline{AB} \). Let \( G \in \overline{AC} \) with \( AF = AG \).

Claim: \( \triangle AFC \cong \triangle AGB \).

Proof of claim: Apply SAS using the following:

- \( AF = AG \) (choice of \( G \))
- \( \angle A = \angle A \)
- \( AC = AB \) (isosceles assumption).

Claim: \( \triangle BFG \cong \triangle CGF \).

Proof of claim: Apply SAS using the following:

- \( BG = CF \) (consequence of \( \triangle AFC \cong \triangle AGB \))
- \( \angle FBG = \angle GCF \) (same)
- \( BF = CG \) (subtract \( AF = AG \) from \( AB = AC \))

Claim: \( \triangle BCG \cong \triangle CFG \).
Proof of claim: Apply SAS using the following:
\[
\overline{BG} = \overline{CF} \quad \text{(consequence of } \triangle AFC \cong \triangle ABG) \\
\angle BFC = \angle CGB \quad \text{(subtract equal angles from equal angles)} \\
\overline{BF} = \overline{CG}.
\]

Now we can conclude \( \angle CBF = \angle BCG \).

\[\square\]

Example 4. Note that the Pons Asinorum is stated exactly how Euclid stated it, but for most of us, we don’t care about the angles under the base. Let’s see if we can leave that out. Does the main result change if we move \( F \) and \( G \) closer to \( B \) and \( C \)? Does the proof change if we move \( F \) and \( G \) closer to \( B \) and \( C \)? Can we apply “\( \lim_{F \to A} \)” to every step of the proof? What happens if we do? Rewrite the proof, eliminating parts that are now redundant, simplifying parts that basically amount to “add 0” and see what you have left.

Let \( \triangle ABC \) be isosceles with \( \overline{AB} = \overline{AC} \). By Postulate 2 we extend \( \overline{AB} \) into a ray \( \overrightarrow{AB} \), and then we let \( F \) be any point on the ray past \( B \). Similarly, we extend \( \overline{AC} \) to a ray \( \overrightarrow{AC} \), and then let \( G \) be a point on the ray with \( \overrightarrow{AF} = \overrightarrow{AG} \).

Claim: \( \triangle AFC \cong \triangle AGB \).

Proof of claim: Apply SAS using the following:
\[
\overline{AF} = \overline{AG} \quad \text{(choice of } G \text{)} \\
\angle A = \angle A \\
\overline{AC} = \overline{AB} \quad \text{(isosceles assumption)}.
\]

Claim: \( \triangle BFC \cong \triangle CGB \).

Proof of claim: Apply SAS using the following:
\[
\overline{CF} = \overline{BG} \quad \text{(CPCTC, } \triangle AFC \cong \triangle ABG) \\
\angle BFC = \angle CGB \quad \text{(same as previous)} \\
\overline{BF} = \overline{CG} \quad \text{(subtract } \overline{AB} = \overline{AC} \text{ from } \overrightarrow{AF} = \overrightarrow{AG}).
\]

Now we can conclude \( \angle CBF = \angle BCG \), i.e. the angles under the base are equal, half of what we wanted to prove.

Furthermore, the last triangle congruence shows that \( \angle CBG = \angle BCF \), and the first shows that \( \angle ABG = \angle ACF \), whence
\[
\angle ABC = \angle ABG - \angle CBG = \angle ACF - \angle BCF = \angle ACB.
\]

Solution:

Proof.

Let \( \triangle ABC \) be given with \( \overline{AB} = \overline{AC} \).

Claim: \( \triangle ABC \cong \triangle ACB \) with the corresponding parts given by \( A \to A, B \to C \) and \( C \to B \).

Proof of claim: Apply SAS using the following:
\[
\overline{AB} = \overline{AC} \quad \text{(isosceles assumption)} \\
\angle A = \angle A \\
\overline{AC} = \overline{AB} \quad \text{(isosceles assumption)}.
\]

Now we see that
\[
\angle ABC = \angle ACB
\]
by “CPCTC” i.e. corresponding parts of congruent triangles are congruent. \[\square\]
Comments. Our next goal is Euclid, Prop. I.13, mostly because it is relevant to one of the problems on Homework 2 (namely Exercise 2.32). To prove this, we need also Euclid, Prop I.11.

**Proposition 2.3.1** (Euclid, Prop. I.11, Erecting Perpendiculars). Given a line and a point on the line, one can construct a line through the given point and perpendicular to the given line.

**Proof.**

Let \( \ell \) be the given line, and \( C \in \ell \) the given point. Let \( D \in \ell \) be another point on the line. Let \( E \) be another point on the line with \( D - C - E \) and \( CE = CD \).

Apply Prop. I.1 to construct an equilateral triangle on \( DE \), let \( F \) be the third point of the triangle.

Note that \( DF = EF \) (\( \triangle DEF \) is equilateral)

\[ \angle D = \angle E \] (pons asinorum applied to \( \triangle DEF \))

\[ CD = CE \] (by construction of \( E \))

Therefore, SAS implies see that \( \triangle CDF \cong \triangle CEF \). Then \( \angle DCE = \angle ECF \), and so both are right angles.

**Proposition 2.3.2** (Euclid, Prop. I.12, Dropping Perpendiculars). Given a line and a point note on the line, one can construct a line through the given point and perpendicular to the given line.

**Lemma 2.3.1** (Exercise). Perpendicular bisectors lemma: If \( X \) is equidistant from \( A \) and \( B \) then \( X \) is on the perpendicular bisector of \( AB \).

**Example 5.** Suppose you have a TI-84 calculator that can only handle numbers up to size 9,999,999,999 (ten 9's). You have been hired by the state of Maryland to help compare two alternative budgets, one produced by the Green Party and one by the Libertarian Party.

The Green Party proposes a budget of

- Environmental cleanup $4,567,891,230
- Teacher raises $7,777,787,769

The Libertarian party proposes a budget of

- Law Enforcement $3,457,890,123
- Business incentives $8,887,788,876

The Green Party accuses the Libertarians of spending more than them, and the Libertarians accuse the Green party wanting to spend more than them. But, the Democrats and Republicans both accuse the Green and Libertarian parties of proposing to spend the exact same amounts! They have called you in to settle the score. But your calculator can’t add these numbers in full accuracy (it approximates them to 8 digits and uses an exponential notation \(1.2345679E10\)).

Figure out a way to do it.

**Solution:** Your calculator can handle any number with 10 digits or less. So, we split the numbers up into common amounts:

\[
4,567,891,230 + 7,777,787,769 = \left( \frac{3,457,890,123 + 1,110,001,007}{4,567,891,230} \right) + 7,777,787,769
\]

\[
= 3,457,890,123 + \left( \frac{1,110,001,007 + 7,777,787,769}{8,887,788,876} \right)
\]

\[
= 3,457,890,123 + 8,887,788,876
\]
It’s easy to verify the addition within each pair of parentheses. Given the part within each pair of parentheses, the two sums must be equal.
Example 6. [Euclid I.13] Translate the following statement and proof into modern language.

If a straight-line stood on another straight-line makes angles, it will certainly either make two right-angles, or (angles whose sum is) equal to two right-angles.

For let some straight-line $AB$ stood on the straight-line $CD$ make the angles $CBA$ and $ABD$. I say that the angles $CBA$ and $ABD$ are certainly either two right-angles, or (have a sum) equal to two right-angles.

In fact, if $CBA$ is equal to $ABD$ then they are two right-angles [Def. 1.10]. But, if not, let $BE$ have been drawn from the point $B$ at right-angles to [the straight-line] $CD$ [Prop. 1.11]. Thus, $CBE$ and $EBD$ are two right-angles. And since $CBE$ is equal to the two (angles) $CBA$ and $ABE$, let $EBD$ have been added to both. Thus, the (sum of the angles) $CBE$ and $EBD$ is equal to the (sum of the) three (angles) $CBA$, $ABE$, and $EBD$ [C.N. 2]. Again, since $DBA$ is equal to the two (angles) $DBE$ and $EBA$, let $ABC$ have been added to both. Thus, the (sum of the angles) $DBA$ and $ABC$ is equal to the (sum of the) three (angles) $DBE$, $EBA$, and $ABC$ [C.N. 2]. But (the sum of) $CBE$ and $EBD$ was also shown (to be) equal to the (sum of the) same three (angles). And things equal to the same thing are also equal to one another [C.N. 1]. Therefore, (the sum of) $CBE$ and $EBD$ is also equal to (the sum of) $DBA$ and $ABC$. But, (the sum of) $CBE$ and $EBD$ is two right-angles. Thus, (the sum of) $ABD$ and $ABC$ is also equal to two right-angles.

Thus, if a straight-line stood on another straight-line makes angles, it will certainly either make two right-angles, or (angles whose sum is) equal to two right-angles. (Which is) the very thing it was required to show.
Proposition 2.3.3 (Euclid, Prop. I.13). The sum of the angle measures of two supplementary angles is equal to the sum of two right angles.

Comments. The proof that follows involves angles that add to 180°. This sort of addition is not defined according to our earlier definition of angles (always less than 180°) and angle sums (you add two angles by finding congruent pieces in a larger angle). What Euclid does to verify that larger combinations are equal is exactly like in the previous example: he finds a common decomposition. In other words, he breaks the one angle up into two pieces, and then looks at different combinations of the three pieces. Each combination is less than 180°, and taking them in pairs we can see that they are equal.

Heath explains this pretty well, by calling sums of angles that are 180° or bigger “formal sums” meaning that it isn’t something that has a specific meaning, or equal to a specific element of the usual set, but that we manipulate strings formally. Then the follow-up question is: how do we declare two formal sums to be equal. Euclid answers it (implicitly) thus: if we find a common decomposition of the terms in each formal sum.

Proof.
Let B, C and D be collinear with D—B—C. Let A be a point not on the line CD. The angles \( \angle ABC \) and \( \angle ABD \) are supplementary, and we will prove that their sum is equal to two right angles.

Case 1: \( \angle ABD = \angle ABC \). Then both angles are right angles and we are done.

Case 2: \( \angle ABD \neq \angle ABC \). Apply Prop. I.11 to construct a perpendicular, \( BE \perp CD \). Without loss of generality, let A be on the same side of \( BE \) as C. Then A is interior to \( \angle CBE \), and so \( \angle CBE = \angle CBA + \angle ABE \). Similarly, D is interior to \( \angle ABD \) and so \( \angle ABD = \angle ABE + \angle DBE \). Now we have \( \angle CBE \) and \( \angle DBE \) are right angles, and

\[
\angle CBE + \angle DBE = \angle CBA + \angle ABE + \angle DBE = \angle CBA + \angle ABD. \quad \Box
\]

Comments. Our next goal is to prove Euclid, Prop. I.20, the Triangle Inequality, mostly because of the pervasiveness of this inequality in other mathematics courses, such as Analysis. To do this, we need Euclid, Prop. I.18, which in turn requires Prop. I.16, which in turn requires Prop. I.15 and Prop. I.14.

Proposition 2.3.4 (Euclid, Prop. I.14). If the angle measures of two adjacent angles sum to 180°, then the non-shared sides of these angles form a straight line.

Proposition 2.3.5 (Euclid, Prop. I.15 (Kinsey et al., Exercise 2.35, Homework)). If two lines intersect, then the vertical angles are equal.

Definition 2.3.2. A set \( S \) is convex if the following holds: for all \( A, B \in S \) we have \( AB \subseteq S \).

Postulate 2 (Plane Separation Axiom). Let \( \mathbb{E} \) be the plane, \( \ell \) a line in \( \mathbb{E} \) and \( \ell^c \) the complement of \( \ell \). In other words, \( \ell^c \) is the set of all points in the plane that are not on the line. Then \( \ell^c \) can be written as \( \ell^c = S_1 \cup S_2 \) where \( S_1 \) and \( S_2 \) are nonempty, disjoint and convex. Furthermore, if \( A \in S_1 \) and \( B \in S_2 \) then \( AB \) intersects \( \ell \).

Let \( \ell \) be a line, let \( S_1 \) and \( S_2 \) be as in the postulate, and let \( A, B \in \ell^c \) and assume \( A \in S_1 \).

Lemma 2.3.3. \( B \in S_1 \) if and only if \( AB \) is disjoint from \( \ell \).
For later use we define the following common use of language: We call $S_1$ and $S_2$ the sides of the line $\ell$. If $A \in S_1$ we can call $S_1$ the $A$-side of $\ell$. If $A, B \in S_1$ we say they are on the same side of $\ell$. If $A \in S_1$ and $B \in S_2$ we say they are on opposite sides of $\ell$.

**Proof.** Homework.

**Example 7.** [Euclid I.16: Weak Exterior Angle Theorem] Translate the following statement and proof into modern language. (I've numbered the sentences to make it easier to see how they transform into the new version.)

1. For any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles.

2. Let $ABC$ be a triangle, and let one of its sides $BC$ have been produced to $D$. 

3. I say that the external angle $ACD$ is greater than each of the internal and opposite angles, $CBA$ and $BAC$.

4. Let the (straight-line) $AC$ have been cut in half at (point) $E$ [Prop. 1.10].

5. And $BE$ being joined, let it have been produced in a straight-line to (point) $F$.

6. And let $EF$ be made equal to $BE$ [Prop. 1.3], and let $FC$ have been joined, and let $AC$ have been drawn through to (point) $G$.

7. Therefore, since $AE$ is equal to $EC$, and $BE$ to $EF$, the two (straight-lines) $AE$, $EB$ are equal to the two (straight-lines) $CE$, $EF$, respectively.

8. Also, angle $AEB$ is equal to angle $FEC$, for (they are) vertically opposite [Prop. 1.15].

9. Thus, the base $AB$ is equal to the base $FC$, and the triangle $ABE$ is equal to the triangle $FEC$, and the remaining angles subtended by the equal sides are equal to the corresponding remaining angles [Prop. 1.4].

10. Thus, $BAE$ is equal to $ECF$.

11. But $ECD$ is greater than $ECF$.

12. Thus, $ACD$ is greater than $BAE$.

13. Similarly, by having cut $BC$ in half, it can be shown (that) $BCG$—that is to say, $ACD$—(is) also greater than $ABC$.

14. Thus, for any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles.

15. (Which is) the very thing it was required to show.

**Solution:** (1) Extend one side of a triangle, call the exterior angle $\gamma$, and the opposite interior angles $\alpha$ and $\beta$.

Then $\gamma > \alpha, \beta$.

**Proof.** (2) Let $\triangle ABC$ be given. Extend $BC$ to any point $D$.

(4) Bisect $AC$ at $E$ [I.10].

(5) & (part of 6) Extend $BE$ and cut off a length $EF$ equal to $BE$ [I.10].
(part of 9) Claim: $\triangle AEB \cong \triangle CEF$.
(rest of 9) Proof of claim: Apply SAS to the following:

- (part of 7) $AE = EC$ (E bisects $AC$),
- (8) $\angle AEB = \angle CEF$ (I.15 Vertical Angles are Equal),
- (rest of 7) $BE = EF$ ($(F$ cuts off equal length).

(GAP!) Claim: point $F$ is in the interior of $\angle ECD$. Proof of claim: homework.

(GAP!) Since $F$ is in the interior, $\angle ECF$ is part of $\angle ECD$ (definition of part of angle).

Therefore we have

- (10) $\angle BAE = \angle ECF$ (CPCTC)
- (11) $< \angle ECD$ (Whole is Greater than the Part, CN 5)
- (12) $= \angle ACD$ (A is on the ray $\overrightarrow{CE}$).

(12) This shows that $\angle ACD$ is greater than the opposite interior angle that’s on the same line.

(13) For the other angle, extend line $AC$ to point $G$, apply the same reasoning to get $\angle BCG$ is greater than the opposite interior angle on line $AC$, and note that $\angle ACD = \angle BCG$ by Vertical Angles are Equal (I.15). 

$\square$
Proposition 2.3.6 (Euclid, Prop. I.18, (Kinsey et al., Exercise 2.39, Homework)). In any triangle, if one side is greater than a second side, then the angle opposite the first side is greater than the angle opposite the second side.

Proposition 2.3.7 (Euclid, Prop. I.19). In any triangle, if one angle is greater than a second angle, then the side opposite the first angle is greater than the side opposite the second angle.

Proposition 2.3.8 (Euclid, Prop. I.20, Triangle Inequality). In any triangle, the sum of the lengths of any two sides is greater than the length of the remaining side.

Proof. Let \( \triangle ABC \) be any triangle. Extend \( BA \) and let \( E \) be a point on the extension such that \( AE = AC \) (Prop. 3). Then

\[
\angle BEC = \angle AEC = \angle ACE \quad (B \text{ on ray } EB) \\
< \angle BCE \quad \text{(Whole is Greater than the Part, CN 5).}
\]

Now consider the triangle \( \triangle BCE \). By what we have just shown, \( \angle BEC < \angle BCE \). Apply “Larger Angle Implies Larger Side”, Prop. 19, to conclude we have \( BE > BC \).

Thus

\[
BC < BE = AB + AE = AB + AC \\
\text{(above)} \quad \text{(A is on line segment } AE) \\
= AB + AC \quad \text{(} AC = AE \text{, by construction of } E)\text{.}
\]

Note that \( BC < AB + AC \) is what we wanted to prove. \( \square \)

Comments. Our next goal is the Playfair Axiom, and this requires the familiar results about parallel lines and various angles in a transversal.

Proposition 2.3.9 (Euclid, Prop. I.23, Copy the Angle Theorem). Given a line segment and an angle, one can construct an angle using the given line segment as one of the sides, congruent to the given angle.

Lemma 2.3.4 (CATE: Corresponding Angles in a Transversal are Equal). Let two lines and one transversal be given, and let the angles be labelled \( \angle 1, \angle 2, \angle 3, \angle 4, \angle 5, \angle 6, \angle 7, \angle 8 \) as shown in the figure below.\(^2\)

\[
\begin{array}{c}
1 \quad 2 \\
3 \\
\text{3} \quad 4 \\
5 \\
6 \\
7 \quad 8
\end{array}
\]

Then the following are equivalent

\(^2\)For sticklers in the audience, the labels can be defined without relying on a picture. Suppose the original lines are \( AB \) and \( CD \), that the transversal crosses the these lines between \( A \) and \( B \) and between \( C \) and \( D \) and that \( A \) and \( C \) are on the same side of the transversal. Let \( E \) and \( F \) be the intersections of the transversal with \( AB \) and \( CD \) respectively. Let \( G \) and \( H \) be points on the transversal with \( G \) on the side of \( AB \) that is opposite \( F \), and with \( H \) on the side of \( CD \) that is opposite \( E \). Then \( \angle 1 = \angle AEG, \angle 2 = \angle BEG, \angle 3 = \angle AEF, \angle 4 = \angle BEF, \angle 5 = \angle CFE, \angle 6 = \angle DFE, \angle 7 = \angle CFH \) and \( \angle 8 = \angle DFH \).
1. \( \angle 1 = \angle 5 \),
2. \( \angle 2 = \angle 6 \),
3. \( \angle 3 = \angle 6 \),
4. \( \angle 4 = \angle 5 \),
5. \( \angle 3 = \angle 7 \),
6. \( \angle 4 = \angle 8 \),
7. \( \angle 3 + \angle 5 = 180^\circ \),
8. \( \angle 4 + \angle 6 = 180^\circ \).

**Proof.** Homework.

**Proposition 2.3.10** (Euclid, Prop. I.27, If CATE then parallel). *If two lines are cut by a transversal so that CATE holds, then then the first two lines are parallel.*

\[ \begin{align*}
\alpha &= \beta \\
\beta &= \angle ABC = 180^\circ - \alpha
\end{align*} \]

\( \alpha = \beta \implies \parallel \)

**Proof.** By CATE, we can use assume any equality of corresponding angles that we like. What we will use, is the alternate interior angles are equal.

Let \( \ell \) and \( m \) be the two given lines, let \( A \) and \( B \) be points of intersection of the transversal with \( \ell \) and \( m \), let \( \alpha \) and \( \beta \) be two alternate interior angles at \( A \) and \( B \) respectively.

Assume, for contradiction, that \( \ell \) and \( m \) intersect at a point \( C \). Then we have a triangle \( \triangle ABC \) and \( \beta \) is an exterior angle. Therefore, by Prop. I.16, \( \beta > \alpha \). This contradicts our hypothesis that the angles are equal, and so the lines cannot meet.

**Proposition 2.3.11** (Euclid, Prop I.31, Construction of Parallel). *Given a straight line and a point not on the line, one can construct a line through the point and parallel to the given line.*

**Proof.** Let \( A \) be the given point and \( BC \) the given straight line. Let \( D \) be any point between \( B \) and \( C \). Construct the angle \( \angle DAE \) such that \( \angle DAE = \angle ADC \). Then the line \( AD \) is a transversal of the lines \( AE \) and \( BC \), with the alternate interior angles equal. Thus, \( AE \) is parallel to \( BC \).
2.3.2 Propositions that depend on the Parallel Postulate

Proposition 2.3.12 (Euclid, Prop. I.29, Parallel implies corresponding angles Equal). If two lines are parallel, then CATE holds.

Comments. The crucial difference between this proposition and I.27 (and I.28) is this: I.27 and I.28 say that if certain angles are equal, then lines are parallel. I.29 is the converse: if lines are parallel, then certain angles are equal.

Proof. We prove this by contrapositive: suppose the alternate interior angles are not equal. Then one is smaller than another. Then the smaller one, and the angle adjacent to it, add to less than 180. Therefore the lines intersect by Postulate 5.

By the preceding Lemma (HW), if any angles are not equal or add to anything besides 180 then the lines will be not parallel. □

Proposition 2.3.13 (Euclid, Prop. I.32, Strong Exterior Angle Thorem). In any triangle, if one side is extended, then the exterior angle is equal to the sum of the two opposite interior angles. Furthermore, the sum of the interior angles in the triangle is $180^\circ$.

\[
\gamma = \alpha + \beta, \quad \gamma + \delta = 180
\]

Proof.

Let $\triangle ABC$ be given and extend $BC$ to the point $D$. Apply Prop. I.31 to construct a line $CE$ parallel to $AB$. Then

\[\angle ACB + \angle ACE + \angle DCE = 180^\circ\]

since the angles form a straight line.

Now, I.29 implies $\angle DCE = \angle ABC$ (opposite interior) and $\angle ECD = \angle ABC$ (exterior and opposite interior).

\[\angle ACB + \angle BAC + \angle ABC = 180^\circ\]

by replacing the last two angles in the equation above by the congruent ones used here. □

Proposition 2.3.14 (Euclid Prop. I.30: Homework). Lines parallel to the same line are parallel to each other.

Proof. Homework □

Definition 2.3.5. Playfair’s Postulate: Given a line and a point not on the line, there exists a unique line through the point and parallel to the given line.

Theorem 2.3.6 (Playfair’s Postulate (Kinsey et al., Exercise 2.53 Homework)). We assume in all cases postulates 1–4. Then postulate 5 is equivalent to Playfair’s Postulate. In other words, postulates 1–5 imply Playfair’s Postulate, and postulates 1–4 together with Playfair’s Postulate imply Postulate 5.

Proof. To re-iterate, we assume Postulates 1–4 in all cases, as well as the missing postulates about circles intersecting and the plane separation axiom. Therefore we assume also Euclid Propositions I.1–I.28 as well as I.31. When we assume Postulate 5, we can also use I.29 and I.30.
“⇒” We assume Euclid’s Parallel Postulate, and prove Playfair’s Postulate.

Let \( \ell \) be a given line and \( C \) a point not on \( \ell \). We claim that there exists a unique line through \( C \) that is parallel to \( \ell \). By Prop. I.31 there exists a line \( m \) through \( C \) and parallel to \( \ell \). To prove uniqueness, let \( n \) be a line through \( C \) with \( n \neq m \). We will show that \( n \) is not parallel to \( \ell \).

Let \( t \) be a line through \( C \) that intersects \( \ell \). Since \( n \neq m \), by the plane separation postulate, one half of \( n \) is on the same side of \( m \) as \( \ell \). Label angles \( \alpha \), \( \beta \) and \( \gamma \) as shown:

Since \( m \parallel \ell \) we have \( \alpha + \beta = 180 \) (Equal Angles Implies Parallel, i.e. I.29). Since \( \gamma \) is interior to \( \beta \), we have \( \gamma < \beta \). Therefore \( \alpha + \gamma < 180 \). Therefore, Postulate 5 shows that \( n \) and \( \ell \) intersect.

“⇐” We assume Playfair’s Postulate and prove Postulate 5. Let \( m \) and \( \ell \) be two lines and \( t \) a transversal, with \( \alpha \) and \( \beta \) as shown:

so that \( \alpha + \beta < 180 \). To prove Postulate 5, we need to prove that \( m \) and \( \ell \) intersect.

Construct a second line \( n \) through \( C \) by copying \( \alpha \) to an alternate interior angle:

Then \( n \parallel \ell \) by Equal Angles Implies Parallel.

Note that \( \alpha + \beta \neq 180 \) but \( \beta + \gamma = 180 \), so \( \alpha \neq \beta \). Therefore \( n \neq m \). Since \( n \parallel \ell \), and since parallels are unique, and since \( n \neq m \), we have that \( m \) is not parallel to \( \ell \). \( \Box \)
Chapter 3

Hyperbolic Geometry

Comments. In this chapter we explore geometry where we make one change to our postulates: we change the parallel postulate (and drop the postulate about area). Instead of Euclid’s Parallel Postulate, we use Playfair’s statement, and then change this.

3.1 Neutral Geometry

Definition 3.1.1. By neutral geometry we mean those propositions that are true using only postulates 1–4, 6–12.

Comments. In neutral geometry, the only things that are true are the theorems that don’t use Euclid’s Parallel Postulate 5. So, the following are not true: Propositions I.29, I.30 and I.32–I.47, I.48. These propositions are, respectively: Parallel Lines Implies Equal Angles, Transitivity of Parallel lines, Interior Angles add to $180^\circ$, and finally the Pythagorean Theorem and its converse.

Here is a short (and incomplete) list of propositions that we may use in neutral geometry: Propositions I.1–I.28, I.31. Here are two more that may or may not be familiar.

Theorem 3.1.2. Let $\triangle ABC$ and $\triangle DEF$ have sides of length $a, b, \ldots, f$, and angles of measure $\alpha, \beta, \gamma, \delta, \varepsilon, \eta$ as shown:

1. Hypotenuse-Side Congruence (Theorem 3.20 in the textbook). If $\alpha = \delta = 90^\circ$, and $a = d$ and $b = e$, then $\triangle ABC \cong \triangle DEF$.

2. SSA+ Congruence (Theorem 3.21 in the textbook). If $a = d$ and $b = e$ and $\alpha = \beta$ and either $\beta, \varepsilon > 90^\circ$ or $\beta, \varepsilon < 90^\circ$, then $\triangle ABC \cong \triangle DEF$.

Comments. Hypotenuse-Side Congruence is “obvious” in Euclidean geometry: simply apply the Pythagorean Theorem and solve for the third side! However, we are no longer allowed to use the Pythagorean Theorem! The result is still true however, and is easy to prove by contradiction.
Table 3.1: Neutral Geometry Postulates

**Postulate 1** Given any two distinct points there is a unique line through them.

**Postulate 2** One can extend a given line segment from either end to form a ray, or from both ends to form a line.

**Postulate 3** Given a point and a length, there exists a unique circle with the given point as a center and the given length as a radius.

**Postulate 4** All right angles are congruent.

**Postulate 6 (Incidence)** 1: There exists at least one plane. 2: Every plane contains at least three noncollinear points. 3: Every line contains at least two points.

**Postulate 7 (Betweenness)** There exist a relation “betweenness” on sets of points, written as $A — B — C$, that satisfies the following: 1: If $B$ is between $A$ and $C$ then $B$ is also between $C$ and $A$, i.e. if $A — B — C$ then $C — B — A$. 2: Given any two distinct points, $A$, $B$, there exists a third distinct point $C$ such that $A — C — B$. 3: Given any two distinct points, $A$, $B$, there exists a third distinct point $C$ such that $A — B — C$. 4: Given any three distinct collinear points, exactly one of them is between the other two.

**Postulate 8 (Plane separation property)** Given a plane $E$ a line $\ell \in E$, $\ell$ divides $E$ into two disjoint convex sets, $S_1 \cup S_2$, called the **sides of the line**, so that if $A \in S_1$ and $B \in S_2$ then the line segment $AB$ intersects $\ell$.

**Postulate 9 (Congruence)** 1: Congruence of line segments is an additive equivalence relation. 2: Congruence of angles is an additive equivalence relation. 3: Equality of area is an additive equivalence relation on polygons. 4: Congruent triangles have equal areas.

**Postulate 10 (Archimedes’ Axiom)** If $a$ and $b$ are positive real numbers with $a < b$, then there exists $n \in \mathbb{N}$ such that $na > b$. In particular, the following hold. 1: Given line segments $\overline{AB}$ and $\overline{CD}$, there exists $n \in \mathbb{N}$ and a point $X$ on $\overline{CD}$ so that $\overline{CX} = n \cdot \overline{AB} > \overline{CD}$, and thus $C — D — X$. 2: Given angles $\angle ABC$ and $\angle DEF$, there exists $n \in \mathbb{N}$ and a ray $\overrightarrow{EX}$ so that $\angle XEF = n \cdot \angle ABC$ and $\angle XEF > \angle DEF$.

**Postulate 11 Circular Continuity Principle** 1: A line segment with one endpoint outside a given circle, and the other endpoint inside the circle will intersect the circle exactly once. 2: Given any circle, if a second circle contains a point inside the first and a point outside the first, then the two circles intersect twice.

**Postulate 12 (SAS)** If $\triangle ABC$ and $\triangle A'B'C'$ are two triangles with $\overline{AB} = \overline{A'B'}$, $\angle ABC = \angle A'B'C'$ and $\overline{BC} = \overline{B'C'}$, then the triangles are congruent.
Definition 3.1.3. 1. Given any triangle $\triangle ABC$ we write $\sum ABC$ for the sum of interior angles of the triangle.

2. Given a triangle $\triangle ABC$, we define the **defect** as

$$\text{def } \triangle ABC = 180 - (\angle A + \angle B + \angle C).$$

Lemma 3.1.4. Given any triangle $\triangle ABC$, there exists a triangle $\triangle DEF$ such that $\angle D \leq \frac{1}{2} \angle A$ and $\text{def } \triangle DEF = \text{def } \triangle ABC$.

**Proof.**

Let $X$ be the midpoint of $BC$ and extend $AX$ to a point $Y$ such that $AX = XY$. We will show that $\triangle ABY$ has the desired properties.

To be more specific, if we label the angles as shown in the figure, then let $D$ equals $A$ or $Y$, whichever vertex has the corresponding minimal angle out of $\angle 1$ and $\angle 6$.

It's easy to see that SAS implies $\triangle BXY \sim \triangle ACX$. Then we have $\angle 3 = \angle 5$ and $\angle 2 = \angle 6$, which implies that

$$\sum ABC = \angle 1 + \angle 4 + \angle 3 + \angle 2 = \angle 1 + \angle 4 + \angle 5 + \angle 6 = \sum ABY.$$

Since $\angle 1 + \angle 2 = \angle A$ we have that either $\angle 1 \leq \frac{1}{2} \angle A$ or $\angle 2 \leq \frac{1}{2} \angle A$.

This means that either $\angle 1 \leq \frac{1}{2} \angle A$ or $\angle 6 \leq \frac{1}{2} \angle A$. Let $D$ be the vertex of $\triangle ABY$ that has the smaller angle, and let $E$ and $F$ be the other two vertices of $\triangle ABY$.

Corollary 3.1.5. Given any triangle $\triangle ABC$ and any $k \in \mathbb{N}$, there exists a triangle $\triangle DEF$ with $\sum DEF = \sum ABC$ and $\angle D \leq \frac{1}{2^k} \angle A$.

**Proof.** (Sketch) Apply the previous lemma $k$ times.

**Proposition 3.1.1** (I.17: Corollary of WEAT). In any triangle the sum of two interior angles is less than $180^\circ$.

**Theorem 3.1.6** (Saccheri-Legendre Theorem). If $\triangle ABC$ is any triangle in neutral geometry, then

$$\text{def } \triangle ABC \geq 0.$$

**Proof.** Suppose for contradiction that $\triangle ABC$ has $\text{def } \triangle ABC = -\alpha$ with $\alpha > 0$. Apply Archimedes Postulate to let $k$ be any natural number with $2^k \alpha > \angle A$. Then we have $\frac{1}{2^k} \angle A < \alpha$. Apply the previous corollary to get a triangle $\triangle DEF$ with $\text{def } \triangle DEF = \text{def } \triangle ABC$ and $\angle D \leq \frac{1}{2^k} \angle A < \alpha$. Then

$$-\alpha = 180 - (\angle D + \angle E + \angle F),$$

$$180 + \alpha = \angle D + \angle E + \angle F,$$

$$180 + \alpha < \alpha + \angle E + \angle F,$$

$$180 < \angle E + \angle F.$$

But this contradicts Euclid I.17 (which says, quite simply, that the sum of any two angles in a triangle is $< 180^\circ$).
3.2 Hyperbolic Geometry

Comments. We start our exploration of hyperbolic geometry by developing a very small amount of theory, by which I mean axioms, definitions, proofs, etc. An alternative is to start by finding some physical model of hyperbolic geometry and exploring it. This approach has some advantages, after all, this is the way we learned Euclidean geometry. The world around us is, roughly, Euclidean, and we learned about straight lines, triangles, circles and planes before we ever started proving theorems about them.

However, I've chosen to put a small amount of theory first for two reasons. One, the physical model that we will construct later, might, possibly, not be as convincingly real, consistent, and meaningful as the physical world which surrounds us, which we have explored for our entire lives, and which has shaped the very nature of our body and minds. Secondly, I think the section that follows is a nice illustration of the power of a theoretical, mathematical, approach. There are many situations in which our physical intuition fails, which we can not picture, which we cannot build or touch, but which we can still explore, following paths laid down by theory. To some degree, this is one of those situations.

3.2.1 A Physical model of the Hyperbolic plane

Activity 17. Make the equilateral triangle based model of the hyperbolic plane.

Cut a slit in each hexagon and insert an equilateral triangle into the slit, taping two edges of the triangle to the two edges of the slit; try to tape the whole edge very firmly with no gaps. You should have 7 equilateral triangles coming together in the center. You now have a regular 7-gon, but it is not planar: it cannot be flattened out on the Euclidean plane.

Once you have a bunch of 7-gons, start to join them together as follows. You can tape two 7-gons together along an edge. Then you should join a third 7-gon, but also add an equilateral triangle so that you again have 7 triangles coming together.

Once you have three 7-gons, add three more along one “side”. Your goal should be to make your shape roughly the same size in all directions (don’t add all the hexagons only at the ends to make something that stretches out in one direction but not the other).

Continue until you could have at least a foot in all directions.

(a) Draw a triangle with sides at least a foot long, smoothing the surface as you draw each line. Measure the angles and add them together. What is the angle sum?

(b) Draw two parallel lines, at least one triangle apart, and as long as possible. Measure the distance between them at a variety of different places. What do you notice?

(c) Draw one line, and one point not on the line, at least one triangle distance apart. Draw lines through the given point that are parallel to the given line. How many parallel lines could you draw through the given point?

(d) See if you can figure out how many triangles one needs to build this model. In other words, suppose your goal is to make a 2 foot diameter hexagon-circle, how many triangles do you need?
dashed line = fold,  
solid line = cut
Comments. There are other physical models of the hyperbolic plane, some made out of curved strips of paper taped together, and others by crochet. There are also lots of examples of hyperbolic shapes in nature, such as lettuce leaves, holly leaves, and folded coral. Hyperbolic shapes show up in architecture in roofs, and towers.

We'll see later how much of the work of Escher can be understood using hyperbolic geometry.

But for now, I'll say that my favorite hyperbolic shape is the mighty Pringle!

3.2.2 The Hyperbolic Postulate

Comments. Now we take neutral geometry, which makes no assumptions about the parallel postulate, and we add one more statement: the negation of Playfair's Postulate. As you recall, Playfair's postulate makes two conclusions: there is a parallel line, and it is unique. In neutral geometry, Proposition I.31 still holds, this asserts that there is a parallel line. So, if we wish to negate Playfair, and not change anything else, we cannot negate the part that says “there is a parallel line.” The part we have to negate is the assertion about uniqueness. Thus, we have arrived at our hyperbolic postulate:

Postulate 5 (Hyperbolic Postulate). Given any line and any point not on the line, there are at least two distinct lines through the given point and parallel to the given line.

Comments. As strikingly hard to believe as the previous postulate is, the first consequence we derive from it is equally hard to believe. I think also that many students are surprised that the following theorem is a consequence of a postulate about parallel lines. But this should not be a surprise, because the main memorable part of the Strong Exterior Angle Theorem is about the angles in a triangle adding to $180^\circ$, and the essential fact needed to prove that theorem was about a parallel line having equal corresponding angles.
Example 1. True or False: Given $\triangle ABC$, line $m$, with two right angles as shown, there exists a ray $\ell$ in the interior of $\angle XBC$, such that $\beta < \alpha$.

Solution: In Euclidean geometry, this would definitely be false, for the following reason. Since $AB$ makes perpendiculars with both $m$ and $r$, we have that $m \parallel r$ (Equal Angles Implies Parallels). Then we would apply Parallel Implies Equal Angles to get that $\angle ACB = \angle CBX$. Then $\alpha$ is part of an angle that's equal to $\beta$, which would imply $\beta > \alpha$.

In Hyperbolic geometry, the first conclusion is still true: $m \parallel r$ since Equal Angles Implies Parallels is still true. However it's not the case that this implies Parallel Implies Equal Angles. So, all we can say for sure is that it's not necessarily false in Hyperbolic geometry, although this doesn't imply that it's true either. We will see soon that it is true.
3.3 Poincaré disc model

Comments. Recall that in all geometry, we have things called “points” and “lines” and “circles” that satisfy the postulates. On one level, we don’t assume anything about them at all except that they satisfy the postulates. On another level, it’s useful to have a model of that geometry: something that exists that satisfies the postulates. For Euclidean geometry, the usual model is $\mathbb{R}^2$, i.e. the $(x, y)$-plane. Then each point is a single $(x, y)$ pair of coordinates, lines satisfy certain equations, etc.

Similarly, we want to have a model of Hyperbolic geometry. One model of can be made (approximately) with paper, but we want one that is more exact, more mathematically precise, etc.

Definition 3.3.1. The Poincaré disc model of the Hyperbolic Plane is defined as follows. Fix a Euclidean circle, $C$, perhaps the unit circle if we want.

1. Hyperbolic Points equal the points in the interior of $C$,
2. Hyperbolic Lines equal one of the following: (i) diameters of $C$, (ii) circular arcs that are orthogonal to $C$,
3. Hyperbolic Circles equal Euclidean circles (but the Hyperbolic Center is not equal to the Euclidean center).

Note that in all cases no points on the circumference are included. A “circular arc” is part of a circle.
Activity 18. Given a line $AB$ and a point $C$ not on $AB$, draw five lines through $C$ and parallel to $AB$, make two of them asymptotic parallels (these are parallels that get infinitely close together). Use the program NonEuclid to help you with this example.

![Diagram showing five lines parallel to AB and two asymptotic parallels]

Solution: The picture below shows four lines made with dashes: these are all parallel to $AB$. It also shows two lines that are blue: these are asymptoticall parallel to $AB$.

Figure 3.1: Poincaré disk models, large number of parallel lines

Comments. We assume, in the Euclidean world, that congruence is a primitive concept, it is just something that we “know” or can “see”. But in the Poincaré Disc model, this is no longer the case.

In particular, if we work on a single line, we know can add copies of a line segment along that line, going infinitely far (adding line segments used only Euclid I.1–I.28). Thus, in the Poincaré model, distance must change from what our eyes tell us as we get closer to the edges of the circle: we imagine that the edge of the circle is infinitely far away. Thus, as we “move” closer to the edge, distances must become compressed, so that a congruent line segment looks shorter nearer the edge.

Activity 19. On the hyperbolic line shown below, mark off a line segment $AB$ with $B$ to the right of $A$, with $AB$ about a third of the Euclidean length of the line shown. Use the program NonEuclid to mark off three or four hyperbolic copies of $AB$ along the same line. You’ll probably want to draw Hyperbolic circles to mark of the distances.
Solution: As described above, $B$ is chosen somewhat randomly. Once we have $B$, draw a circle with center $B$ that goes through $A$. Let $C$ be the second point where this circle intersects the line. Now draw a circle with center $C$ that goes through $B$, and let $D$ be the second point where this circle intersects the line. Keep repeating this step and the result may be as shown below.

Note the following lesson from this: segments that are congruent in hyperbolic geometry look smaller as they get closer to the edge.

Activity 20. Which of the following line segments are congruent to each other? Are they all line segments? Can you explain? Use NonEuclid to help you out.
Solution: It should be easy to tell that most of these segments are not congruent. Remember from the previous activity: Segments that are congruent in hyperbolic geometry look smaller as they get closer to the edge. Turning this around: segments that look the same and are closer to the edge, cannot be congruent. In this picture, some of these segments that I've shown you look the same, and therefore cannot be congruent.

In particular, we can say that $CD$ and $EF$ cannot be congruent to $AB$. Similarly, $IJ$ cannot be congruent to $GH$.

It turns out, though this is harder to see, that $CD$, $EF$ and $IJ$ are all not really line segments. We know that a line segment is part of a line. We know that lines are arcs of circles that are orthogonal to the outside circle. But, if you extend the curves for $CD$, $EF$ and $IJ$ but use the same curvature as they start with, then the results will not be orthogonal to the outside.

Finally, it turns out that $AB \cong GH$. Again: segments that are congruent look smaller when they are closer to the outside edge.

Activity 21. On a given line segment, construct an equilateral triangle. Make sure your construction passes the “drag test.”
Solution: Shown below is the solution I made in NonEuclid. I followed the same steps as in Euclid I.1: draw a circle with center $A$ that goes through $B$. Draw a circle with center $B$ that goes through $A$. Let $C$ be one of the intersections of these two circles. Construct the line segment $AC$ and $BC$ (note these are of course, hyperbolic line segments, i.e. segments of circular arcs that would be, if extended, orthogonal to the outside circle.

The “drag test” means this: you should be able to select the “move” tool, then click on $A$ or $B$ and move them, and have the whole construction move and adjust, maintaining an equalateral triangle as you go. To do this, you have to make sure that you construct the objects correctly: to construct the first circle you have to click on the center and then click on the other point, you can’t just make a circle the right size. To construct $C$ you have to have NonEuclid find the intersection, not just put a point there yourself.

Activity 22. From the point $C$, drop a perpendicular to $AB$. Make sure your construction passes the “drag test.”
Solution: Shown below is the solution I made in NonEuclid. I followed roughly the same steps in Euclid I.12: Draw a circle with center $C$ that goes through $A$. Let $D$ be the second point where this circle intersects $AB$. Now we draw two circles with centers $A$ and $D$ and the same radius, $\overline{AD}$. So one circle has center $A$ and goes through $D$, the other has center $D$ and goes through $A$. Let $E$ and $F$ be the intersections of these two circles. Then the line $EF$ is perpendicular to $AB$.

The “drag test” means the same thing as before: you should be able to select the “move” tool, then click on $A$ or $B$ and move them, and have the whole construction move and adjust.
Activity 23. See if you can construct $\triangle ABC$, line $m$, with two right angles as shown, and a ray $\ell$ in the interior of $\angle XBC$, such that $\alpha > \beta$ (Yes, $\alpha \geq \beta$).

Solution: It’s definitely possible to do this. I did it just by drawing the line segments $AC$, $AB$, $BX$, and then moving $X$ and $C$ around to make the angles $90^\circ$ (I used the “measure angle” tool to watch them as I moved points.) Then I moved $C$ towards the edge so that $\beta$ got pretty small. I made the line $\ell$ and it was easy to make $\alpha > \beta$.

Activity 24. In the pictures below, the figure in the middle has three line segments that are all the same length, and the figure in the lower left is an equilateral triangle. In each figure which angles are equal to $60^\circ$?

Solution: The picture in the upper right is not of an angle or multiple angles: it does not consist of straight line segments.

The picture in the middle is of an angle, or rather multiple angles: H-lines include E-diameters, which these are. The divide $360^\circ$ into 6 equal parts, so they must be $60^\circ$ degrees each.

The picture in the bottom left consists of a triangle. The angles are not $60^\circ$.

Activity 25. In NonEuclid, draw a triangle roughly as shown below. What is $\sum ABC$? Move the points around: how big can you make $\sum ABC$? How small can you make $\sum ABC$?
Solution: (a) As shown, the triangle is close to Euclidean. When I created this in NonEuclid, I found $\angle ABC = 46.7^\circ$, $\angle BCA = 63.6^\circ$ and $\angle CAB = 55.6^\circ$. In this case $\sum ABC = 165.9^\circ$.

(b) If we make the triangle smaller:

I found $\angle ABC = 59.6^\circ$, $\angle BCA = 60.0^\circ$ and $\angle CAB = 59.9^\circ$. In this case $\sum ABC = 179.5^\circ$.

A similar result holds for this triangle as well:

To make the angle sum smaller, we should make the triangle larger, like this
At the most extreme, we can have a triangle like this:

For this triangle, NonEuclid shows each angle as being $< 0.1$. Thus, the angle sum is $< 0.3$.

3.3.1 Constructing a line segment in the Poincaré disc

Comments. In this section we show how construct a line segment through two points $A$ and $B$, using Euclidean tools.

Example 2. Find a proof of the following: (Euclidean) Given any three noncollinear points $A$, $B$ and $C$, there exists a circle that goes through them.

You can take your pick if you want to find a synthetic solution, arguing about points and lines in a Euclidean sense, or an analytic solution, looking at equations with $x$ and $y$.

Synthetic Hint: Justify why the perpendicular bisector of $AB$ intersects the perpendicular bisector of $BC$. The intersection is the center of the circle; justify the rest of the construction.

Analytic Hint: Suppose the given points have coordinates $A = (a_1, a_2)$ and $B = (b_1, b_2)$ and $C = (c_1, c_2)$. Explain why we know it’s possible to solve for $h$, $k$ and $r$ such that three equations are simultaneously true:

$$\begin{align*}
(a_1 - h)^2 + (a_2 - k)^2 &= r^2 \\
(b_1 - h)^2 + (b_2 - k)^2 &= r^2 \\
(c_1 - h)^2 + (c_2 - k)^2 &= r^2
\end{align*}$$

(3.1)  (3.2)  (3.3)
Start by explaining why we can find the center \((h, k)\) (probably by subtracting these equations from each other to get two \textit{linear} equations in \(h\) and \(k\)).

\textbf{Solution:}

\textit{Proof.} Construct the line segments \(\overline{AB}\) and \(\overline{BC}\). Since \(A\), \(B\) and \(C\) are noncollinear, the angle \(\angle ABC\) is not 0° or 180°. Construct the perpendicular bisector of \(\overline{AB}\) and \(\overline{BC}\). These lines are not parallel, so they intersect at a point \(O\). Let \(C'\) be the circle with center \(O\) and radius \(AO\). Since \(O\) is on the perpendicular bisector of \(\overline{AB}\) it is equidistant from \(A\) and \(B\). Since \(O\) is on the perpendicular bisector of \(\overline{BC}\), it is equidistant from \(B\) and \(C\). Thus, \(C'\) also goes through \(B\) and \(C\). \(\square\)
Definition 3.3.2. (Euclidean) Let $C$ be a circle with center $O$ and radius 1. Let $P$ be any point, distinct from $O$. The inverse of $P$ through $C$ is the unique point $P' \in \overrightarrow{OP}$ such that

$$\frac{OP'}{OP} = \frac{1}{OP}.$$ 

If $P$ is interior to $C$ the picture looks like this

If $P$ is moved closer to $O$, then $P'$ moves farther to the right, and vice versa. If $P$ is on $C$ then $P = P'$. If $P$ is exterior to $C$ then simply reverse the labels in the picture above.

The point $P$ is also called the reflection of $P$ in $C$. The process of taking $P$ to $P'$ is called inversion.

We can apply the whole idea above to a circle of radius $r$ as well. In that case we require

$$\frac{OP'}{r} = \frac{r}{OP}.$$ 

Example 3. Let $C$ be a the unit circle in $\mathbb{R}^2$. Find the inverse, in cartesian coordinates, of each of the following points.

(a) $P = (1, 0)$
(b) $P = (1/2, 0)$
(c) $P = (1/10, 0)$
(d) $P = (1/2, 1/2)$
(e) $P = (3/4, 3/4)$
(f) $P = (2/5, 2/5)$

Solution: See the next example.
Example 4. Find an explicit formula for the function

\[
\text{Inv.} : \text{int}(\mathcal{C}) \longrightarrow \mathbb{R}^2 \\
(x, y) \longmapsto (x', y')
\]

Solution: Let’s write a point \((x, y)\) as a vector \(\vec{x}\). Then

\[
\frac{1}{\|\vec{x}\|} \vec{x}
\]

is a unit vector, pointing in the same direction as \(\vec{x}\). Now we want to change that vector and make it have length a certain length. The way we change the length of a unit vector is to multiply it by the length we want it to have. Thus

\[
\frac{1}{\|\vec{x}\|} \cdot \frac{1}{\|\vec{x}\|} \vec{x}
\]

has the length we want. The formula is

\[
\text{Inv.}\vec{x} = \frac{1}{x^2 + y^2} \vec{x}.
\]
Example 5. Make a rough sketch of the inverse image of the letter “R” shown below

![R](image)

Solution: We start by drawing radial lines from the center of the circle out through R:

We've marked radial distances on the picture to make it easier to estimate inversions. For instance, the lower end of the left leg of the letter R has a distance from the origin of about 0.375, and so that point should move outwards along the radial line to a distance of \( \frac{1}{0.375} \approx 2.7 \). Similar calculations can be done for other points:

- top left corner \( 0.7 \mapsto \frac{1}{0.7} \approx 1.45 \)
- bottom of left leg \( 0.35 \mapsto \frac{1}{0.35} \approx 2.8 \)
- middle points on left edge \( 0.4 \ldots 0.55 \mapsto 2.5 \ldots 1.8 \)
- lower end right leg \( 0.6 \mapsto \frac{1}{0.6} \approx 1.66 \)
- right leg intersecting upper loop \( 0.6 \mapsto \frac{1}{0.6} \approx 1.66 \)

In fact, the whole right leg of R is roughly along the distance 0.6: a little more than that at the bottom and a little less than that in the middle. Thus, that leg, when we invert it, will be roughly along the distance 1.66, a little less than that at the bottom, and a little more than that in the middle.

The result is as follows:
And here's how it looks without the extra information:
**Lemma 3.3.3.** (Euclidean) Let \( C \) be any circle with radius 1 and let \( P \) and \( P' \) two points that are inverses of each other with respect to \( C \). Let \( C_2 \) be any circle through \( P \) and \( P' \). Then \( C_2 \) is orthogonal to \( C \).

**Proof.** Let \( C, C_2, P \) and \( P' \) be as pictured below. Suppose that \( O \) is the center of \( C \), \( M \) is the center of \( C_2 \). Let \( X \) be one of the intersections of \( C \) and \( C_2 \). Let \( B \) be the intersection of \( OP' \) with the perpendicular through \( M \).

We are assuming that \( C \) has unit length, so \( OX = 1 \) and therefore \( OP \cdot OP' = 1 \) because

\[
O^2 = 1
= OP \cdot OP'
= (OB - BP)(OB + BP')
= (OB - BP)(OB + BP)
= OB^2 - BP^2
= (OM^2 - BM^2) - BP^2
= OM^2 - MP^2
= OM^2 - MX^2
\]

This shows that \( \angle OXM = 90^\circ \) because \( \triangle OXM \) satisfies the Pythagorean Theorem.

Therefore \( C \perp C_2 \) because \( OX \perp MX \) and the lines \( OX \) and \( MX \) are tangent to \( C_2 \) and \( C \) respectively.
Corollary 3.3.4. Given two distinct points in the Poincaré Disc we can construct the hyperbolic line through them.

Proof. Let $A$ and $B$ be the two points. Let $A'$ be the inverse of $A$. Let $\mathcal{C}_2$ be the unique circle that goes through $A, A'$ and $B$. By the lemma, $\mathcal{C}_2$ is orthogonal to $\mathcal{C}$. Therefore $\mathcal{C}_2$ is a hyperbolic line in the Poincaré Disc.

3.4 Theory

Theorem 3.4.1 (Pasch’s Theorem: Neutral Euclidean). Let $\triangle ABC$ be given and let line $\ell$ intersects $\triangle ABC$ at some point $D$ on the line segment $AB$, with $D \neq A, B$. Then $\ell$ intersects the triangle at a second distinct point. In fact, $\ell$ intersects contains of the following: a point between $A$ and $C$, a point between $B$ and $C$, or the vertex $C$.

Proof. Since $A - D - B$ we have that $A$ and $B$ are on opposite sides of $\ell$. By the line separation axiom, we have one of the following cases: $C$ lies on $\ell$, $C$ lies on the same side of $\ell$ as $A$, $C$ lies on the same side of $\ell$ as $B$. Case 1: If $C$ lies on $\ell$ then we are done. Case 2: If $C$ lies on the same side of $\ell$ as $A$ then $C$ is on the opposite side from $B$. Then $BC$ intersects $\ell$, let $E = BC \cap \ell$. Since neither $B$ or $C$ lie on $\ell$, we have $E \neq B, C$. So $B - E - C$. Case 3: If $C$ lies on the same side of $\ell$ as $A$, the proof is similar.

Theorem 3.4.2 (Cross-bar Theorem: Neutral Euclidean). Let $\triangle ABC$ be given and let $\overrightarrow{AD}$ be between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. then $\overrightarrow{AD}$ intersects the segment $\overline{AB}$ at a point between $B$ and $C$.

Proof. Let $E$ be a point on line $AC$ such that $A$ is between $E$ and $C$. Then we can apply Pasch’s Theorem to the line $AD$ intersecting side $EC$. So $AD$ intersects either $BC$ or $BE$, and we just need to prove that it doesn’t intersect $BE$.

Note that $D$ is interior to $\triangle ABC$ and so $D$ and $E$ lie on opposite sides of line $AB$. Therefore $AD$ and $BE$ lie on opposite sides of line $AB$. Therefore $AD$ and $BE$ cannot intersect. Therefore, $AD$ intersects $BC$.

Lemma 3.4.3. Let $\triangle ABC$ be any right triangle, with right angle at $A$. We define a sequence of points $D, E, F, \ldots$ recursively: let $D$ be the point on $\overrightarrow{AC}$ with $CD = BC$; Let $E$ be the point on $\overrightarrow{CD}$ with $DE = BD$; etc.
Label the angles at $B$

\[ \angle 1 = \angle ABC, \quad \angle 2 = \angle CBD, \quad \angle 3 = \angle DBE, \ldots. \]

For all $n \in \mathbb{N}$, we have $\angle 1 + \cdots + \angle n < 90^\circ$.

**Proof.** Construct a perpendicular to $AB$, creating point $X$. Then $BX$ is parallel to $AC$ by Neutral Euclidean Geometry.

Note that $\angle 1 + \cdots + \angle n = \angle ABM$ for some point $M$ in the sequence $C, D, E$, etc. We claim that $M$ is interior to $\angle ABX$. If this wasn’t the case then $M$ would be on $BX$ or on the opposite side of $BX$ from $A$. Then $BX$ would enter the triangle $\triangle ABM$ at vertex $B$. Then the Cross-Bar Theorem would imply that $\overline{BX}$ intersects $\overline{AM}$, contrary to the fact that $m$ is parallel to $r$. Since $M$ is interior to $\angle ABX$ we have that $\angle ABM < \angle ABX = 90^\circ$. \qed

**Corollary 3.4.4** (Neutral Euclidean). Let $\ell$ be a second line through $B$, distinct from $m$. Let $\alpha$ be the angle between $m$ and $AC$. There exists $n \in \mathbb{N}$ such that $n\alpha > 90^\circ$.

**Proof.** This is Archimedes Axiom, applied to $\alpha$ and $90^\circ$. \qed

**Corollary 3.4.5** (Neutral Euclidean). With $\angle 1, \angle 2, \ldots$, as in the previous lemma, and $n$ fixed as in previous corollary, there exists $\angle i$ such that $1 \leq i \leq n$ and $\angle i < \alpha$.
Proof. If this is not the case then all of the angles \( \angle 1, \angle 2, \ldots, \angle n \) are \( \geq \alpha \), and so
\[
\angle 1 + \cdots + \angle n \geq \alpha + \cdots + \alpha = n\alpha > 90°.
\]
This contradicts the lemma before the corollary.

Corollary 3.4.6 (Neutral Euclidean). With everything as in the previous corollary, suppose \( \angle i \) satisfies \( 1 \leq i \leq n \) and \( \angle i < \alpha \). Let \( K \) be the point on \( AC \) that defines one leg of \( \angle i \). Then \( \angle AKB < \alpha \).

Proof. With \( i, K \) fixed as above, and applying Pons Asinorum, we have the following figure:

Note that \( \angle ABK = \angle 1 + \cdots + \angle i \) by definition of the labels \( 1, \ldots, i \). Also, \( BJ = JK \) by the recursive definition of the points \( D, E, \ldots K \). Therefore, by Pons Asinorum, we have \( \angle K = \angle i < \alpha \).

Corollary 3.4.7 (Hyperbolic). Assume now that \( \ell \) is a second line through \( B \) parallel to \( AC \). Then \( Y \) is exterior to \( \triangle ABK \).

Proof. We claim that \( Y \) is exterior to \( \triangle ABK \), for otherwise \( \ell \) would intersect \( \overline{AK} \) by Cross-Bar Theorem, but this would contradict the fact that \( \ell \parallel r \).

Theorem 3.4.8 (Hyperbolic). As we constructed above, we assume the following:

- \( \triangle ABK \) is a right triangle with \( \angle A = 90° \)
- lines \( m \) and \( \ell \) both go through \( B \), are both parallel to \( AK \),
- Both \( m \) and \( \ell \) are exterior to \( \angle B \), with \( \ell \) between \( m \) and \( BK \),
- \( \angle K < \alpha \) where \( \alpha \) is the angle between \( m \) and \( \ell \)

Then \( \sum AKB < 180° \).

Comments. In simplified words: If you can stretch one leg of a right triangle to make the angle there smaller than the angle between two parallel lines that go through the opposite vertex, then the angles in the triangle add up to less than 180°.

Proof. Let \( \gamma \) be the angle between \( BK \) and \( \ell \):
Since \( \ell \) is exterior to \( \triangle ABK \) we have
\[
\alpha + \beta + \gamma = 90.
\]
Therefore \( \alpha + \beta < 90 \), and \( \beta < 90 - \alpha \). Now we add
\[
\beta < 90 - \alpha \\
\gamma < \alpha \\
\alpha = 90
\]
to get
\[
\alpha + \beta + \gamma < 180
\]
\( \square \)

### 3.5 Defect and sum of interior angles

**Corollary 3.5.1.** In Hyperbolic Geometry, there exists a right triangle \( T \) with \( \text{def } T > 0 \).

**Proof.** By the previous theorem, there is a right triangle \( T \) with \( \sum T < 180 \). Therefore \( \text{def } T > 0 \). \( \square \)

**Lemma 3.5.2.** The defect of a triangle is additive. In other words, if we divide \( \triangle ABC \) into sub-triangles \( T_1, T_2, T_3, \ldots \), then
\[
\text{def } ABC = \text{def } T_1 + \text{def } T_2 + \text{def } T_3 + \ldots.
\]

**Proof.** Homework. \( \square \)

**Theorem 3.5.3.** In Hyperbolic geometry, all triangles have defect \( \gg 0 \). In other words, all triangles have angle sum \( < 180^\circ \).

We will break the proof into a couple of lemmas.

**Lemma 3.5.4 (Hyperbolic).** Suppose for contradiction that some right triangle \( X \) exists for which the angles add to 180. Fill in the missing steps below to show that a triangle \( \tilde{X} \) exists with sides twice as large as \( X \), but the same angles as \( X \).

**Sketch.** Represent \( X \) with \hspace{1cm} . We can construct four copies of \( X \) (or one original \( X \) and three new copies if you want) and arrange them into a single triangle thus:

\[
\tilde{X} =
\]

**Proof.** Represent \( X \) with \hspace{1cm} . Extend the base of \( X \) and copy the angles to make two triangles
Since the angles in $X$ add to 180, and since the measure of a straight line is 180, we have that the angle between these two triangles is the same as one of the angles in $X$. Thus, SAS implies that

and so we have three copies of $X$

Now we extend the leg on the left

We now have two sides and a right angle. Thus we can make the triangle on top, and SAS shows that it's congruent to the original

Since the angles in $X$ add to 180, the hypotenuse of the last triangle added makes a $180^\circ$ angle with the hypotenuse of the triangle on the right, and so it makes a straight line. Thus we have a single triangle, $\tilde{X}$, decomposed into 4 copies of $X$, and all the corresponding angles are the same as in $X$.

**Corollary 3.5.5** (Hyperbolic). Suppose for contradiction that some right triangle $X$ exists for which the angles add to 180. For each $k \in \mathbb{N}$, there exists a triangle $\tilde{X}$ exists with each side $2^k$ times as large as $X$, but the same angles as $X$.

**Proof.** Short version: Apply the previous lemma $k$ times.

The previous lemma shows that we can construct a triangle with sides 2 times as large as $X$, and with the same angles as $X$. Induction shows that we can apply the lemma repeatedly, and construct, for all $k \in \mathbb{N}$, a triangle that is $2^k$ times as large as $X$, and with the same angles as $X$.

**Corollary 3.5.6** (Hyperbolic). Suppose for contradiction that some right triangle $X$ exists for which the angles add to 180. Let $T$ be a right triangle with angles that add to $< 180$. There exists a right triangle $\tilde{X}$ with the same angles as $X$ and legs larger than the legs of $T$.

**Proof.** Short version: Combine the previous corollary and Archimedes' Postulate.

Let $X$ have legs $x$ and $y$, and let $T$ have legs $t$ and $s$. By Archimedes' Postulate there exists some natural numbers $n$ and $m$ such that $nx > t$ and $my > s$. Let $k \in \mathbb{N}$ such that such that $2^k > m$ and $2^k > n$. By the previous corollary, for all $k \in \mathbb{N}$, we can construct a triangle $\tilde{X}$ with sides of length $2^k x$ and $2^k y$. Then $2^k x > nx > t$ and $2^k y > my > s$.

**Corollary 3.5.7** (Hyperbolic). Suppose for contradiction that some right triangle $X$ exists for which the angles add to 180. Let $T$ be a right triangle with angles that add to $< 180$. Let $\tilde{X}$ be a right triangle whose legs contain the legs of $T$ (or a copy of $T$) and that has the same angles as $X$. Then we have a contradiction.
Proof. We apply the previous results to make \( X \) sufficiently large so that its legs are longer than the legs of \( T \). Then we can copy \( T \) inside of \( X \) as shown above. Label the vertices of \( T \) with \( A, B \) and \( C \) and the additional vertices of \( X \) with \( D \) and \( E \).

Add the line segment \( CD \) and then divide \( X = \triangle ADE \) into triangles \( T, T_2, \) and \( T_3 \).

Then we have

\[
0 = \text{def } X = \text{def } T + \text{def } T_1 + \text{def } T_2
\]

which is a contradiction since \( \text{def } T > 0 \) and \( \text{def } T_1 + \text{def } T_2 \geq 0 \).

Proof of Theorem. Let \( X \) be any right triangle. If \( \text{def } X = 0 \) then the previous lemmas and corollaries show that

\[
0 = \text{def } X = \text{def } X > \text{det } T > 0
\]

which is a contradiction. Therefore, \( \text{def } X = 0 \) is impossible. Therefore \( \text{def } X > 0 \) is the only possibility.

Corollary 3.5.8. Every triangle has angle sum \( < 180 \).

Proof. Let \( \triangle ABC \) be given. Suppose that \( \angle A \) is the largest angle in the triangle. Drop the perpendicular from \( A \) to \( BC \), intersecting at the point \( D \).

We claim that \( D \) lies between \( B \) and \( C \). Suppose, for contradiction, that this does not happen. Then \( D \) is on the line \( BC \) and makes an exterior angle to the triangle.

Apply Proposition I.16, the Weak Exterior Angle Theorem, to the triangle \( \triangle ADB \) and the angle \( \angle ABC \). Then \( \angle ABC \) is greater than \( \angle ADB \), i.e. \( \angle ABC > 90 \). But this can’t happen since \( \angle ABC < \angle BAC \) and \( \triangle ABC \) can have only one angle greater than 90.

Then \( \triangle ABC \) is divided into two right triangles. Each of the right triangles has nonzero defect, and \( \text{def } ABC \) is the sum of these defects, so \( \text{def } ABC > 0 \).

Corollary 3.5.9. The interior angles inside of any quadrilateral add to less than \( 360^\circ \).

Corollary 3.5.10. Rectangles do not exist.
3.6 Angle of Parallelism

Lemma 3.6.1. Let \( r \) be a line and \( C \) a point not on \( r \) and \( A \in r \). Let \( \ell, m, \) and \( n \) be distinct lines through \( C \). Suppose that on each side of \( AC \) we have that \( m \) is between \( \ell \) and \( n \). If \( \ell \) and \( n \) are parallel to \( r \) then so is \( \ell \).

Proof. Suppose for contradiction that \( m \) intersects \( r \) at a point \( B \). On the same side of \( AC \) as \( B \) we have that \( m \) is between \( \ell \) and \( n \). Without loss of generality, suppose that \( n \) is the line closest to \( r \).

Then \( n \) is a line entering the triangle \( \triangle ABC \) and so, by Pasch’s Theorem \( n \) would intersect \( r \). \( \square \)

Corollary 3.6.2. Let \( r \) be a line and \( C \) a point not on \( r \) and \( A \in r \). Let \( \ell, m, \) and \( n \) be distinct lines through \( C \). Let \( X, Y \) and \( Z \) be points on \( \ell, m \) and \( n \) all on the same side of \( AC \). Let \( \alpha = \angle ACX, \beta = \angle ACY \) and \( \gamma = \angle ACZ \). Suppose \( \alpha < \beta < \gamma \). If \( \ell \) and \( n \) are parallel to \( r \) then so is \( \ell \).

Proof. The assumption that \( \alpha < \beta < \gamma \) implies that line \( m \) is between lines \( \ell \) and \( n \). Therefore we can apply the previous lemma. \( \square \)

Corollary 3.6.3. Given any line \( r \) and any point \( C \) not on \( r \), there are an infinite number of lines through \( C \) and parallel to \( r \).

Proof. By the Hyperbolic Postulate, there are at least two lines, \( m \) and \( n \) that go through \( C \) and are parallel to \( r \). There exists a third line \( \ell_1 \) between \( m \) and \( n \). By the previous lemma, \( \ell_1 \) is parallel to \( r \). Now there exists another line \( \ell_2 \) between \( m \) and \( \ell_1 \). By the previous lemma, \( \ell_2 \) is parallel to \( r \). Induct to see that for all \( n \in \mathbb{N} \), there exists a line \( \ell_n \) that is parallel to \( r \). \( \square \)

Theorem 3.6.4. Let \( r \) be any line, \( C \) any point not on \( r \), let \( A \in r \) such that \( AC \perp r \), and \( B \) any other point on \( r \). Define a real number \( \Pi \) as follows:

\[
\Pi = \lim_{X \to \infty} \frac{\angle ACX}{AB}
\]

Let \( \ell \) be any line through \( C \) and let \( \alpha \) be the angle between \( AC \) and \( \ell \) on the same side as \( X \). Then

\[
\ell \text{ is } \begin{cases} \parallel r & \text{if } 90 \geq \alpha \geq \Pi \\ \text{intersects } r & \text{if } \alpha < \Pi \end{cases}
\]
Definition 3.6.5. With the set-up as in the previous theorem, we call $\Pi$ the **Angle of Parallelism**

```
\begin{center}
\begin{tikzpicture}
\coordinate (A) at (0,0);
\coordinate (B) at (1,0);
\coordinate (C) at (0,1);
\coordinate (X) at (1.5,0);
\draw (A) -- (B) -- (C) -- cycle;
\draw (C) -- (X);
\draw (C) -- (A) -- (X);
\draw (A) -- (X) node [right] {$\alpha$};
\end{tikzpicture}
\end{center}
```

**Proof.** Before we prove the actual statements of the theorem, we should discuss why the limit exists. Basically, there are two facts that are being combined. (1) If $X_2$ is on the opposite side of $X_1$ from $B$, then $\angle ACX_1 < \angle ACX_2$. (2) The set of all real numbers $\angle ACX$ is bounded above. Given these two facts, then we can construct a sequence of points $X_n$ that are monotonically “approaching infinity”, and then the sequence of real numbers $\angle ACX_n$ is a bounded monotonic sequence and it converges by fairly deep properties of the real numbers. Property (1) follows from the fact that $X_1$ is in the interior of $\angle ACX_2$. Property (2) follows from the fact that $\sum ACX < 180$, and so $\angle ACX < 90$. Note that property (1) implies that $\Pi > \angle ACX$ for any $X$.

So now we assume that the limit exists and so $\Pi$ is defined.

Suppose $\alpha < \Pi$. Since $\alpha$ is less than the limit, there exists some $X$ such that $\alpha < \angle ACX < \Pi$. Then $\ell$ enters the triangle $\triangle ACX$ and so it intersects $r$ by Pasch’s Theorem.

Now we prove “if $90 \geq \alpha \geq \Pi$ then $\ell$ is parallel to $r$” by contrapositive: if $\ell$ intersects $r$ then $\alpha < \Pi$. Suppose that $\ell$ does intersect $r$ at point $D$. Let $X$ be another point on $r$, on the opposite side of $D$ from $A$.

```
\begin{center}
\begin{tikzpicture}
\coordinate (A) at (0,0);
\coordinate (B) at (1,0);
\coordinate (C) at (0,1);
\coordinate (D) at (1.5,0);
\coordinate (X) at (2,0);
\draw (A) -- (B) -- (C) -- cycle;
\draw (C) -- (D) -- (X);
\draw (C) -- (A) -- (X);
\draw (A) -- (X) node [right] {$\alpha$};
\end{tikzpicture}
\end{center}
```

Then $D$ is in the interior of $\angle ACX$, so $\angle ACX > \alpha$. On the other hand, as stated above, $\Pi > \angle ACX$, so $\Pi > \alpha$.

**Definition 3.6.6.** A line that makes the angle $\Pi$ is called the **asymptotic parallel** (through $C$ and in the direction of $B$). A line through $C$ and parallel to $AB$ that is not asymptotic is called a **divergent parallel**.

**Corollary 3.6.7.** $\Pi < 90$.

### 3.7 Triangles

**Comments.** In Euclidean Geometry we have Angle-Angle-Angle similarity: If two triangles have equal corresponding angles, then they are similar. But a far stronger statement is true in hyperbolic geometry.

**Theorem 3.7.1** (AAA congruence). If two triangles have equal corresponding angles, then they are congruent.

**Proof.** Homework. Hint: Proceed by contradiction. Justify the construction in the following picture, then apply SAS, then show the angles in the quadrilateral add to 360. This is a contradiction in hyperbolic geometry.
3.8 Biangles

Comments. This section describes rather amazing objects that exist only in hyperbolic geometry: biangles. These are things that have many of the same properties as triangles, the difference is that they are missing one vertex. In other words, two of the sides are parallel, are asymptotically parallel. This is easy to picture in the Poincaré disc: the vertex is on the perimeter of the disc, and can thus be thought of as infinitely far away.

Comments. We can get a feel for biangles, and their congruent properties by looking at two regular triangles in Poincaré that look congruent.

Then we can imagine what happens as we stretch one point on each triangle farther away.

We can bring it almost all the way to the circumference and we still have regular triangles.
But, once we drag the third point all the way to the circumference, we no longer have triangles because the third point is no longer part of the hyperbolic plane.

But it still looks like a triangle. So what do we have? Since the figure in the final stage is so close to a triangle, shouldn't we be able to prove things about it? Yes.

**Definition 3.8.1.** A **biangle** consists of two distinct rays $m$ and $n$, and two distinct points $A$ and $B$ such that $A$ is the endpoint of $m$, $B$ is the endpoint of $n$ and $m$ and $n$ are asymptotic parallels. We call $AB$ the **base** of the biangle.

![Biangle Diagram](image)

**Definition 3.8.2.** We say that two biangles $\angle mABn$ and $\angle rCDs$ are **congruent** if

$$m \cong r$$

$$\overline{AB} \cong \overline{CD}$$

$$n \cong s$$

$$\angle A \cong \angle C$$

$$\angle B \cong \angle D$$

**Comments.** The following theorem is the analogue of SAS congruence, where we cross off the first “$S$”
**Theorem 3.8.3** (SAS Congruence). Given two biangles, if one pair of corresponding angles are equal, and the bases are equal, then the biangles are congruent.

**Proof.** Let $\angle mABn$ and $\angle rCDS$ be given with $\angle A = \angle C$ and $\overline{AB} = \overline{CD}$.

Fix points $X$, $Y$, $W$ and $Z$ on the rays $m$, $n$, $r$ and $s$. We need to prove $\angle XAB \cong \angle WCD$. In other words, we need to prove that the remaining pair of angles are equal.

Suppose, for contradiction, that $\angle ABY > \angle CDZ$.

Copy angle $\angle CDZ$ to $AB$ to create the point $E$ such that $\angle ABE = \angle CDZ$ and with $E$ interior of $\angle ABY$ (since we assumed for contradiction that $\angle ABY > \angle CDZ$).

(The rest of the proof follows the same argument as the proof of Theorem 3.10.1, that the angle of parallelism does not depend upon the side of the transversal.)

Since $\angle ABE$ is less than the $\angle ABY$, and since $BY$ is the asymptotic parallel to $AX$, the line $BE$ intersects $AX$. Let $F$ be the intersection. Let $G$ be the point on $\overline{CW}$ with $\overline{CG} = \overline{AF}$.

Now we apply SAS to the two triangles $\triangle ABF$ and $\triangle CDG$:

\[
\begin{align*}
\overline{AB} &= \overline{CD} & \text{The “S” in our given SAS} \\
\angle XAB &= \angle WCD & \text{The “A” in our given SAS} \\
\overline{AF} &= \overline{CG} & \text{definition of } G
\end{align*}
\]

Therefore $\triangle ABF \cong \triangle CDG$. Therefore $\angle CDG \cong \angle ABF$. But $\angle ABF = \angle ABE$ and $E$ was constructed by copying $\angle CDZ$. So $\angle CDG = \angle CDZ$. This implies that $DG = DZ$, which is a contradiction since $DG$ is not parallel to $CW$ and $DZ$ is.

**Theorem 3.8.4** (WEATB: Weak Exterior Angle Theorem for Biangles). Given any biangle, if we extend the base, then the exterior angle is greater than the opposite interior angle.

**Theorem 3.8.5** (AA$S$ congruence, Homework). If two biangles have equal interior angles, then they are congruent.

**Proof.**
3.9 Mathematical Models of the Hyperbolic Plane

Comments. Now we have a few theoretical results about the Hyperbolic Plane, but they are probably hard to understand. How can the angles in a triangle add up to $< 180^\circ$? How can rectangles be impossible? What do asymptotic parallels look like?

The answers to these questions cannot be only mathematical, for they are not answered by proving that a logical statement is true or false. We will provide some answers by studying two mathematical models of the hyperbolic plane.

In a sense, each model is really an extended example. After all, we already have our axioms, our definitions, and a few theorems. So, what we will see next are examples that make the axioms true, that illustrate the definitions and theorems.

3.9.1 Beltrami-Klein model

Comments. Recall that the Hyperbolic Plane $\mathbb{H}$ consists of a set of “points” and “lines” that satisfy the postulates we have made for hyperbolic geometry. One model of these can be made (approximately) with paper. We consider now a model that is more mathematically precise than paper.

Definition 3.9.1. The Beltrami-Klein model of the Hyperbolic Plane $\mathbb{H}$ is defined as follows.

1. We fix a Euclidean circle $C$,
2. Points in $\mathbb{H} = \text{the Euclidean points in the interior of } C$, not including $C$ itself,
3. Lines in $\mathbb{H} = \text{Euclidean line segments that are chords across } C$, not including the endpoints on $C$ itself.

Comments. Starting with the above definitions, we can then apply all of our other definitions: segments consist of those points on a line and between two endpoints; rays are defined similarly; angles consist of the union of two rays with a common endpoint, etc.

Example 6. [Activity] Given a line $AB$ and a point $C$ not on $AB$, draw five lines through $C$ and parallel to $AB$, make two of them asymptotic parallels.

Solution: The dashed lines are ordinary parallels. The blue lines are asymptotic parallels; although they “appear” to intersect $AB$, the intersection would be on $C$ itself, and these points are not in $\mathbb{H}$. Thus, the blue lines get infinitely close to $AB$. 
Example 7. [Activity] Which of the following line segments are congruent to each other? Can you explain?

Figure 3.6: Five line segments: are they congruent?

Solution: It turns out that only $\overline{AB}$ and $\overline{GH}$ are congruent. Although we cannot prove this at present, the next example will justify it to some degree.

Comments. We assume, in the Euclidean world, that congruence is a primitive concept, it is just something that we “know” or can “see”. But in the Beltrami-Klein model, this is no longer the case.

In Euclidean geometry the easiest way to “see” distance, is to work on a single line. We know can add copies of a line segment along that line, going infinitely far:

This shows us a “ruler” of unit lengths, going off into the distance.

Example 8. [Activity] On the hyperbolic line shown below, mark off a line segment $\overline{AB}$ about half or a third of the Euclidean length of the radius. Mark off three or four hyperbolic copies of $\overline{AB}$ along the same line (in other words, the distances should be equal in hyperbolic geometry, but not in Euclidean geometry), as we did above for a Euclidean line.
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Solution: We should be able to put an infinite number of segments to the right of $A$, without hitting the circumference of $C$, so the segments must get shorter and shorter. So in the Beltrami-Klein model, distance must change from what our eyes tell us as we get closer to the edges of the circle: we imagine that the edge of the circle is infinitely far away. Thus, as we “move” closer to the edge, distances must become compressed, so that a congruent line segment looks shorter nearer the edge.

Figure 3.7: Congruent line segments in Beltrami-Klein model

Comments. The previous picture should convince the reader that distances in the Beltrami-Klein model cannot possibly be the same as distances that we can see with our eyes. By analogy, the Beltrami-Klein model is like a map of the globe: when we look at the map, the distances we can see with our eyes are the distances on the piece of paper, but these are not the “real” distances on the surface of the earth.

Like projections that make maps, we can ask if about features aside from distances: When we compare areas in the Hyperbolic sense are they the same as areas in the Euclidean sense? What about angles?

Example 9.[Activity] In the pictures below, which angles are equal to $60^\circ$? Can you explain?
Solution: We have not required this in our definition of the Beltrami-Klein model, but it is usually assumed that the center of this model is geometrically isotropic. In other words, we assume that, geometrically speaking, everything directly above the center is the same as everything directly to the side, and similarly in every direction. This principle should apply to both distances and angles, but only to those that are based at the center. For instance, two line segments, each with one endpoint at the center, should be congruent in hyperbolic geometry if and only if they are congruent in Euclidean Geometry. The same should apply to angles, each with one vertex at the center.

Thus, all the angles at the center are congruent. Are they all equal to 60°? In some sense, 60° is a meaningless number; what does it mean? In a classical Euclidean sense, all angles start with 90°, a right angle. From there we define the angle of a straight line to be two right angles, or 180°. Then 60° is one third of a straight line. This is still the case for the angles around the center. So, the hyperbolic measure of each angles at the center is indeed 60°.

What about the other \( \neq \) shaped figure? So far we have not said enough to be convincing one way or the other. Let's move on to the triangle.

We have proven that the angles in every triangle do not add up to 180°. But the angles that we see with our eyes do add up to 180°. Therefore, at least one (maybe all) of the angles that we are looking at does not have the measure that we think it does.

Now we can return to the last \( \neq \) shaped figure. Since we know that at least one angle in the triangle is not 60°, and since the angles in the last figure look the same, and since both are pretty far away from the center of the circle, there's a good chance that none of these angles are 60°.

Example 10. Recall that line segments appear to compress as we approach the edges of the circle. See if you can guess what an equilateral triangle will look like with one segment as shown below. What does your picture tell you about the angles in the triangle?
Solution:

Pons Asinorum is still true. Thus, the angles are equal, although they appear not to be. The angle near the origin is what it appears to be, maybe 25°? So all angles are about 25°. The outside angles look bigger than this, so (at least some) angles near the border are smaller than they appear.

Example 11. It can be shown that regular 3-gons (i.e. equaliteral triangles) can cover the Euclidean plane (i.e. without overlapping and without gaps):
But it's a theorem that it is not possible to cover the Hyperbolic plane with equilateral triangles. Hm, but doesn't the same picture as above work here? Why doesn't the picture below show how to start covering the Beltrami-Klein plane with equilateral triangles?

Solution:

**Example 12.** Finally, we show below a tiling of the plane by regular 7-gons and equilateral triangles, as found on Wikipedia (http://en.wikipedia.org/wiki/Beltrami-Klein_model)
Figure 3.9: Tiling of the Beltrami-Klein Hyperbolic plane by regular 7-gons and equilateral triangles

Comments. From these pictures we draw the following conclusions: the Beltrami-Klein model does represent the points, lines, angles, etc. of the Hyperbolic Plane, but we have to learn to ignore the distance and angles that we see. In the next section we learn about a model of the Hyperbolic Plane that fixes one of these deficiencies.

3.9.2 Distance in Beltrami-Klein

Comments. We can approach distance in the Beltrami-Klein model of $\mathbb{H}$ the same way we approach distance in the Euclidean model $\mathbb{E}$. We fix a line segment in $\mathbb{H}$ and declare that it has length 1. Suppose we extend this line segment to a line, and then want to mark off a second segment on this line that is congruent to the first.

We know that the second segment looks shorter than the first to our eyes. A quick, and accurate, way of describing this is that the hyperbolic distance of the second line segment is not the same as the euclidean distance. To some degree there is choice in how long the second line segment should look. To put it differently, there is choice in how quickly the shrinking should occur. Once we mark the second segment, that is, once we set the rate of shrinkage, then all the other lengths are determined.

There is one other choice to be made before all distances in Beltrami-Klein are determined. It was not defined, or required, in the description above, but our sense of symmetry suggests that, starting from the center of the circle, every direction should be the same as every other direction. In other words, the hyperbolic distances on a vertical line through the center should be the same as the hyperbolic distances on a horizontal line through the center.

The same argument, that things should be symmetric around the center of the circle, shows that angles with vertex at the center should have hyperbolic measure equal to euclidean measure. However, angles change as we move towards the edge of the circle.

Comments. It is possible to assign real numbers to line segments in the Beltrami-Klein plane in a way that is consistent with our postulates. Thus, the distances should add up, satisfy the congruence postulates, extend to infinity, etc. There is more than one way to do this, although the different ways are (I think) similar to each other (they probably differ by a scaling constant or two).
Fact. Let $A$ and $B$ be two points in the Beltrami-Klein model. Let $X$ and $Y$ be the points on $\mathcal{C}$ where the line $AB$ intersects, with $X \rightarrow A \rightarrow B \rightarrow Y$. Then
\[
d(A, B) = \frac{1}{2} \ln \left( \frac{XB \cdot AY}{XA \cdot BY} \right)
= \frac{1}{2} \ln \left( \frac{XY \cdot BY}{AY} \right).
\]

3.10 Asymptotic Parallels

Theorem 3.10.1. The angle of parallelism does not depend upon the side of the transversal. In other words, given $A \rightarrow D \rightarrow B$ and a point $C$ not on the line, with $AB \perp CD$, the angle of parallelism on the $A$-side of $CD$ is equal to the angle of parallelism on the $B$-side of $CD$.

Proof. Let $WX$ be the asymptotic parallel through $C$ on the $A$-side of $CD$, with $W \rightarrow C \rightarrow X$ and $W$ on the $B$-side of $CD$. Let $YZ$ be the asymptotic parallel through $C$ on the $B$-side of $CD$ with $Y \rightarrow C \rightarrow Z$ and $Z$ on the $B$-side of $CD$.

We prove that $\angle DCW = \angle DCZ$.

Suppose $\angle DCW \neq \angle DCZ$. Then one of these angles is smaller than the other: we assume $\angle DCW < \angle DCZ$. Copy the angle $\angle DCW$ to the other side of $CD$, creating a point $E$ with $\angle ECD = \angle DCW$.

Since the angle $\angle DCE$ is less than the angle of parallelism on the $B$-side, the line $CE$ intersects $AB$, at a point, say $F$.

Let $G$ be a point on $AB$ on the opposite side of $D$ from $F$, such that $DG = DF$, and draw $CD$. Then SAS implies $\triangle CDG \cong \triangle CDF$. Then $\angle DCG = \angle DCE = \angle DCW$. But then $CW = CG$, contradicting the assumption that $CW$ is asymptotic parallel to $AB$.

Theorem 3.10.2. The angle of parallelism is less than $90^\circ$. 

Proof. Consider two distinct parallel lines and the angle they make with the transversal. One of the parallel lines might make an angle of 90°, but the other one must make an angle not equal to 90°. If this angle is > 90° on one side of the transversal, then it must make an angle < 90° on the other side of the transversal. The smaller angle is the one that defines the angle of parallelism.

\[ \text{Theorem 3.10.3.} \quad \text{The angle of parallelism through a point depends only on the distance of the point from the given line.} \]

Proof. (Omitted)

\[ \text{Theorem 3.10.4.} \quad \text{Given a line, a second line that is asymptotic parallel to the first at some point. Then the second line is asymptotic parallel to the first and at any other point on the line.} \]

Proof. Let \( AB \) be the given line. Let \( YZ \) be asymptotic parallel to \( AB \) at the point \( C \), in the direction of \( B \). Let \( E \) be a second point on the line \( YZ \). We need to show that \( YZ \) is asymptotic parallel to \( AB \) at the point \( E \), in the direction of \( B \).

To prove this, recall what the definition of asymptotic parallel is: any line through \( E \) with a smaller angle must intersect \( AB \). Let \( G \) be a point in the interior of \( \angle FEZ \). There are two cases: Case 1: \( E \) is on the \( B \)-side of \( CD \) and Case 2: \( E \) is on the \( A \)-side of \( CD \). We prove Case 1 here and leave case 2 to the reader.

Case 1: Suppose that \( E \) is on the \( B \)-side of \( CD \). Then \( G \) is also in the interior of \( \angle DCZ \). Therefore the line \( CG \) intersects \( AB \) at a point, say \( H \).

Then \( EG \) is a line intersecting one side of the triangle \( \triangle CDH \). Pasch’s Theorem implies that \( EG \) has to intersect another side of the triangle. Since \( G \) is on the \( B \)-side of \( EF \) the intersection of \( EG \) and \( AB \) also has to be on the \( B \) side of \( EF \). Therefore, \( EG \) intersects along \( DH \), which implies that \( EG \) intersects \( AB \).

Case 2: (Sketch) Suppose, for contradiction that \( EG \) is an interior line that does not intersect \( AB \). Applying Pasch, one can show that \( EG \) intersects \( CD \) at a point, \( G \). Let it make angle \( \alpha \) and let \( \beta \) be the angle of \( YZ \). Let \( CH \) be a line through \( C \) with angle \( \alpha \). Then \( CH \) and \( EG \) have transversal \( CD \) with two corresponding angles \( \alpha \). Thus \( CH \) and \( EG \) are parallel. Thus \( CH \) is parallel to \( AB \) (not by transitivity of parallels, but by Pasch again, and the fact that \( EG \) is between \( AB \) and \( CH \) and parallel to \( CH \)). Therefore, by the definition of \( \beta \) as the angle of parallelism, we must have \( \alpha \geq \beta \).

On the other hand, applying vertical angles we see that \( \beta \) is an exterior angle to \( \triangle EGC \) with opposite interior angle of \( \alpha \), and so \( \beta > \alpha \), a contradiction.

\[ \text{Theorem 3.10.5.} \quad \text{Asymptotic parallels are transitive. In other words, if } \ell \text{ is asymptotically parallel to } m, \text{ and } m \text{ is asymptotically parallel to } n, \text{ all in the same direction (i.e. all one side of a common transversal), then } \ell \text{ is asymptotically parallel to } n. \]
Proof. (Omitted)

**Theorem 3.10.6.** Asymptotic parallels become arbitrarily close together in the direction of parallelism, and arbitrarily far apart in the other direction.

Proof. (Omitted)

### 3.10.1 Angle of Parallelism

**Comments.** Recall that we “proved” (or at least stated) that the angle of parallelism does not depend on which point or line is used, but only the distance between them. The next result makes this statement much more precise: we give an exact formula for the angle of parallelism, and the only input to the formula is the distance between the point and the line.

**Theorem 3.10.7.** Let $\Pi(d)$ be the angle of parallelism between any point and line whose distance apart is $d$. Then $\Pi(d) = 90^\circ - 2 \tan^{-1} \left( \frac{e^d - 1}{e^d + 1} \right)$.

Proof. Since $\Pi(d)$ does not depend on which point and which line we use, as long as they have a distance $d$ apart, we choose a convenient point and line in the Poincaré Disc.

Let $O$ be the center of the disc, fix a diameter $OZ$ with $Z$ on the perimeter of the circle, and fixed a perpendicular diameter $OP$ with $P$ also on the perimeter of the circle. Let $A$ be a point on $OP$ with $AO = d$. The line through $A$ that is asymptotic parallel to $OZ$ is represented by the circle through $A$ and $Z$ and perpendicular to the disc.

We wish to calculate the angle between $AO$ and $AZ$. This angle is the same as the angle between $AO$ and the tangent line $AB$. Furthermore, measuring this angle in the hyperbolic sense, is the same as measuring it in the Euclidean sense. Therefore, we now translate everything into Euclidean geometry.

Let $\mathcal{C}$ be a circle with radius $r$ and center $O$. Let $OZ$ and $OP$ be perpendicular radii. Let $A \in OP$ and let $Q$ be the second intersection of $OP$ with $\mathcal{C}$. Let $\mathcal{C}'$ be the circle through $A$ and $Z$ and perpendicular to $\mathcal{C}$. Let $\theta$ be the angle between $AO$ and the circular arc. In other words, $\theta$ is the angle between $AO$ and $AB$ where $AB$ is tangent to $\mathcal{C}'$ at $B$. We calculate $\theta = \angle BAO$. 
The orthogonality of radii and tangents shows that $AO' \perp AB$ and $O'Z \perp BZ$. Thus, triangles $\triangle ABO'$ and $\triangle BZO'$ are right triangles, with a common hypotenuse and one pair of equal sides. Applying the Pythagorean Theorem, we see that the third sides are equal. Thus, $AB = BZ$ and so $\angle BZA = \angle BAZ$. Now, $\angle ABO$ is an exterior angle to $\triangle BAZ$, so

$$\angle ABO = \angle BZA + \angle BAZ = 2\angle BZA.$$ 

Thus, looking at the interior angles in $\triangle ABO$, we get

$$180 = \theta + \angle ABO + 90^\circ = \theta + 2\angle BZA + 90^\circ$$

which implies that

$$\theta = 90 - 2\angle BZA.$$ 

Let $a = AO$ and we get

$$\theta = 90 - 2 \tan^{-1}(a/r).$$

Now we translate $a/r$ back into Hyperbolic distances. Recall that the distance $AO$ is given by a formula involving logs:

$$d = AO = \ln \left( \frac{AQ^E}{AP^E} \cdot \frac{OP^E}{OQ^E} \right)$$

where $AQ^E$ means the Euclidean distance between points. Note that

$$AQ^E = r + a, \ AP^E = r - a, \ OP^E = r, \ OQ^E = r.$$
Therefore,

\[ d = \ln \left( \frac{r + a}{r - a} \right) \]

\[ e^d = \frac{r + a}{r - a} \]

\[ e^d r - e^d a = r + a \]

\[ -a - e^d a = r - e^d r \]

\[ a(-1 - e^d) = r(1 - e^d) \]

\[ a = \frac{e^d - 1}{e^d + 1} \]

Therefore,

\[ \theta = 90 - 2 \tan^{-1}\left( \frac{a}{r} \right) = 90 - 2 \tan^{-1}\left( \frac{e^d - 1}{e^d + 1} \right) \]

\[ \square \]

**Corollary 3.10.8.** \( \lim_{d \to \infty} \Pi(d) = 0 \) and \( \lim_{d \to 0} \Pi(d) = 90^\circ \).

### 3.11 Asymptotic Triangles

**Definition 3.11.1.** A *singly asymptotic triangle* is a biangle. We can think of them as being just like triangles with one point at infinity, or one point on the boundary of the Poincaré Disc. A *doubly asymptotic triangles* is a triangle with two points at infinity or on the boundary of the Poincaré Disc.

**Proposition 3.11.2.** Given an angle \( \angle ABC \), there exists a line \( XY \) that is asymptotically parallel to both \( AB \) and \( BC \). Thus, \( ABCYX \) is a doubly asymptotic triangle.

**Proof.** This can be proven directly from the axioms of Hyperbolic Geometry, but it is much simpler using the Poincaré Disc model. Let \( A' \) and \( C' \) be the points where \( \overline{BA}^H \) and \( \overline{BC}^H \) intersect \( \mathcal{C} \), the circle defining the Poincaré Disc.

The rest of the proof is Euclidean. If \( A' \) and \( C' \) are on a Euclidean diameter, let \( XY \) equal this diameter. Otherwise, we will show that there exists a circle through \( A' \) and \( C' \) that is orthogonal to \( \mathcal{C} \). Once this is done, let \( X \) and \( Y \) be two points on this circle and interior to \( \mathcal{C} \).

We have that \( A' \) and \( C' \) are two points on \( \mathcal{C} \) and are not on a single diameter of \( \mathcal{C} \). Let \( \ell \) and \( m \) be the tangent lines of \( \mathcal{C} \) through the points \( A' \) and \( C' \). Then \( \ell \) and \( m \) intersect at a point, \( O' \). Let \( \mathcal{C}' \) be the circle with center \( O \) going through \( A' \). To see that \( \mathcal{C}' \) also goes through \( C' \), note that an easy application of SAS shows that \( \triangle A'O'O' \cong \triangle C'O'O' \). Thus \( \overline{AO'} = \overline{CO'} \). Now, the tangent lines are equal to radii of \( \mathcal{C}' \) and and perpendicular to the radii of \( \mathcal{C} \), whence \( \mathcal{C}' \) is perpendicular to \( \mathcal{C} \). \( \square \)
Chapter 4

Elliptic Geometry

4.1 Spherical Geometry

Activity 26.

- Explore postulate 1: pick two points, and use a piece of string to find the shortest distance between them. Extend your path until it goes all the way around the sphere. Can you define what “line” should mean? Is there always a line between two points? Is it always unique?

- Can you find two lines that are parallel? Does Playfair’s Postulate hold?

- Explore Postulate 2: Given any line segment, can we extend it infinitely far in either direction?

- Explore Postulate 8: the plane separation postulate. Is it true on the sphere? Is the line separation property true? What about betweenness?

- Let $A$, $B$ and $C$ be collinear, with none of the points antipodal. Is it always true that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$? Explain.

- Let $A$, $B$ and $C$ be collinear, with none of the points antipodal. If $\overrightarrow{AB} = \overrightarrow{BC}$, is $B$ necessarily the midpoint of $\overrightarrow{AC}$? Explain.

- Can you construct an equilateral triangle as in Euclid Prop. I.1? (i.e. start with $\overrightarrow{AB}$, draw two circles, with centers at $A$ and $B$ and radii $\overrightarrow{AB}$.)

- Give an example to show that there can be more than one perpendicular from a point to a given line.

- Explore angles in a triangle. Do they add to 180? How big can you make the sum? How small? Does either version of the exterior angle theorem hold?

- Download and/or run in your web browser the program Spherical Easel. Explore some of the above ideas in the program.

Definition 4.1.1. A great circle on a sphere equals the intersection of the sphere with a plane that goes through the center of the sphere.

Comments. There are other ways to define great circles; the definition above is due to Euclid. Equivalent definitions: they are circles the same length as the equator; they are circles that have the same radius as the sphere.

Activity 27.
Explore postulate 1.

First, we have to figure out what “straight lines” are. If a straight line must be the shortest distance between two points, then it is given by a “great circle”, a circle such as the equator, that has the same center as the center of the sphere, and that has the same radius as the sphere.

Now, it is true that given any two distinct points there is “line” through them. (To see this note that given the two points, and the center, there is a unique plane through all three points. Intersecting this plane with the sphere gives the great circle.) However, it is not the case that the line through two points is always unique. In particular, if we pick any two antipodal points (i.e. points that are directly opposite each other on the sphere), then there are an infinite number of lines through them. For instance, on the globe of the earth, the north and south poles are opposite each other, and each circle of longitude goes through both poles.

Can you find two lines that are parallel? Does Playfair’s Postulate hold?

It is not possible to find two parallel lines. Every pair of great circles intersects. (To see this, note that each great circle equals the intersection of a plane with the sphere. Thus, finding the intersection of the great circles is equivalent to finding the intersection of the planes. In Euclidean space, every pair of planes through the origin has an intersection that is a one dimensional line through the origin. This line intersects the sphere in two antipodal points, the points that are on both great circles.

Explore Postulate 2: Given any line segment, can we extend it infinitely far in either direction?

This depends upon what we mean by “extend it infinitely”. It is true that we can move along a great circle without end: an ant walking along the circle could walk infinitely far. However, the line itself is not infinite in terms of length: the ant will be eventually covering the same ground over and over again. The conventional answer to this question is to adopt the view that the line is not infinitely extensible.

Explore Postulate 8: the plane separation postulate. Is it true on the sphere? Is the line separation property true? What about betweenness?

It is true that a great circle separates the surface of the sphere into two disjoint subsets.

It is not the case that given a line (i.e. great circle), that one point on the line separates the line into two disjoint subsets. If $A$ and $B$ are not antipodal, we let $AB$ be the great circle through them. Then $A$ and $B$ divide $AB$ into two disjoint arcs. As a matter of convenience, we denote by $\tilde{AB}$ the shorter of these two arcs.

It is fairly clear that no canonical definition of betweenness exists when two points are antipodal. Even if $A$ and $B$ are not antipodal, there are problems with betweenness. We should be tempted to say that $C$ is between $A$ and $B$ if $C \in \tilde{AB}$. This definition will satisfy some of our requirements for “betweenness” but not others. For instance,

$A - B - C$ and $B - C - D \not\equiv A - B - D$.

(Homework: how does the Euclidean proof of the above proposition that we gave earlier break down in the present context?)
• Let $A$, $B$ and $C$ be collinear, with none of the points antipodal. Is it always true that $AB + BC = AC$? Explain.

This is not always true. If $AB$ and $BC$ each cover, say, $1/3$rd of the circle, then $AC$ should not cover $2/3$rd of the circle, but should go around the other way.

• Let $A$, $B$ and $C$ be collinear, with none of the points antipodal. If $AB = BC$, is $B$ necessarily the midpoint of $AC$? Explain.

This is not always true, with the counter example being the same as above: if $AB$ and $BC$ are each $1/3$ of the circle, then the midpoint of $AC$ should be on the other arc of the circle.

• Can you construct an equilateral triangle as in Euclid Prop. I.1? (i.e. start with $AB$, draw two circles, with centers at $A$ and $B$ and radii $AB$.)

Sometimes this works and sometimes it doesn’t: it depends on how close together $A$ and $B$ are. If they are too far apart, the two circles either don’t intersect, or intersect in just one point.

• Give an example to show that there can be more than one perpendicular from a point to a given line.

If we fix as our point the north pole, then every line of longitude goes through the north pole and as is perpendicular to the equator.

• Explore angles in a triangle. Do they add to 180? How big can you make the sum? How small? Does either version of the exterior angle theorem hold?

It’s easy to see that we can make a triangle that adds to $270^\circ$, or $3\pi/2$ radians. In fact, we can make triangles whose angle sum is as close to $540^\circ$, or $3\pi$ radians as we want.

### 4.2 Single Elliptic Geometry

**Definition 4.2.1.** The **single elliptic model** (also known as the real projective plane) consists of the following:

1. A fixed Euclidean sphere, $S$.
2. Points = pairs of antipodal points on $S$,
3. Lines = great circles on $S$.

**Comments.** We can usually picture the real projective plane, $\mathbb{P}$, as follows: all the points on the upper half sphere, together with the points on the equator, but where we “identify” opposite points on the equator. In other words, we view opposite points as the same point. Thus, if an ant walked down towards the equator, at the moment the reached it, they would reappear on the opposite side going up away from the equator.

Straight lines in $\mathbb{P}$ consist of portions of great circles that are in the upper half plane, together with the equator itself (a great circle is a circle on the sphere with the same center and radius as that of the sphere).

We’ll point out two useful facts about this model:

1. Every two points are contained in a unique line.

To see this, note that on a sphere, every two nonantipodal points are contained in a unique great circle. The only points on a sphere that are not in a unique great circle are antipodal points. But in our model $\mathbb{P}$, there are no antipodal points.
2. Every two lines intersect (in one point).

To see this, note that on a sphere, every two great circles intersect, in two points. Most of the time, exactly one of these points will be in the upper half sphere. The only exception, is if the two points are both on the equator. But in this case, in $\mathbb{P}$, the two points are identified as being the same.

There are also two unuseful (or unfortunate, or bewildering, or . . . ) facts about $\mathbb{P}$:
1. The plane separation property fails. To see this, we cannot picture just the upper half hemisphere: think about the whole hemisphere with great circle. Now, remind yourself that opposite points on the sphere are the same point in $\mathbb{P}$: thus, there are not two sides of the line, there is only one side. In this sense, and in others, the projective plane is almost exactly like some version of the Möbius strip.

2. The projective plane cannot be pictured in three dimensional space. We can picture part of it, say the upper hemisphere, but we cannot picture the identification along the equator, without ripping part of our picture, or having part of our picture intersect itself.

Comments. There are many other ways of describing $\mathbb{P}^2$. The multitude of descriptions reflects the wide range of contexts, uses, and roles that the projective plane plays in mathematics.

### 4.3 Synthetic Elliptic Geometry

**Definition 4.3.1.** We take as primitive the following

- point, line, separation, congruence.

**Postulate 1.** Given any two distinct points there is a unique line through them.

**Postulate 7 (Separation).** 1: If $A$ and $B$ separate $C$ and $D$, then $A$, $B$, $C$ and $D$ are distinct points and lie on a line. 2: If $A$ and $B$ separate $C$ and $D$, then $C$ and $D$ separate $A$ and $B$, and $B$ and $A$ separate $C$ and $D$. 3: If $A$ and $B$ separate $C$ and $D$, then $A$ and $C$ do not separate $B$ and $D$. 4: Given any four distinct points $A$, $B$, $C$, and $D$ on a line, then $A$ and $B$ separate $C$ and $D$, or $A$ and $C$ separate $B$ and $D$, or $A$ and $D$ separate $B$ and $C$. 5: Given three distinct points $A$, $B$, and $C$ on a line, then there is a point $D$ so that $A$ and $B$ separate $C$ and $D$. 6: Given any five distinct points $A$, $B$, $C$, $D$, and $E$ on a line, if $A$ and $B$ separate $D$ and $E$, then either $A$ and $B$ separate $C$ and $D$, or $A$ and $B$ separate $C$ and $E$.

**Definition 4.3.2.** Given $AB$, the line $AB$ through them, and a point $C \in AB$, we define the arc $ABC$ as the points, $A$, $B$ and all the set of all points $D$ such that $A$ and $B$ do not separate $C$ and $D$.

Given $A$, $B$ and the line $AB$, we have that $A$ and $B$ define two arcs in $AB$. If these arcs are not of the same length, then one is shorter; we call the shorter arc a line segment and denote it by $\overrightarrow{AB}$. If $\overrightarrow{AB}$ is given and $C \in AB$ then we say that $C$ is between $A$ and $B$ (note that elliptic betweenness does not satisfy all of the properties of euclidean betweenness).

Given $\overrightarrow{AB}$ we can talk about the ray $\overrightarrow{AB}$. As a set of points, $\overrightarrow{AB} = AB$, the ray is equal to the line, but we can think of $\overrightarrow{AB}$ as having a direction: starting with $A$ and moving along $\overrightarrow{AB}$ to $B$, and then continuing.

**Postulate 2.** One can extend a given line segment to form a line.
Table 4.1: Elliptic Geometry Postulates

**Postulate 1** (p.228) Given any two distinct points there is a unique line through them.

**Postulate 7 (Separation)** (p.228) 1: If $A$ and $B$ separate $C$ and $D$, then $A$, $B$, $C$ and $D$ are distinct points and lie on a line. 2: If $A$ and $B$ separate $C$ and $D$, then $A$ and $C$ do not separate $B$ and $D$. 4: Given any four distinct points $A$, $B$, $C$, and $D$ on a line, then $A$ and $B$ separate $C$ and $D$, or $A$ and $C$ separate $B$ and $D$, or $A$ and $D$ separate $B$ and $C$.

**Postulate 2** (p.229) One can extend a given line segment to form a line.

**Postulate 3** (p.230) Given a point and a length less than $\frac{\pi}{2}$, there exists exactly one circle with the given point as center and radius equal to the given length.

**Postulate 4** (p.230) All right angles are congruent.

**Postulate 6 (Incidence)** (p.230) 1: There exists at least one elliptic plane. 2: Every plane contains at least three noncollinear points. 3: Every line contains at least two points.

**Elliptic Postulate 8, Pasch’s Postulate, p.230** If $A$, $B$ and $C$ are noncollinear points and a line $\ell$ intersects $AB$ at a single point $D$ between $A$ and $B$, then $\ell$ must intersect exactly one of the following: $\overline{BC}$ between $B$ and $C$, $\overline{AC}$ between $A$ and $C$, or the vertex $C$.

**Postulate 9, Congruence, p.230** 1: Congruence in lengths is an additive equivalence relation on arcs. 2: Congruence in angle measure is an additive equivalence relation on angles. 3: Equality in area is an additive equivalence relation on polygons. 4: Congruent triangles have equal areas.

**Elliptic Postulate 10, Archimedes Axiom, p.231** If $a$ and $b$ are positive real numbers with $a < b$, then there exists $n \in \mathbb{N}$ so that $na > b$. In particular, 1: Given arcs $\overline{AB}$ and $\overline{CD}$ with $\overline{CD}$ longer than $\overline{AB}$, there exists $n \in \mathbb{N}$ and a point $X$ on $\overline{CD}$ such that $\angle CDX = n \cdot \angle AB$ or $\angle CDX$ exceeds $\pi r$. 2: Given angles $\angle ABC$ and $\angle DEF$, there exists $n \in \mathbb{N}$ and a ray $\overrightarrow{EX}$ so that $\angle XEF = n \cdot \angle ABC$ and $\angle XEF > \angle DEF$.

**Postulate 11, Circular Continuity Principle, p.231** 1: An arc with one endpoint outside a given circle and the other endpoint inside the circle will intersect the circle exactly once. 2: A circle passing through a point inside a given circle and a point outside that circle will intersect the given circle twice.

**Postulate 12, SAS, p.231** If two triangle have two sides equal to two sides respectively and have the angles contained by the equal sides equal, then the triangles will be congruent.

**Elliptic Postulate 5, p.231** Any two distinct lines in a plane meet in exactly one point.
Comments. We will prove later that all lines have the same length, $\pi r$ in the Projective Plane Model. Note that in Euclidean Geometry, if we start with a circle, there is a unique center and radius of the circle. But in Spherical geometry, there are two possible centers (antipodes) and radii. In Elliptic Geometry, there is again one center, and only the shorter of the two radii. Thus, since all lines have length $\pi r$, the maximal radius is $\pi r/2$ (we will prove this more carefully later).

Postulate 3. Given a point and a length less than $\frac{\pi r}{2}$, there exists exactly one circle with the given point as center and radius equal to the given length.

Postulate 4. All right angles are congruent.

Postulate 6. 1: There exists at least one elliptic plane. 2: Every plane contains at least three noncollinear points. 3: Every line contains at least two points.

Comments. As discussed earlier, the plane separation property fails in the Projective Plane. Thus, we do not expect, or even want it, to be a postulate in our work. But we still need some way of separating things. We will use triangles. In Euclidean Geometry, we define the interior of a triangle using plane separation. If we had some other way of defining the interior of a triangle, we could use this in place of plane separation. The following result gives us a different way of defining interior of a triangle.

Postulate 8 (Pasch's Postulate). If $A$, $B$ and $C$ are noncollinear points and a line $\ell$ intersects $\overline{AB}$ at a single point $D$ between $A$ and $B$, then $\ell$ must intersect exactly one of the following: $\overline{BC}$ between $B$ and $C$, $\overline{AC}$ between $A$ and $C$, or the vertex $C$.

Comments. What do we look for in a definition of interior? A triangle, or a circle, should define the plane into two disjoint sets. One of these sets should be the interior, and the other the exterior. Our definition should allow us to specify which set is the interior, in a manner which is both unambiguous and in agreement with whatever intuition we have. The interior should be convex. It should satisfy the property that a line segment connecting an interior point and an exterior point must intersect the triangle.

Definition 4.3.3. Let $A$, $B$ and $C$ be three noncollinear points such that the line segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are all defined. The $\triangle ABC$ consists of $A$, $B$, $C$, $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$. The interior of the triangle $\triangle ABC$ is the set of all points contained in line segments such that the line segment has endpoints on two distinct sides of the triangle.

Postulate 9 (Congruence). 1: Congruence in lengths is an additive equivalence relation on arcs. 2: Congruence in angle measure is an additive equivalence relation on angles. 3: Equality in area is an additive equivalence relation on polygons. 4: Congruent triangles have equal areas.

Postulate 10 (Archimedes's Axiom). If $a$ and $b$ are positive real numbers with $a < b$, then there exists $n \in \mathbb{N}$ so that $na > b$. In particular: 1: Given arcs $\overline{AB}$ and $\overline{CD}$ with $\overline{CD}$ longer than $\overline{AB}$, there exists $n \in \mathbb{N}$ and a point $X$ on $\overline{CD}$ such that $\overline{CDX} = n \cdot \overline{AB}$ or $\overline{CDX}$ exceeds $\pi r$. 2: Given angles $\angle ABC$ and $\angle DEF$, there exists $n \in \mathbb{N}$ and a ray $\overrightarrow{EX}$ so that $\angle XEF = n \cdot \angle ABC$ and $\angle XEF > \angle DEF$.

Postulate 11 (Circular Continuity Principle). 1: An arc with one endpoint outside a given circle and the other endpoint inside the circle will intersect the circle exactly once. 2: A circle passing through a point inside a given circle and a point outside that circle will intersect the given circle twice.

Postulate 12 (SAS). If two triangle have two sides equal to two sides respectively and have the angles contained by the equal sides equal, then the triangles will be congruent.
Postulate 5 (Elliptic Parallel Postulate). Any two distinct lines in a plane meet in exactly one point.

**Theorem 4.3.4.** The following hold in Elliptic Geometry.

1. Given a point on a line and a length less than or equal to $\frac{\pi r}{2}$, one can cut off a segment of the given length.
2. The base angles of an isosceles triangle are equal.
3. If the base angles of a triangle are equal, then the triangle is isosceles.
4. Vertical angles are equal.
5. Two adjacent angles form a line if and only if their sum is $180^\circ$.
6. Any arc can be bisected.
7. Any angle can be bisected.
8. Given a line and a point on the line, a unique perpendicular can be constructed to the given line and through the given point.
9. Given a line and a point not on the line, at least one perpendicular can be constructed to the given line and through the given point.
10. Given an angle and a segment, one can construct on that segment, at either endpoint, an angle equal to the given angle.
11. If two triangles satisfy the SSS criterion, then they are congruent.
12. If two triangles satisfy the ASA criterion, then they are congruent.

**Proof.** Most of these can be proven using the same euclidean proofs we gave earlier: one just needs to double check each and every justification given. For instance, to prove (2), use the proof we gave earlier of Euclid Prop. I.5, *Pons Asinorum* (see page 75).

(2) Let $\triangle ABC$ be isosceles with $\overline{AB} = \overline{AC}$. Let $F$ be a point on $AB$ not on the arc $\overline{AB}$. Pick a point $G$ on $AC$, not on $\overline{AC}$ such that $\overline{AF} = \overline{AG}$. Note that

$\overline{AC} = \overline{AB}$ (isosceles assumption).

We apply SAS to see that $\triangle AFC \cong \triangle ABG$. Note that

$\overline{CF} = \overline{BG}$ (consequence of $\triangle AFC \cong \triangle ABG$)

$\angle BFC = \angle CGB$ (same)

$\overline{BF} = \overline{CG}$ (subtract $\overline{AB} = \overline{AC}$ from $\overline{AF} = \overline{AG}$)

We apply SAS to see that $\triangle BFC \cong \triangle CGB$.

Now we can conclude $\angle CBF = \angle BCG$, i.e. the angles under the base are equal, half of what we wanted to prove.

Furthermore, the last triangle congruence shows that $\angle CBG = \angle BCF$, and the first shows that $\angle ABG = \angle ACF$, whence

$\angle ABC = \angle ABG - \angle CBG = \angle ACF - \angle BCF = \angle ACB$. 

**Theorem 4.3.5.** Let $\ell$ be a line. There is a point $P$ such that every line from $P$ to $\ell$ is perpendicular to $\ell$, and such that all points on $\ell$ have the same distance to $P$.

**Definition 4.3.6.** The point $P$ in the theorem is called the **pole** of $\ell$. If $X$ is a point in $\ell$ then the length $\overline{PX}$ does not change if we move $X$ to a different point in $\ell$. We call this distance the **polar distance**
Comments. Note that the proof we give does not depend on the model we discussed in the previous section. There are two reasons for this. (1) It is good idea to give synthetic proofs when possible. This ensures that you haven’t made hidden assumptions about your subject. (2) To give a proof that is both rigorous and based on a model, one has to show that the model is categorical: that is, only theorems that are true in the synthetic sense are also true for the model. (3) The model that we gave above is hardly so “obvious” or even understandable, that one feels fully confident in proving anything based on it!

Proof. Let \( A \) and \( X \) be any points on \( \mathcal{C} \). Apply Theorem 4.3.4(2) to construct a point \( B \in \overrightarrow{AX} \) such that \( \overrightarrow{AX} = \overrightarrow{BX} \). Apply Theorem 4.3.4(8) to construct perpendiculars \( \ell \) and \( m \) at \( A \) and \( B \) respectively. By Postulate 5, we know that \( \ell \) and \( m \) intersect at a point, say \( P \). Apply Postulate 1 to define the line \( PX \). Apply Postulate 7 to get points \( C_A, D_A, C_X, D_X, \) and \( C_B, D_B \) that separate \( A \) and \( P \), \( X \) and \( P \) and \( B \) and \( P \) respectively.

Now \( \angle PAX = 90^\circ \) and \( \angle PBX = 90^\circ \), so Theorem 4.3.4(3) shows \( \overrightarrow{PC_A A} = \overrightarrow{PC_B B} \). Then SAS (with \( PX \) in the middle) implies \( \triangle APX = \triangle BPX \). Therefore \( \angle AX P = \angle BX P = 90^\circ \) and so \( \overrightarrow{PC_X X} \perp \mathcal{C} \). This shows too that \( \overrightarrow{PC_X X} = \overrightarrow{PC_A A} = \overrightarrow{PC_B B} \).

Sub-claim: \( \overrightarrow{PC_X X} = \overrightarrow{PC_D X} \). Suppose for contradiction that this claim is not the case. Apply Theorem 4.3.4(1) to construct a point \( P' \) on \( \overrightarrow{PD_X X} \) such that \( \overrightarrow{PC_X X} = \overrightarrow{P'D_x X} \) (abusing the location of \( D_x \) a little: it may not be “between” \( P' \) and \( X \), but we want the arc from \( P' \) to \( X \) in the same direction as \( \overrightarrow{PD_X X} \)). Draw \( \overrightarrow{PA} \) and \( \overrightarrow{PB} \).

By SAS we have \( \triangle AP'X \cong BP'X \) (the equal angles are at \( X \)) and \( \triangle BP'X \cong \triangle BPX \) (because \( \overrightarrow{PX} = \overrightarrow{P'X} \)) and \( \triangle AP'X \cong \triangle APX \). Therefore \( P'A \) and \( P'B \) are perpendicular to \( \mathcal{C} \). By the uniqueness statement in Theorem 4.3.4(8), we must have \( P'A = PA \) and \( P'B = PB \). Since \( PA \) and \( PB \) intersect at \( P \), and since two lines intersect at a unique point, we conclude that \( P = P' \). This finishes the sub-claim.

Now we show that every perpendicular from \( \mathcal{C} \) goes through \( P \) and that the distance is always equal. We start the same way as with \( A \) and \( B \) above. Let \( Y \) be any point on
Theorem 4.3.5. We need to show that \( \widehat{PY} = \widehat{PX} \) and \( \widehat{PY} \perp \mathcal{C} \). Cut a point \( Z \) so that \( \widehat{XZ} = \widehat{YZ} \). Erect perpendiculars at \( Y \) and \( Z \), let them intersect at \( Q \).

As before we get \( \angle QXZ = \angle QXY = 90^\circ \). Therefore \( \angle AXQ = 90^\circ \), but this means that \( Q \) lies on \( XP \). As before, \( Q \) and \( X \) divide \( PX \) into two equal line segments, and so \( Q = P \). As above we also have \( \widehat{QY} = \widehat{QX} = \widehat{QZ} \), and so \( \widehat{PY} = \widehat{PX} = \widehat{PB} \). Thus, we have shown that \( \widehat{PY} \) doesn’t depend on \( Y \) (or \( X \)) and that \( PY \) is a perpendicular. \( \square \)

Scholium 4.3.7. If \( \mathcal{C} \) is any line, \( P \) the pole of \( \mathcal{C} \), and \( X \) any point on \( \mathcal{C} \), the \( P \) and \( X \) divide the line \( PX \) into two equal arcs.

Proof. This result is exactly the subclaim within the previous proof. \( \square \)

Corollary 4.3.8. Let \( P \) be a point. There is a line \( \mathcal{C} \) such that every line from \( P \) to \( \mathcal{C} \) is perpendicular to \( \mathcal{C} \).

Definition 4.3.9. The line \( \mathcal{C} \) is called the polar line of \( P \).

Proof. Let \( P \) be any point. Let \( A \) be any point distinct from \( P \). Cut off \( \widehat{PB} \) so that \( \widehat{PA} = \widehat{PB} \). Let \( C \) be in the arc of \( AB \) that does not contain \( P \). Apply Theorem 4.3.4(6) to bisect \( \widehat{ACB} \), and let \( X \) be the midpoint. Since \( \widehat{PA} = \widehat{PB} \) and \( \widehat{AX} = \widehat{BX} \), we conclude that \( \widehat{PA} = \widehat{PB} \).

Now let \( \mathcal{C} \) be the line through \( X \) that is perpendicular to \( PX \).

Let \( A \) and \( B \) be two points on \( \mathcal{C} \), with \( AX = BX \). Then, as before, we show that \( PA \perp \mathcal{C} \), etc. \( \square \)

Comments. We are now in the position to prove one of our prior assertions: that all lines have the same, finite length. This is kind of amazing: although we thought that this should be the case based on our single elliptic model, there’s nothing in the postulates that has an obvious connection to lines having finite length.
Corollary 4.3.10. All lines have the same polar distance, \( k \).

Proof. Let \( \mathcal{C} \) and \( \mathcal{C}' \) be any two lines, with poles \( P, P' \) and polar distances \( k \) and \( k' \) respectively. Let \( A, B \in \mathcal{C} \) and choose two points \( A', B' \in \mathcal{C}' \) such that \( A'B' = AB \).

Then ASA implies \( \triangle ABP \cong \triangle A'B'P \), whence \( k = k' \).

Corollary 4.3.11. All lines have the same, finite length, \( 2k \).

Proof. Let \( \mathcal{C} \) be any line and \( P \) its pole, with polar distance \( k \). Let \( \ell \) and \( m \) be any two lines through \( P \), with \( \ell \perp m \).

Then \( \ell \) and \( m \) intersect \( \mathcal{C} \), by Postulate 5. They are perpendicular to \( \mathcal{C} \) by the definition of pole. Let \( A \) and \( B \) be the points where they intersect \( \mathcal{C} \), so that \( \mathcal{C} = AB \). By the scholium, \( P \), \( A \) divide \( \ell \) into two equal parts, and similarly for \( P, B \) and \( m \). Since \( \ell \perp m \), and since \( A \) and \( P \) divide \( \ell \) into two equal arcs, we have that \( A \) is the pole for \( m \) (this is the construction we gave for starting with a point and constructing the polar line).

Thus \( \widehat{AB} = k \) and \( A,B \) divide \( AB \) into two equal parts. Therefore, \( \mathcal{C} = AB \) has length \( 2k \).

Definition 4.3.12. Let \( r \) be defined by \( k = \pi r \), where \( k \) is the constant line length. We call \( r \) the radius of the elliptic plane.
Theorem 4.3.13. Let \( \triangle ABC \) be a right triangle with \( \angle B = 90^\circ \). Then

\[
\angle C = \begin{cases} 
< 90^\circ & \text{if } \hat{AB} < \frac{\pi}{2} \\
= 90^\circ & \text{if } \hat{AB} = \frac{\pi}{2} \\
> 90^\circ & \text{if } \hat{AB} > \frac{\pi}{2}
\end{cases}
\]

Proof. (Omitted) \( \square \)

Definition 4.3.14. A \textit{saccheri} quadrilateral is a quadrilateral with two base angles equal to \( 90^\circ \), and the sides next to the base being equal to each other.

Theorem 4.3.15. In elliptic geometry, the summit angles of a saccheri quadrilateral are obtuse.

Proof. Let \( ABCD \) be a saccheri quadrilateral with \( \angle B = \angle C = 90^\circ \). Let \( F \) be the midpoint of \( \hat{BC} \), and let \( E \) be the midpoint of \( \hat{AD} \). It is easy to show that \( EF \) is perpendicular to both \( AD \) and \( BC \). Let \( P \) be the pole of \( BC \) and \( Q \) the pole of \( EF \).

Then since all polar distances are the same, \( \hat{QAE} = \hat{QBF} = \frac{\pi}{2} \). Then \( \hat{QAB} < \frac{\pi}{2} \).

Applying this to the right triangle \( \triangle QBA \), and applying the previous theorem, we get \( \angle QAB < 90 \). Then \( \angle DAB > 90 \) (since these two angles add to 180).

Corollary 4.3.16. The angles in a right triangle add to greater than 180.

Proof. Let \( \triangle ABC \) be a right triangle, with \( \angle B = 90^\circ \). Let \( D \) be the midpoint of \( \hat{BC} \), and drop the perpendicular \( DE \) from \( D \) to \( BC \). One can show that \( E \) is on the arc \( \hat{BC} \).

Extend \( DE \) to the point \( F \) so that \( \hat{DEF} = \hat{DFER} \), i.e. \( D \) bisects \( EF \). Then vertical angles imply that \( \angle ADF = \angle CDE \), whence SAS implies congruence triangles \( \triangle ADF \cong \triangle CDE \). Congruent triangles imply that \( \angle DAF = \angle DCE \).

Construct a perpendicular through \( C \), let \( G \) be a point on this perpendicular such that \( \hat{CG} = \hat{AB} \). Then \( ABCG \) is a saccheri quadrilateral. Then the midline of this quadrilateral is perpendicular to \( BC \), whence the midline coincides with \( EF \). Whence, \( F \) is on the edge of the saccheri quadrilateral, and so the angle \( \angle BAF \) is greater than 90. But \( \angle BAF \) is the sum of two angles, one of which is in \( \triangle ABC \) and the other of which is equal to the other angle in \( \triangle ABC \). Whence, angles in the triangle \( \angle C + \angle A > 90 \). \( \square \)

4.4 Gauss-Bonnet

Activity 28. • (p. 225) Consider a sphere of radius 5. Find the angle sum and area of a 90, 90, 90 triangle (you may want to recall the area of a sphere is \( 4\pi r^2 \)).
• (p. 225) Consider a sphere of radius $r$. Consider a triangle made by the equator and two longitudes meeting at the north pole at $30^\circ$. Find the angle sum and the area.

• (p. 226) Repeat the previous problem with a $120^\circ$ angle between the longitudes.

• Make a conjecture relating the angle sum of a triangle to the area of the triangle and the radius of the sphere.

**Theorem 4.4.1** (Gauss-Bonnet). *Within any geometric plane, there is a constant $\kappa$ such that*

$$\text{def}(\triangle ABC) = \kappa \text{Area}(\triangle ABC).$$

*for all triangles $\triangle ABC$. For Euclidean Geometry, $\kappa = 0$. For a Hyperbolic Plane, $\kappa > 0$. For an Elliptic Plane, $\kappa < 0$.*

**Example 1.** On a sphere of a radius $r$, a $90, 90, 90$ triangle has area $\frac{1}{8}$ of the total. Thus,

$$\text{Area} = \frac{1}{8} \cdot 4\pi r^2 = \frac{1}{\kappa}(-90)$$

whence

$$\kappa = -\frac{2 \cdot 90}{\pi} \cdot 1r^2 = -4 \cdot \frac{180}{\pi} \cdot 1r^2.$$  

Thus

$$\text{Area}(\triangle ABC) = \frac{\pi}{180} \cdot r^2 \cdot -\text{def}(\triangle ABC).$$

Note that the formula here looks a little more complicated than needed. In particular, if we use radians instead of degrees, then we can erase the factor of $\frac{\pi}{180}$.

$$\text{Area}(\triangle ABC) = -r^2 \text{def}(\triangle ABC).$$

**Activity 29.** • (p. 241) Choose a partner and fit your triangle to various body parts. Take note of the angle made by the two flaps, and describe the local geometry of that body part as Euclidean, Hyperbolic, or Elliptic. (Some suggestions: the top of the head, under the chin, shoulder, back, etc.)
The remaining chapters in these notes include material that I have covered in some courses, but not this term (Spring, 2012). I've kept the material here for reference, interested readers, and for possible use in future courses.
Chapter 5

Non Obvious Euclidean Geometry

Definition 5.0.1. Three or more lines are said to be concurrent if they all intersect in a single point.

Our main results about concurrent lines will involve lines derived from triangles.

Definition 5.0.2. A cevian of a triangle is a line that passes through one vertex of the triangle and that intersects the line through the opposite vertices of the triangle.

Note, our definition includes the case that the cevian intersects the other line outside of the triangle.

Definition 5.0.3. Let $L$ be any line, and apply Proposition ??, to choose a positive direction on $L$. With this direction fixed, we define for $A, B$ on $L$ the function $|AB|^\pm$, called the directed distance between $A$ and $B$, as follows: $|AB|^\pm = |AB|$ if $B$ is in the positive direction from $A$, and $|AB|^\pm = -|AB|$ if $B$ is in the negative direction from $A$. Note that $|AB|^\pm = -|BA|^\pm$.

Proposition 5.0.4. Let $L$ be any line, choose a positive direction on the line, and let $A, B$ and $Z$ be any three points. Then we have

$$|AZ|^\pm + |ZB|^\pm = |AB|^\pm.$$

Proof. Homework.

Theorem 5.0.5 (Ceva’s Theorem\footnote{Named after Giovanni Ceva, 1647–1734. Ceva attended a Jesuit College and became a professor of mathematics, serving at various Italian universities. He came up with new geometric results, discovered some parts of Calculus before the subject became widely known, and published some of the first works in mathematical economics.}). Let $\triangle ABC$ be given. Fix a direction on each line along the side of the triangle. Let $X, Y$ and $Z$ be points on the lines through the side opposite $A, B$ and $C$, respectively. Then we have

$$AX, BY, and CZ are concurrent \iff \frac{|BX|^\pm |CY|^\pm |AZ|^\pm}{|XC|^\pm |YA|^\pm |ZB|^\pm} = 1$$

Note: this theorem holds both when the cevians intersect in the triangle, and when they intersect outside. The pictures below show the intersection on the inside, but this should not be relied upon. Also, the use of signed distance is required for the case where the intersection is on the outside.
Proof. We have two directions to prove: If the lines are concurrent, then the product above equals 1; conversely, if the product above equals 1, then the lines are concurrent. We note for the reader who is bothered by signed distances, we only really need them in two places, towards the end of the proof.

Suppose that the lines are concurrent, and intersect in a point $P$. Note that $\triangle ABX$ and $\triangle AXC$ have the same height, $h$. Thus

$$\frac{|\triangle ABX|}{|\triangle AXC|} = \frac{\frac{1}{2}h|BX|^\pm}{\frac{1}{2}h|XC|^\pm} = \frac{|BX|^\pm}{|XC|^\pm}.$$  

Similarly,

$$\frac{|BX|^\pm}{|XC|^\pm} = \frac{|\triangle BPX|}{|\triangle CPX|}.$$  

Now, for any rational numbers $\frac{a}{b} = \frac{c}{d}$ and $\frac{e}{f} = \frac{g}{h}$ we may conclude that $ad = bc$, $af = be$. Subtraction yields $ad - af = bc - be$, and factoring gives $a(d - f) = b(c - e)$. Rewriting this as a fraction, we obtain $\frac{a}{b} = \frac{d-f}{e-c}$. Applying this to the triangle we have above, we find

$$\frac{|BX|^\pm}{|XC|^\pm} = \frac{|\triangle ABX| - |\triangle BPX|}{|\triangle ACX| - |\triangle CPX|} = \frac{|\triangle ABP|}{|\triangle ACP|}.$$  

Thus, the ratio of the two line segments on the bottom of the triangle, is given by the ratio of the areas of the triangles that don’t touch the bottom. Similarly, we have

$$\frac{|CY|^\pm}{|YA|^\pm} = \frac{|\triangle BCP|}{|\triangle ABP|} \quad \text{and} \quad \frac{|AZ|^\pm}{|ZB|^\pm} = \frac{|\triangle ACP|}{|\triangle BCP|}.$$  

Combining these, we find

$$\frac{|BX|^\pm}{|XC|^\pm} \cdot \frac{|CY|^\pm}{|YA|^\pm} \cdot \frac{|AZ|^\pm}{|ZB|^\pm} = \frac{|\triangle ABP|}{|\triangle ACP|} \cdot \frac{|\triangle BCP|}{|\triangle ACP|} \cdot \frac{|\triangle BCP|}{|\triangle ACP|} = 1.$$  

Now we turn to the converse proof. Suppose that $\frac{|BX|^\pm}{|XC|^\pm} \cdot \frac{|CY|^\pm}{|YA|^\pm} \cdot \frac{|AZ|^\pm}{|ZB|^\pm} = 1$. Suppose that $AX$ and $BY$ intersect at $P$, and construct an additional cevian through $P$, with $Z'$ the intersection of this line with the opposite site of the triangle.
Since $AX$, $BY$ and $CZ'$ are concurrent, we have (using the direction we have already proven)

$$\frac{|BX|^{\pm}}{|XC|^{\pm}} \cdot \frac{|CY|^{\pm}}{|YA|^{\pm}} \cdot \frac{|AZ'|^{\pm}}{|Z'B|^{\pm}} = 1.$$ 

Combining this with our assumption about the other product of fractions equalling 1, we conclude

$$\frac{|AZ|^{\pm}}{|ZB|^{\pm}} = \frac{|AZ'|^{\pm}}{|Z'B|^{\pm}}.$$ 

Now we add 1 to both sides of this equation, and get common denominators

$$\frac{|AZ|^{\pm} + |ZB|^{\pm}}{|ZB|^{\pm}} = \frac{|AZ'|^{\pm} + |Z'B|^{\pm}}{|Z'B|^{\pm}}.$$ 

By Proposition 5.0.4, the top of each fraction equals $|AB|^{\pm}$ (this is the first place we really needed signed distances). Thus, we conclude that

$$\frac{|AB|^{\pm}}{|ZB|^{\pm}} = \frac{|AB|^{\pm}}{|Z'B|^{\pm}}$$

and so $|ZB|^{\pm} = |Z'B|^{\pm}$ and $Z = Z'$ (this is the second place we needed to use signed distance). \hfill \Box

**Proposition 5.0.6.** The three cevians that go to the midpoint of the opposite side are concurrent.

**Proof.** Look at the picture for Ceva’s Theorem. Note that $|AZ| = |ZB|$, etc. So the product of fractions is 1. \hfill \Box

**Proposition.** An altitude of a triangle is a cevian that is perpendicular to the side of the triangle that it intersects. The three altitudes of a triangle are concurrent.

**Proposition.** The three cevians that bisect the interior angle are concurrent.

**Duality**

Geometry has the following form of duality\(^2\). Take any geometric statement and switch every use of “point” with “line”. Of course, some of the verbs need to be switched as well. For instance, consider the following phrase

Two points determine a line.

The dual of this statement is “Two lines determine a point”. Perhaps there the verb “determine” isn’t ideal, so one could say instead “Two lines intersect in a point.” Note that this dual statement is not universally true, unlike the original one. None the less, it is a sensible statement, and it is usually true. As another example, if $P$ is point, and $\ell$ is a line, then the dual of the statement $P \in \ell$ would be $L \in p$ where $L$ is a point and $p$ is a line. In general, when taking duals we reverse containments.

Here are some more dual statements.

1. Find the dual of the definition of “concurrent lines”.

   We start by repeating the definition, and switching just the words “point” and “line”, and leaving a question mark wherever there’s a word we don’t know how to change.

---

\(^2\)Duality is something of a vague concept, but what it generally means in mathematics, is that there is some way of switching two sets of objects, the objects involve are sort of opposites, and the switching can, at least some of the time, produce sensible results. Some examples include switching column vectors and row vectors in Linear Algebra; switching trigonometric functions and their co-functions in trigonometry, etc.
Three or more lines are said to be concurrent if they all intersect in a single point.
Three or more points are said to be \( \ldots \) if they all \( \ldots \) in a single line.

For the second question mark, we need to replace “intersect”. In the first sentence, the word “intersect” means “contain”. Thus, in the second we should have “are contained in”. Thus, we have “Three or more points are said to be ___ if they all are contained in a single line.” Finally, we see that the last word to fill in is one that we already know, “collinear”.

2. Find the dual of “triangle”.

A triangle is a set of three non-collinear points, together with the three lines determined by the points.

The dual statement: “A \( \ldots \) is a set of three non-concurrent lines, together with the three points determined by the points.” Note,

**Definition 5.0.7.** A **menelaus** point of a triangle is a point lying on an extended side of the triangle.

**Theorem 5.0.8** (Menelaus’s Theorem\(^3\)). Let \( \triangle ABC \) be given. Fix a direction on each line along the side of the triangle. Let \( D, E \) and \( F \) be points on the lines through the side opposite \( A, B \) and \( C \), respectively. Then we have

\[
D, E, \text{ and } F \text{ are collinear } \iff \frac{|BD|^\pm |CE|^\pm |AF|^\pm}{|DC|^\pm |EA|^\pm |FB|^\pm} = -1
\]

**Proof.** Suppose that the points are collinear, on line \( L \). Let \( p, q, r \) be perpendicular segments from \( A, B \) and \( C \) respectively to \( L \).

Then similar triangles gives

\[
\frac{|BD|}{|DC|} = \frac{q}{r}, \quad \frac{|CE|}{|EA|} = \frac{r}{p}, \quad \frac{|AF|}{|FB|} = \frac{p}{q}.
\]

Therefore, the fraction we are interested in equals \( \pm 1 \), and we need to eliminate the case \( +1 \). Note that the number of external intersections (i.e. intersections of \( L \) with the extended line on the side of the triangle, where the intersection is not on the line segment \( AB, BC \) or \( AC \)) of the line \( L \) equals the number of minus signs that appear

\(^3\)Named after Menelaus of Alexandria, c. 70–140 AD. This Menelaus is *not* the same as the ancient Greek king who lead an army to the battle of Troy. Rather, this Menelaus lived about 8 centuries later, and was a mathematician and astronomer. Interestingly, Menelaus is reported to have gotten into a dispute with Lucius, in which the latter does not believe that the law of reflection is correct (cf Corollary 6.2.2).
in the product. Therefore, we need to show that $L$ has an odd number of external intersections. The number of intersections could (theoretically) be 0, 1, 2 or 3.

Since $L$ does intersect each of the extended lines, we cannot have 0 external intersections (otherwise all the intersections are internal, which is to say one line hits three sides of a triangle on the inside). Similarly, a line can’t have two external intersections (otherwise it has one internal intersection, and somehow enters the triangle through one side, but does not leave through the other two sides).

Therefore, there are 1 or 3 external intersections, and 1 or 3 minus signs in the product.

For the converse direction, the same proof as for Ceva holds. Let $D'$ be the intersection of $EF$ and $BC$. Then the fraction with $D'$ equals $-1$. Set this equal to the other fraction, and cancel to get

$$\frac{|BD|\pm}{|DC|\pm} = \frac{|BD'|\pm}{|D'C|\pm}$$

which implies $D = D'$.

\[\square\]

**Theorem 5.0.9** (Pappus Theorem\(^4\)). Let $A, C, E$ be collinear, and $B, D, F$ collinear. Suppose that certain (extended) lines intersect as shown:

$$AB \cap DE = L, \ CD \cap FE = M, \ EF \cap BC = N.$$ 

Then $L, M$ and $N$ are collinear.

**Proof.** We ignore the possibility that any lines are parallel to any others. In particular, let $AC, CD$ and $EF$ make a triangle $\triangle UVW$ as pictured.

\[^4\text{Named after Pappus of Alexandria, c. 290 – 350 AD. Pappus was the last great mathematician of ancient Greece. He made the most important contributions to geometry after Euclid, and before the Renaissance, 1000 years later.}\]
Note that \( \triangle UVW \) has a number of menelaus points: \( A, B, C, D, E, F, L, M, N \). We apply Menelaus's Theorem to all the ones points that are collinear (excluding the lines that form the side of \( \triangle UVW \), such as \( AB \), and excluding the line that we are trying to prove exists, through \( L, N, M \)). We know that the following are collinear:

\[
L, D, E, \quad A, M, F, \quad B, C, N, \quad A, C, E, \quad B, D, F.
\]

Menelaus gives the following fractions (in each case, we will insert the three letters that form the side of \( \triangle UVW \), such as \( AB \), and excluding the line that we are trying to prove exists, through \( L, N, M \)). We know that the following are collinear:

\[
\frac{|VL|}{|LW|} \pm \frac{|WD|}{|DU|} \pm \frac{|UE|}{|EV|} = -1 \quad \frac{|VA|}{|AW|} \pm \frac{|WM|}{|MU|} \pm \frac{|UF|}{|FV|} = -1 \quad \frac{|VB|}{|BW|} \pm \frac{|WC|}{|CU|} \pm \frac{|UN|}{|NV|} = -1
\]

The first three products come from the criss-crossing lines (i.e. the colored lines in the first picture). The last two come from the original black lines (i.e. the top and bottom of the first picture). We divide the first three by the last two and cancel a lot to get

\[
\frac{|VL|}{|LW|} \pm \frac{|WM|}{|MU|} \pm \frac{|UN|}{|NV|} = -1
\]

which shows that \( L, M \) and \( N \) are collinear.

**Theorem 5.0.10** (Viviani’s Theorem\(^5\)). Let \( \triangle ABC \) be equilateral, and let \( P \) be an interior point of this triangle. Let \( X, Y \) and \( Z \) be Menelaus points opposite \( A, B \) and \( C \) respectively with \( PX, PY \) and \( PZ \) perpendicular to \( BC, AC \) and \( AB \) respectively. Then \( |PX| + |PY| + |PZ| \) equals the height of \( \triangle ABC \).

\[\text{Proof.}\]

\[
\frac{1}{2} h |BC| = \frac{1}{2} |PZ| \cdot |AB| + \frac{1}{2} |PX| \cdot |BC| + \frac{1}{2} |PY| \cdot |AC|
\]

\[
|BC| = |AB| = |AC|
\]

\[
h = |PZ| + |PX| + |PY|.
\]

\(^5\)Named after Vincenzo Viviani (1622-1703). The following description should convince the modern reader of the value of a Jesuit education and of studying Euclid. It is taken from the Mac-Tutor math history website. “Vincenzo studied at a Jesuit school where he learnt the humanities. He was taught mathematics by Clemente Settimi, a Scapolian friar and a friend of Galileo. Settimi quickly saw Viviani’s exceptional intelligence and, in 1638, presented him at court to Ferdinando II de’ Medici, Grand Duke of Tuscany. The court was in Livorno, so Viviani had to make the long journey from Florence. While on the road he did not waste his time but spent many hours studying the first three books of Euclid’s Elements. He made a presentation to the Tuscan Court explaining the first sixteen propositions in the first book of the Elements. Famiano Michelinii, Ferdinando’s Court mathematician, then set him a problem to which he responded confidently. Ferdinando was greatly impressed and provided a monthly salary for Viviani that enabled him to purchase mathematical books.” Viviani went on to become a famous scientist. Among other accomplishments he helped obtain a fairly accurate estimate for the speed of sound, became famous as an engineer and mathematician, and helped restore Galileo’s reputation in the aftermath of his persecution by the Catholic church.
Chapter 6

Transformations

6.1 Elementary rigid motions

A transformation is a function \( f : \mathbb{E} \to \mathbb{E} \). We will not be interested in all such functions, only ones that preserve certain features of our geometry. As described above, we call such a function a rigid motion if it does not change any angles (in absolute values) or distances. Rigid motions underlie our definition of congruence.

It turns out that we can describe (1) all possible rigid motions and (2) that we can describe them synthetic geometry (axioms, propositions as above), as well as using linear algebra. We will do this now.

**Definition 6.1.1** (Translation: Synthetic Approach). Given a line \( \ell \) in \( \mathbb{E} \), and a directed distance \( r \), we can define a **translation** \( f \) as follows. For any \( A \in \mathbb{E} \), draw the line \( m \) through \( A \) parallel to \( \ell \). Then we let \( f(A) \) be the point on \( m \) with \( |Af(A)| = r \).

Note: this definition specifies what \( f \) should do. To construct such a function \( f \) using our simple axioms in Chapter 2, we would apply the rigid motion axiom, as we did in Homework 1, #4. This would show that \( f \) preserves distances, although we will give an alternate proof below of this fact.

**Alternative** (Translation: Linear Algebra Approach). Use \( \mathbb{R}^2 \) as our model of \( \mathbb{E} \). Given \( \vec{v} \in \mathbb{R}^2 \), we can define a **translation** \( f \) as follows. For any \( \vec{x} \in \mathbb{R}^2 \) let \( f(\vec{x}) = \vec{x} + \vec{v} \).

Based on background knowledge, and our experience with geometry software, we will define the following types of rigid motions, and see that the following properties hold.

<table>
<thead>
<tr>
<th>Isometry</th>
<th>Reverses orientation</th>
<th>Number of fixed points</th>
</tr>
</thead>
<tbody>
<tr>
<td>translation</td>
<td>No</td>
<td>0</td>
</tr>
<tr>
<td>rotation</td>
<td>No</td>
<td>1</td>
</tr>
<tr>
<td>reflection</td>
<td>Yes</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

**Proposition 6.1.2.** A nontrivial translation preserves distances, has 0 fixed points and preserves orientation.

*Proof.* Let \( f \) be a nontrivial translation, with a distance of \( r \), parallel to a line \( \ell \). The fact that \( f \) is nontrivial means that \( r > 0 \).

Let \( A \) and \( B \) be given. Let \( m \) and \( n \) be lines through \( A \) and \( B \), and parallel to \( \ell \). Then \( f(A) \) and \( f(B) \) are on the lines \( m \) and \( n \), and the four vertices \( A, f(A), B, f(B) \) form a parallelogram.
Thus, $|f(A)f(B)| = |AB|$. To show that $f$ has 0 fixed points, note that $|Af(A)| = r \neq 0$. Thus, $A \neq f(A)$. To see that $f$ preserves orientation, note that if $\angle ABC$ is, say, clockwise, then $\angle f(A)f(B)f(C)$ is as well.

\begin{definition}[Reflection: Synthetic Approach] Given a line $\ell$ we define a reflection $f$ across $\ell$ as follows. Let $A$ be any point in $E$. If $A \in \ell$ then let $f(A) = A$. If $A$ is not on $\ell$, then let $f(A)$ be the point such that $\ell$ is the perpendicular bisector to the line segment $Af$.
\end{definition}

As before, we now give an alternative way of defining a rotation, again using Linear Algebra. However, this time the definition is somewhat more complicated.

\begin{alternative}[Reflection: Linear Algebra approach] Use $R^2$ as our model of $E$. Given two points $\vec{v}, \vec{w} \in R^2$ we define the reflection across the line through $\vec{v}$ and $\vec{w}$ as follows. Let $\vec{v} - \vec{w}$ make the angle $\theta$ with the $x$-axis. Given any $\vec{x} \in R^2$, first apply the translation $-\vec{w}$, then multiply the result by the following matrix $A$

$$
\begin{pmatrix}
\cos(2\theta) & \sin(2\theta) \\
\sin(2\theta) & -\cos(2\theta)
\end{pmatrix}
$$

then apply the translation $\vec{w}$. In other words,

$$
f(\vec{x}) = \vec{w} + \begin{pmatrix}
\cos(2\theta) & \sin(2\theta) \\
\sin(2\theta) & -\cos(2\theta)
\end{pmatrix}(\vec{x} - \vec{w}).
$$

The main step in the previous definition is multiplying by the matrix. The reason we need the extra steps of subtracting, and later adding, $\vec{w}$, is that matrix multiplication always leaves $\vec{0}$ as $\vec{0}$. In other words, $\vec{0}$ is a fixed point of any matrix multiplication. But for a reflection across the line through $\vec{v}$ and $\vec{w}$, we can’t (in general) have $\vec{0}$ fixed. Thus, we subtract $\vec{w}$, which moves the line to one through $\vec{0}$, then reflect across this line, and then move everything back by adding $\vec{w}$.

\begin{proposition} A reflection preserves distances, reverses orientation, and fixes all the points on a line. \end{proposition}

\begin{proof} Exercise. \end{proof}

Now we define rotation.
Definition 6.1.5 (Rotation: Synthetic Approach). Given a point \( B \) and a real number \( \theta \), we can define a rotation \( f \) about \( B \) as follows. For any \( A \in \mathbb{E} \), let \( f(A) \) be the point on the circle with center \( B \) and \( \angle ABf(A) = \theta \).

Alternative (Rotation: Linear Algebra Approach). Use \( \mathbb{R}^2 \) as our model of \( \mathbb{E} \). Given a point \( \vec{v} \in \mathbb{R}^2 \) and a real number \( \theta \) we define a rotation \( f \) about \( B \) as follows. Given any \( \vec{x} \in \mathbb{R}^2 \), first apply the translation \(-\vec{v}\) to \( \vec{x} \), then multiply the result by the following matrix
\[
\begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}
\]
then apply the translation \( \vec{v} \). In other words,
\[
f(\vec{x}) = \vec{v} + \begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}(\vec{x} - \vec{v})
\]

Proposition 6.1.6. A nontrivial rotation preserves distances, has one fixed point, and preserves orientation.

Proof. Exercise.

Proposition 6.1.7. Let \( \ell_1 \) and \( \ell_2 \) be parallel lines. Let \( r_1 \) and \( r_2 \) be two reflections across lines \( \ell_1 \) and \( \ell_2 \) respectively. Then \( r_1 \circ r_2 \) is a translation. The direction of the translation is perpendicular to \( \ell_1 \) and \( \ell_2 \), and the distance of the translation is twice the distance between \( \ell_1 \) and \( \ell_2 \).

Proof. Let \( A \) be any point, let \( m \) be the line through \( A \) and \( r_2(A) \).

1. Then \( m \) is perpendicular to \( \ell_2 \) (definition of reflection).
2. Then \( m \) is perpendicular to \( \ell_1 \) (since \( \ell_1 \) is parallel to \( \ell_2 \)).
3. Then \( r_1(r_2(A)) \) is also on \( m \) (since \( \ell_1 \) is perpendicular to the line segment \( r_2(A)r_1(r_2(A)) \) ).

Let \( d = |Ar_1(r_2(A))| \).

Now, so far we have shown that one point, \( A \), has been translated a distance of \( d \). But this is meaningless: every function, if you look at just one point, looks like a translation. What we need to do is look at any other point, \( B \) and verify that \( r_1(r_2(B)) \) is translated parallel to \( m \) by a distance of \( d \).

Let \( B \) be any point. As before, let \( m_1 \) be the line through \( B \), \( r_2(B) \), \( r_1(B) \).
4. Then \( m_1 \) is parallel to \( m \) (since both are perpendicular to \( \ell_1 \)).

Let \( X_1 \) and \( X_2 \) be the intersections of \( m \) with \( \ell_1 \) and \( \ell_2 \) respectively. Let \( d_1, d_2, d_3, d_4, d_5, d_6 \) be the directed distances shown in the picture,

(5) We have \( d_1 = d_2 \) and \( d_3 = d_4 \) (since \( \ell_2 \) bisects \( Ar_2(A) \) and \( \ell_1 \) bisects \( r_1(r_2(A)) \) )

Recall that Proposition 5.0.4, gives the additive property of directed distances.
6. We have \( d = d_1 + d_2 + d_3 + d_4 \) (Prop 5.0.4).

7. We have \( d = d_3 + d_5 + d_4 \) (combining 6 with \( d_5 = d_2 + d_3 \)).
(8) We have \( d = 2d_5 \) (by 5 we have \( d_1 + d_4 = d_2 + d_3 = d_5 \)).

(9) Now we have shown \( |A r_1 (r_2 (A))| = 2d_5 \). Note that the distance \( d_5 \) does not depend on \( A \) at all, it is the distance between \( \ell_1 \) and \( \ell_2 \). The exact same proof will show that \( |B r_1 (r_2 (B))| = 2d_5 \). Thus, all points are translated, parallel to \( m \), the same distance of \( 2d_5 \).

**Proposition 6.1.8.** Let \( r_1 \) and \( r_2 \) be two reflections across lines \( \ell_1 \) and \( \ell_2 \) respectively. If \( \ell_1 \) and \( \ell_2 \) are not parallel, then \( r_1 \circ r_2 \) is a rotation. The center of rotation is the \( \ell_1 \cap \ell_2 \) and the angle of rotation is twice the angle between \( \ell_1 \) and \( \ell_2 \).

**Proof.** Exercise.

Soon we will study the following

<table>
<thead>
<tr>
<th>Isometry</th>
<th>Reverses orientation</th>
<th>Number of fixed points</th>
</tr>
</thead>
<tbody>
<tr>
<td>glide reflection</td>
<td>Yes</td>
<td>0</td>
</tr>
</tbody>
</table>

### 6.2 Some applications of transformations

**Proposition 6.2.1.** Let \( \ell \) be any line, \( A \) and \( B \) two distinct points on the same side of \( \ell \). Let \( A' \) and \( B' \) be the reflections of \( A \) and \( B \) across \( \ell \). Let \( H = AA' \cap \ell \) and \( K = BB' \cap \ell \).

Let \( X \in \ell \). If \( |\angle HXA^0| = |\angle KXB^0| \) then the line segments \( AX \) and \( XB \) give the shortest path from \( A \), to \( \ell \), and then to \( B \).

**Proof.** Let \( \alpha = |\angle HXA^0| = |\angle KXB^0| \). Looking at the line \( \ell \) and the points \( H, X, K \), we see that \( \angle HXK^0 = 180^\circ \). Thus, \( 2\alpha + \angle AXB^0 = 180^\circ \).

Since reflection is a rigid motion, it preserves absolute values of angles. Thus, we have \( \alpha = |\angle A'XH^0| \). Then \( |\angle A'XB^0| = |\angle AXB^0| + 2\alpha = 180^\circ \). Thus, \( A', X \) and \( B \) are collinear.

Let \( Y \) be any other point on \( \ell \). Then \( |AY| + |YB| = |A'Y| + |YB| \). But \( |AX| + |XB| = |A'X| + |XB| = |A'B| \). Since \( Y \) is not between \( A' \) and \( B \) we have that \( |A'Y| + |YB| > |A'B| \).

**Corollary 6.2.2** (Fermat’s Principle implies the Law of Reflection). If a light ray that strikes a mirror, the angle of incidence, \( \theta_i \), equals the angle of reflection, \( \theta_r \).
Proof. Fermat’s principle states that a light ray travelling between given points will take the path that requires the least amount of time. Assuming that the light is travelling through a single substance, this means that the light will travel along the shortest path. Thus, the light will make the same angle with the mirror on both sides of the reflection, which implies that $\theta_i = \theta_r$. \qed

Three jugs problems

Consider the following problem:

1. We have three jugs $A$, $B$ and $C$, of known volumes, $a$, $b$ and $c$.
2. We have a known quantity of water $q$, in the three jugs.
3. We have a known starting division of the water, $x_0$, $y_0$ and $z_0$ in the three jugs.
4. We are allowed to pour water from one jug into another jug: we must stop only when one jug is all the way full, or the other all the way empty.
5. Our goal is to get some desired division of water $r$, $s$ and $t$.

Example 1. Suppose we start with:

(a) three jugs of size 7, 6 and 3 liters,
(b) a total of 8 liters,
(c) a starting point of 3 liters in the first jug, 3 in the second, and 2 in the third,
(d) a goal of getting 4 liters in the first and 4 liters in the second.

We picture this problem using tri-linear coordinates. We start with an equilateral triangle of height 8. We give any point in the interior coordinates $(x, y, z)$

$$ (x, y, z) \leftrightarrow x = \text{perpendicular distance to } BC $$

$$ y = \text{perpendicular distance to } AC $$

$$ z = \text{perpendicular distance to } AB $$

By Viviani’s Theorem, $x + y + z$ always adds up to the height of the triangle which is 8.

Now, we subdivide the triangle smaller triangles, each with height 1. In this picture, the quantity we have in each jug can be represented by the vertices of the smaller triangles, each of which has integer coordinates. Thus, the initial division of 3 liters, 3 liters and 2 liters corresponds to $(3, 3, 2)$ (shown in abbreviated notation of 332). We
are trying to reach point \((4,4,0)\) (shown as 440). The sizes of the jugs correspond to maximum values of \(x\), \(y\) and \(z\), so

\[
0 \leq x \leq 7, \quad 0 \leq y \leq 6, \quad 0 \leq z \leq 3.
\]

These maximum values define straight lines in the triangle, which in turn define a boundary of possible divisions of water, shown below in red.

Pouring all of the water from one jug to another corresponds to moving to a point on the boundary, along a straight line made by sides of triangles. From the starting position of 332 we can move in one of six directions, until we get to the boundary. If we pour all of \(C\) in \(A\) we move away from \(C\), towards \(A\) and get 530. If we pour \(B\) into \(C\) we move away from \(B\), towards \(C\), but have to stop when \(C\) is full, i.e. 323.

In this way, our first move must take us in a straight line to the boundary of the region. Subsequent moves must always take us in a straight line until a new boundary region is reached. Thus, when we reach a boundary, our path either reflects as a light ray would, or we stay on the boundary. We show a solution of this problem int the picture below.
Example 2. Start with 12 liters, jugs of size 12, 9, and 5, an initial division of 12, 0 and 0. Get to $(6, 6, 0)$.

6.3 A classification of rigid motions

Proposition 6.3.1. Let $A$, $B$ and $C$ be three non-collinear points and $C_A$, $C_B$, $C_C$ three circles centered at $A$, $B$ and $C$ respectively. If $C_A \cap C_B \cap C_C$ is nonempty, then it is a unique point.

Proof. Suppose $D$ and $E$ are in the intersection of all three points. Then the perpendicular bisector of $DE$ equals all the points that are an equal distance from $D$ and $E$. This means that $A$, $B$ and $C$ are all collinear. 

Example 3. Suppose we have three seismic centers, at locations $A = (1, 10)$, $B = (5, 7)$, $C = (6, 1)$. The measure an earthquake at a distance of 9.22, 8.49 and 7.0 respectively. Find the location of the quake.

To solve this algebraically means solving the following three equations simultaneously. Of course, we can solve just two of them, get two points, and then see which of the two points satisfies the third.

A: $$(x - 1)^2 + (y - 10)^2 = 9.22^2$$

B: $$(x - 5)^2 + (y - 7)^2 = 8.49^2$$

C: $$(x - 6)^2 + (y - 1)^2 = 7.0^2$$

Graphically, this is what it looks like
From either the graph, or the algebraic approach, we find that the location is \((-1,1)\).

**Example 4.** Your GPS calculates its distance from three satellites in space. Each distance from a satellite defines a circle on the surface of the earth. Intersecting the three circles pinpoints your location.

**Proposition 6.3.2.** If \(f\) and \(g\) are two rigid motions, and \(A, B\) and \(C\) are three non-collinear points such that \(f(A) = g(A)\), \(f(B) = g(B)\) and \(f(C) = g(C)\), then \(f = g\).

**Proof.** Let \(X\) be any point. Let \(A' = f(A) = g(A)\), etc. Let \(\mathcal{C}_A'\), \(\mathcal{C}_B'\) and \(\mathcal{C}_C'\) be circles centered at \(A'\), \(B'\) and \(C'\) with radius \(|XA|\), \(|XB|\) and \(|XC|\) respectively. Since \(f\) preserves lengths, we have that \(f(X)\) has distance \(|XA|\) from \(A'\), \(|XB|\) from \(B'\), and \(|XC|\) from \(C'\). Thus \(f(X)\) is the unique point in the intersection \(\mathcal{C}_A, \mathcal{C}_B\) and \(\mathcal{C}_C\).

But the same reasoning applies to \(g\) as well: \(g(X)\) must have distance \(|XA|\) from \(A'\), etc. Thus \(g(X)\) is also the unique point in the intersection, and so \(g(X) = f(X)\).

**Proposition 6.3.3.** Every rigid motion can be written as the composition of at most three reflections.

**Proof.** Let \(f\) take \(A, B\) and \(C\) to \(A', B'\) and \(C'\). Note that \(\triangle ABC \cong \triangle A'B'C'\).

Reflection 1 acts across the perpendicular bisector of \(AA'\).

WLOG we assume \(A = A'\) now.

Reflection 2 acts perpendicular bisector of \(BB'\). Crucial fact: \(A = A'\) is equal distance from \(B\) and \(B'\) (congruent triangles), thus \(A = A'\) is not moved.

Reflection 3 acts across the perpendicular bisector of \(CC'\). Again, \(A\) and \(B\) are fixed. In this way we have

\[
\begin{align*}
    f(A) &= r_3(r_2(r_1(A))) \\
    f(B) &= r_3(r_2(r_1(B))) \\
    f(C) &= r_3(r_2(r_1(C)))
\end{align*}
\]

We show below that \(r_3 \circ r_2 \circ r_1\) is a rigid motion. This, together with Proposition 6.3.2, proves that \(f(X) = r_3(r_2(r_1(X)))\).

Note that there is one gap in the previous proof: we need to show that \(r_3 \circ r_2 \circ r_1\) is a rigid motion to apply Proposition 6.3.2. We fill this gap now.

**Proposition 6.3.4.** If \(f\) and \(g\) are rigid motions, then so is the composition \(f \circ g\).
Proof. Let \( A, B \in \mathbb{E} \). Since \( f \) is an isometry we have \( |f(g(A))f(g(B))| = |g(A)g(B)| \). Since \( g \) is an isometry we have \( |g(A)g(B)| = |AB| \). Therefore, \( |f(g(A))f(g(B))| = |AB| \) and \( f \circ g \) is an isometry.

Example 5. Let \( f(x) \) be defined as an isometry of \( \mathbb{R}^2 \) with the following action on three points \( A, B \) and \( C \):

\[
\begin{align*}
  f: A &= (5, 30) \mapsto \left( \frac{5}{2} + 17\sqrt{3}, \sqrt{3} - 13 \right) \\
  B &= (6, 6) \mapsto \left( 3 + 5\sqrt{3}, 3\sqrt{3} - 1 \right) \\
  C &= (25, 4) \mapsto \left( \frac{25}{2} + 4\sqrt{3}, \frac{25}{2}\sqrt{3} \right)
\end{align*}
\]

Decompose \( f \) as the composition of at most three reflections, and identify \( f \) as a translation, glide reflection, etc.

Solution: We apply the steps used in the proof of Proposition 6.3.3. These are shown in Figure 6.1. Note: I strongly recommend that you do these geometrically by drawing line segments, and perpendicular bisectors as accurately as possible, but using a ruler. Then, if you want, you can estimate the equations for the reflecting lines, again, graphically.
Here we start with the original points $A$, $B$ and $C$, as well as our target points. We draw the line segment $AA'$, and then the perpendicular bisector of $AA'$ (shown in black). Each point $A$, $B$ and $C$ moves across the black line when it's reflected. Note that $A$ goes straight to $A'$, so we won't say anything more about it. We also get new points $B_1$ and $C_1$ which is where we will start for the next step.

Here we start with the points $B_1$, $C_1$ from the previous steps. We draw the line segment $B_1B'$, and then the perpendicular bisector of $B_1B'$ (shown in black). The points $B_1$ and $C_1$ move across the black line when reflected. Note that $B_1$ goes straight to $B'$, and we have one more new point $C_2$, which we will move in the next step.

Here we start with the points $C_2$, from the previous steps. We draw the line segment $C_2C'$, and then the perpendicular bisector of $C_2C''$ (shown in black). The point $C_2$ moves across the black line when reflected and goes straight to $C'$. We're done moving the points.
Proposition 6.3.5. Suppose a nontrivial rigid motion fixes two points. Then it is the reflection across the line that goes through the two points.

Proof. Let \( A \) and \( B \) be the two fixed points. Let \( C \) be between \( A \) and \( B \). Then \(|AB| = |AC| + |CB| = |AC'| + |C'B'| = |AC'| + |C'B|\). Therefore \( C' \) is on the line as well. But the distance from \( A \) to \( C \) is the same as the distance from \( A \) to \( C' \), and thus, \( C = C' \).

A similar proof applies to all the points on the line through \( A \) and \( B \), showing that all such points are fixed.

Now, let \( C \) be a point not on the line. We have \( C \neq C' \) since otherwise three points are fixed, and so the rigid motion would be trivial.

Let \( m \) be a line through \( C \) and perpendicular to \( \ell \). Let \( B = m \cap \ell \). Let \( A \) be any other point on \( \ell \). Then \(|\triangle ABC'\rangle = |\triangle ABC''\rangle\) so \( C'' \in m \). Furthermore, \(|BC| = |BC'|\) so \( C \) is one of two points on \( m \). Since \( C \neq C' \) we conclude that \( C' \) is the point on \( m \) on the other side of \( \ell \), and \( \ell \) is the perpendicular bisector of \( CC' \). Thus, \( C' \) is the reflection of \( C \) across \( \ell \).

Proposition 6.3.6. Let \( r_1 \), \( r_2 \) and \( r_3 \) be reflections. Then \( r_1 \) reverses orientation, \( r_1 \circ r_2 \) preserves orientation, and \( r_1 \circ r_2 \circ r_3 \) reverses orientation.

Proof. This proof is essentially the fact that reverses something twice returns things to their original state. We will use the shorthand \( f(\circ) = \circ \) to indicate that \( f \) reverses orientation. Then we have

\[
\begin{align*}
    r_1(\circ) &= \circ & \text{Prop 39} \\
    r_1(r_2(\circ)) &= r_1(\circ) = \circ & \text{Prop 39, twice} \\
    r_1(r_2(r_3(\circ))) &= r_1(r_2(\circ)) = r_1(\circ) = \circ
\end{align*}
\]

Recall earlier we started to categorize rigid motions in terms of their fixed points and orientation properties. We finish this now, starting with a definition.

Definition 6.3.7. Let \( \ell \) and \( m \) be parallel lines. Let \( r \) be the reflection across \( \ell \) and \( \tau \) be a translation along \( m \). Both compositions \( r \circ \tau \) and \( \tau \circ r \) are called a glide reflection.

Proposition 6.3.8. A glide reflection reverses orientation and has 0 fixed points.

Proof. Since \( \tau \) preserves orientation, and \( r \) reverses it, we have that \( \tau \circ r \) reverses orientation as shown

\[
\tau(r(\circ)) = \tau(\circ) = \circ
\]

and similarly for \( r \circ \tau \).

Let \( C \) be a point on \( \ell \). Then \( r(C) = C \), and \( \tau(r(C)) = \tau(C) \neq C \) (since \( \tau \) has no fixed points). Let \( C \) be a point not on \( \ell \). Then \( r(C) = C'' \) is on the opposite site of \( \ell \) from \( C \). Then \( \tau(r(C)) = \tau(C'') \) is on the opposite side of \( \ell \) as \( C'' \) since the line \( C'' \tau(C'') \) is parallel to \( \ell \). Thus, \( \tau(C'') \) is on the opposite side of \( \ell \) from \( C \).

Theorem 6.3.9 (Fundamental Theorem of Rigid Motions). Let \( f \) be a rigid motion. In the following statement each \( r_i \) is a reflection across a line \( \ell_i \).

1. \( f \) is a (single) reflection if and only if it has an infinite number of fixed points and reverses orientation.

2. \( f \) is a translation if and only if it has 0 fixed points and preserves orientation if and only if it is possible to write \( f = r_2 \circ r_1 \) with \( \ell_1 \) parallel to \( \ell_2 \).

3. \( f \) is a rotation if and only if it has exactly 1 fixed point and preserves orientation if and only if it is possible to write \( f = r_2 \circ r_1 \) with \( \ell_1 \) not parallel to \( \ell_2 \).
4. \( f \) is a glide reflection if and only if it has 0 fixed points and reverses orientation if and only if it is possible to write \( f = r_3 \circ r_2 \circ r_1 \) with \( \ell_1 \) parallel to \( \ell_2 \), and \( \ell_3 \) perpendicular to \( \ell_1 \) and \( \ell_2 \).

5. In all cases, \( f \) is equal to a reflection, translation, rotation, or glide reflection, and we may identify which case using the following table:

<table>
<thead>
<tr>
<th>( \geq 1 ) fixed point?</th>
<th>Preserves orientation?</th>
<th>No</th>
<th>Yes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Reflection</td>
<td>Glide reflection</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Rotation</td>
<td>Translation</td>
<td></td>
</tr>
</tbody>
</table>

**Proof.** We label the assertions in 1–4 as 1a, 1b, 2a, 2b, etc.

Prop 39 shows 1a \( \Rightarrow \) 1b.

Prop 49 shows 1b \( \Rightarrow \) 1a.

Prop 38 shows 2a \( \Rightarrow \) 2b.

Assume now 2b. Apply Proposition 6.3.3 to write \( f \) as a product of reflections. Then Proposition 6.3.6 shows we must have used 2 reflections. Then Propositions 6.1.7 and 6.1.8 show than we must have used two reflections across parallel lines, thus we have a translation. This shows that 2b \( \Rightarrow \) 2c.

Proposition 6.1.7 shows 2c \( \Rightarrow \) 2b. This completes the proof of all the parts of 2.

Similar steps prove 3 and 4. Part 5 follows from parts 1 – 4.

**Warning:** In the previous Theorem I use the phrase “it is possible to write”. The reflections referred to in the theorem are not necessarily the ones that would be produced following the proof of Proposition 6.3.3. In particular, the reflections produced in Proposition 6.3.3 for a glide reflection will not necessarily have two of the lines be parallel. In fact, if you compose three reflections, the result will either be a reflection, or a glide reflection, with the former occurring if and only if the three lines are concurrent.

**Example 6.** Consider the patterns shown in Figure 6.2. Assume that each pattern extends infinitely far to the left and to the right, and ignore any minor artistic imperfections. Identify which symmetries are present in each pattern, i.e. which rigid motions can be applied that take the pattern back to itself.

**Solution:** All of the patterns have translation, so we can leave this out from our description. Also, if a pattern has two rigid motions, then it also has their combination: this if it has reflection and it has translation, then it has glide reflection. Note that the converse is not true: a pattern can have glide reflection, but not reflection. Thus, it suffices list the rigid motions, leaving out translations and leaving out combinations of things we’ve listed.

- The pattern marked by 11 has no rigid motions (besides translations).
- 1g has a glide reflection (and translations).
- m1 has a reflection across a vertical line through the center (and translations).
- 1 has rotation by \( 180^\circ \) (and translations).
- mg has reflection across the vertical center, and rotation by \( 180^\circ \) around a point between two white semicircles (and translations).
- 1m has reflection across the horizontal line through the center (and translations).
- mm has reflection across the horizontal center and reflection across the vertical center (and translations)

**Corollary 6.3.10.** There are exactly 7 kinds of one dimensional patterns.
Figure 6.2: San Ildefonso Pueblo Patterns

11

1g

ml

12

mg

1m

mm
Chapter 7

Projective Geometry

7.1 Perspective drawing

7.1.1 Model for perspective drawing

Here's the basic idea of perspective drawing: the canvas that you are drawing on should get an accurate record of what your eye sees. Here's one way to figure out what this means: instead of canvas use a piece of glass and put this between your eye and what you are trying to draw. Record on the glass exactly what you see, where you see it. Where you see a person, draw that person on the glass so that your drawing will now cover up the person. Do this with everything you see. For later use let's take note of exactly how this system works. The real world that you are painting, the position of your eye, and the position of your glass are all fixed. For each point in the real world, imagine a straight line connecting that point to your eye. Where this straight line intersects the glass plane is where the mark should be made for the drawing.

When you are done you will have as accurate a drawing of what you originally saw as is possible given your drawing abilities.

7.1.2 Class activity

As a class we recreated the idea shown in Figure 7.1: projecting a picture onto a clear plane. I built a device with a clear piece of plastic and an eyepiece and mounted this on a tripod. We drew geometrical figures on the ground using chalk and blue painter's tape. We then drew (the projected image of) these figures on a piece of plastic; the result was a clear illustration of perspective.

In Figures 7.2–7.6 we can see the original picture as it looked on the ground and the projected drawing that the students made.
Figure 7.2: Projection of Pythagorean Theorem

Figure 7.3: First projection of parallel lines and transversal
I want to use these drawings to make conclusions and conjectures about what geometric properties are or are not preserved under projection.

1. Each straight line in the real world projects to a straight line in the drawing. This is good, it means we can still talk about straight lines in projective geometry.

2. Similarly, projection preserves points and intersections. Thus, projective geometry can still talk about intersecting lines, triangles, etc. Also, we can still say that two points determine a unique straight line.

3. Most lines that are parallel in the real world are not parallel in the projection. Therefore in projective geometry we will not use the concept of “parallel”. In fact, instead of saying that most lines intersect, we will go all the way and consider the possibility that all lines intersect.

4. Distances and angles change. Thus, in projective geometry we will not refer to distances and angles.

5. The Pythagorean theorem does not hold.

### 7.1.3 Constructing perspective grids

A checkerboard, or tiled floor, provides one of the simplest pictures to practice perspective drawing on, and to learn from.
The basic geometry of this picture is just a grid of squares. If we imagine this grid on the floor, and take or draw a picture we will get something vaguely like one of the following:

Let's learn how to make these drawings.

### 7.1.4 A perspective grid with horizontal
Step 1: Draw any isosceles triangle with the base horizontal.

Step 2: Divide the base into equal length segments, and connect each of these points to the top vertex.

Step 3: Draw a horizontal line to mark the top of the first row of squares.

Step 4: Draw a diagonal on one of the “squares”, extend it to the opposite side of the triangle. This line will be used for construction, but will not be part of the final picture.
Steps 5, 6, . . . : Draw a horizontal at each intersection of the red diagonal with one of the black lines that we created in Step 2.

Last step: after all the horizontals are done, erase the red diagonal, and any remaining part of the original triangle.

### 7.1.5 A perspective grid from a corner

Step 1: Draw an isosceles triangle with the top a horizontal line.

Step 2: Mark off equal lengths on the two legs of the triangle, and connect each of these points to the opposite vertex: call these lines “grid-lines”.
Step 3: Draw the diagonal of the first square, and extend it (shown here in red) to intersect the opposite side of the triangle. Pick another point on one of the legs of the triangle; draw a line (shown here in green) connecting this point to the point where the red diagonal hits the top side of the triangle.

Step 4: Draw a line connecting the right-hand vertex to the point of intersection of the green line and one of the other grid-lines.

Step 5: Draw a new grid-line through the left-hand vertex and the intersection of the red diagonal and the last grid-line.

Steps 6, 7, ...: keep adding grid-lines which connect a vertex to the intersection of the red or green diagonal with the previous grid-line.

Last Step: erase the red and green diagonals, and other lines which are not part of the picture

Based on these considerations, we take the axioms for Euclidean Geometry, and we remove any mention of This also removes any notion of betweenness, and our definition of circles. Similarly, we remove any notion of measure of an angle. This removes right angles. What are we left with? Points and lines.

**History**

In 1410’s the Italian artist Filippo Brenelleschi developed the first fairly complete system of perspective drawing. His first steps in figuring this out involved drawing buildings on mirrors. He extended his techniques and appears to have fairly rigorous, and mathematical approach to them. However, it wasn’t until Leon Battista Alberti in
the 1430’s, that we have the publication of a treatise De Pictura (On Pictures) \[http://www.noteaccess.com/Texts/Alberti/\], for others to use (interestingly, there are no pictures or diagrams in his work, not even lines or triangles!). Alberti was also the first to treat the problem by intersecting lines, from the observer to the object, with a plane, the canvas (prior work had considered cones of light).

The next steps (mathematically speaking) took place by Dürer in the late 1400’s and early 1500’s. He explicated the costruzione legittima (which preexisted him) and discusses a variety of mechanisms to aid in perspective drawing, such as the camera lucida.

The final step along the path from art to mathematics was taken by Desargues. Desargues was known as an engineer and architect. At the time this meant someone who was interested in applying mathematical ideas to the real world in an artistic manner. During the 1630’s Desargue applied geometry to perspective drawing, publishing a number of short, dense works on the subject. His works do not appear to have been for artists, but for for the mathematical basis of what the artists already knew how to do. They included mathematical derivations and proofs. Among his key contributions was the discussion of points at infinity, and giving a unified treatment of all conics.

Poncelet is credited with the modern birth of projective geometry. In 1812 we was in Napoleon’s army as it invaded Russia. He was captured by the Russians and spent the years 1812 – 1814 in prison. During this time, he decided to reconstruct the mathematics he had previously learned, and while doing this extended his previous work and inventing projective geometry!
Exercises

1. (From [9, p91].) This problem works with projections from the $x$-axis to the $y$-axis. Let the eye $E$ be the point $(-1, 1)$. To project a point $(x, 0)$ on the $x$-axis to the $y$-axis you draw a line from $(x, 0)$ to $E$ and see where this line intersects the $y$-axis (see Figure 7.7).
   
   (a) Show that the line from $(-1, 1)$ to $(x, 0)$ intersects the $y$-axis at $\frac{x}{x+1}$.
   
   (b) Show that the perspective images of the points $x = 0, 1, 2, 3, \ldots$ are the points $y = 0, 1/2, 1/3, \ldots$.

2. Draw a $6 \times 6$ checkerboard grid, in perspective from one corner.

3. (From [9, p94].) Draw a perspective “grid” of many triangles tiling the plane (hint: use half of a square).

4. (From [9, 94].) Draw a perspective “grid” of many hexagons tiling the plane (hint: take your solution from the previous problem and erase some of the lines).

5. (Inspired by a mention of “curved vanishing lines” in Mary Kate McNulty’s paper about Escher.)

Repeat the steps applied in Section 7.1.5, but starting with the shape in Figure 7.8.

6. See if you can extend problem 5 in various ways. For instance, you could try replacing your straight ruler with the curved edge of a protractor.
CHAPTER 7. PROJECTIVE GEOMETRY

7.2 Explicit projection calculation

In this section we derive a formula for a function that operates in a similar manner to our sidewalk drawing activity. We work in \( \mathbb{R}^3 \) just like in real life! Let our eyepiece be given by the point \( E = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \) and let \( \pi \) be the projection from the \((x, y)\)-plane to the \((x, z)\)-plane. In other words, given a point \((x, y, 0)\), let \( L \) be the line from \((x, y, 0)\) to \( E \), and let \( \pi(x, y, 0) \) be the point equal to the intersection of \( L \) with the \((x, z)\)-plane.

To find a formula for \( \pi(x, y, 0) \), we work with column vectors, and start by describing \( L \):

\[
L: \quad (1 - t) \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} x \\ y \\ z \end{bmatrix}.
\]

To intersect \( L \) with the \((x, z)\)-plane means to find the element of \( L \) that has \( y \)-coordinate equal to 0:

\[
(1 - t)(-2) + ty = 0 \rightarrow t = \frac{2}{2 + y}.
\]

Figure 7.7: A one dimensional projection

![Figure 7.7: A one dimensional projection](image)

Figure 7.8: An M.C. Escher grid

![Figure 7.8: An M.C. Escher grid](image)
We plug this value for $t$ back into $L$ to get a formula for $\pi$

$$
\pi : \quad (x, y)\text{-plane} \quad \longrightarrow \quad (x, z)\text{-plane}
\begin{bmatrix}
    x \\
    y \\
    0
\end{bmatrix}
\quad \longmapsto \quad \frac{1}{2+y}
\begin{bmatrix}
    2x \\
    0 \\
    y
\end{bmatrix}
$$

Now we find a formula for the inverse of $\pi$. We could find this directly from the
formula of $\pi$ (i.e. given a point $\begin{bmatrix}
    \hat{x} \\
    0 \\
    \hat{z}
\end{bmatrix}$ in $V$ solve for $x$ and $y$ such that $\begin{bmatrix}
    \hat{x} \\
    0 \\
    \hat{z}
\end{bmatrix} = \frac{1}{2+y}\begin{bmatrix}
    2x \\
    0 \\
    y
\end{bmatrix}$),
but the construction with the line is perhaps easier and in any case worth practicing.

Thus, to find $\pi^{-1}$ let $L$ be the line from $E$ to a point $\begin{bmatrix}
    x \\
    0 \\
    z
\end{bmatrix}$

$$
L : \quad (1-t)\begin{bmatrix}
    0 \\
    -2 \\
    1
\end{bmatrix} + t\begin{bmatrix}
    x \\
    0 \\
    z
\end{bmatrix}
$$

Solve for $t$ such that the $z$-coordinate is zero:

$$(1-t) + tz = 0 \quad \rightarrow \quad t = \frac{-1}{z - 1}$$

plug this back into $L$ to get a formula for $\pi^{-1}$

$$
\pi : \quad (x, z)\text{-plane} \quad \longrightarrow \quad (x, y)\text{-plane}
\begin{bmatrix}
    x \\
    0 \\
    z
\end{bmatrix}
\quad \longmapsto \quad \frac{1}{z-1}\begin{bmatrix}
    -x \\
    -2z \\
    0
\end{bmatrix}
$$
Exercises

1. Calculate $\pi$ applied to the points
   $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, ..., $\begin{bmatrix} 1 \\ 10 \\ 0 \end{bmatrix}$

2. Using the points in $V$ from the previous problem, calculate the distances between each successive pair. See if you can find a pattern or formula for how the distance will change from one pair to the next.

3. Show that the projection (under $\pi$, or $\pi_2$, or if you like for any projection) of a line is still a straight line.

4. What does $\pi$ or $\pi_2$, as defined earlier in the section, do to a point in the $(x, y)$-plane which is behind the image plane $V$? For example what is $\pi \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$? What is the geometric description of how you find $\pi \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$.

5. Let $M_2$ and $M_3$ be the lines $x = 2$ and $x = 3$ in the $(x, y)$-plane (for comparison, Figure ?? shows $M_1$). Find $\pi_2(L_1)$ and $\pi_2(L_2)$. What is $\pi(\infty)$? What is the significance of this point?

6. Let $E = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$ be an “eye” and find a formula for $\pi_{a,b}$, the projection from the $(x, y)$-plane to $V$, the $(x, z)$-plane.

7. As shown in this section, there are two distinct sets of three collinear points $P_1$, $P_2$, $P_3$ and $P'_1$, $P'_2$, $P'_3$, all in the $(x, y)$-plane, such that their distances are not equal (i.e. $|P_i - P_j| \neq |P'_i - P'_j|$) for $(i, j)$ equal to $(1, 2), (1, 3), (2, 3)$ and yet they project to the same points in $V$ (i.e. $\pi(P_i) = \pi_2(P_i)$ for $i$ equal to $1, 2, 3$).

However, this still begs the question of whether distances can be determined by three projected points and a fixed ruler, or distance, between two of the original points.

Prove that there exists a projection $\varphi$ and collinear points $Q_1$, $Q_2$, and $Q_3$ with $|P_1 - P_2| = |Q_1 - Q_2|$, $|P_1 - P_3| \neq |Q_1 - Q_3|$ and $|\pi(P_i) - \pi(P_j)| = |\varphi(Q_i) - \varphi(Q_j)|$ for $(i, j)$ equal to $(1, 2), (1, 3)$ and $(2, 3)$.

(Hint: Here’s one way to find these points and projection. Let $P_1$, $P_2$, $P_3$ and $\pi$ be as described earlier in the section. Pick $Q_1$ and $Q_2$, at random if you like, so that $|P_1 - P_2| = |Q_1 - Q_2|$. Let $\varphi$ be the projection from $\begin{bmatrix} 0 \\ a \\ 1 \end{bmatrix}$ (you can find a formula for $\varphi$ from the previous problem) and choose $a$ such that $|\pi(P_i) - \pi(P_j)| = |\varphi(Q_i) - \varphi(Q_j)|$. Geometrically this means that you are looking at $Q_1$ and $Q_2$ from the right distance so that $Q_1Q_2$ appears to have the right length. This is always possible, but if you want $E$ to be on one side of $V$ and $Q_1$ and $Q_2$ to be on the other side of $V$, you will have to choose $Q_1$ and $Q_2$ close to $V$.

Now you should have $Q_1$, $Q_2$ and $E$ fixed. Find the third point $X$ on the line through $\varphi(Q_1)$ and $\varphi(Q_2)$ such that the distance from $X$ to $\varphi(Q_2)$ equals $|\pi(P_1) - \pi(P_2)|$. Now apply $\varphi^{-1}$ to $X$ to get a point in the $(x, y)$-plane, and call this $Q_3$. Unless you have been very unlucky in choosing $Q_1$ and $Q_2$, you can verify that $Q_3$ satisfies all the properties you want.)

8. Take the parabola $y = x^2$, in the $(x, y)$-plane, and see what it projects to under $\pi$.

9. Take the circle $(x - 1)^2 + (y - 1)^2 = 4$ in $V$, apply $\pi_2^{-1}$ to it and describe the shape you get in the $(x, y)$-plane.

10. Take the hyperbola $y = 1/x$ and apply $\pi$ to it and describe the shape you get in $V$. 

7.3 The cross ratio

Before defining the cross-ratio we introduce a new concept: signed length. Let \( L \) be any Euclidean line and pick one direction of \( L \) to be positive; if you like, we pick a point \( O \) on \( L \) and a ray from \( O \) along one half of \( L \). For any points \( X \) and \( Y \) let \(|XY|\) be the usual Euclidean length. For \( X, Y \) in \( L \) we define \(|XY|\) to equal \(|XY|\) if the direction from \( X \) to \( Y \) is positive and define \(|XY|\) to equal \(-|XY|\) if the direction from \( Y \) to \( X \) is positive.

Let \( A, B, C \) and \( D \) be four points on a Euclidean line \( L \) with a fixed positive direction. The cross ratio is written as \([A, B, C, D]\) and defined as

\[
[A, B, C, D] = \frac{AC}{AD} \div \frac{BC}{BD} = \frac{AC}{AD} \cdot \frac{BD}{BC}
\]

**Definition 7.3.1.** Let \( E \) be any point in the Euclidean plane \( \mathbb{E} \) and let \( \ell \) be any line. The central projection determined by \( E \) and \( \ell \) is the function \( \pi \) defined as follows: given \( A \in \mathbb{E} \), we let \( \pi(A) \) equal the intersection of the line \( AE \) with \( \ell \). Let \( m \) be the line through \( E \) and parallel to \( \ell \). Note that \( \pi \) is defined on \( \mathbb{E} \) minus \( m \).

**Theorem 7.3.2.** Let \( \pi \) be any central projection defined on a subset of \( \mathbb{E} \). Then \( \pi \) preserves the cross ratio. In other words, if \( A, B, C, D \) are four distinct collinear points then

\[
[A, B, C, D] = [\pi(A), \pi(B), \pi(C), \pi(D)].
\]

**Proof.** Recall that the area of a triangle \( \triangle ABC \) is given by \( \frac{1}{2}|AC| \cdot |AB| \sin(\angle BAC) \), where \( |AB| \sin(\angle BAC) \) is the height, from \( B \) to the line \( AC \).

Let \( E \) be any point, let \( A, B, C \) and \( D \) be four collinear points, on the line \( L \), and \( A', B', C' \) and \( D' \) be their projection, defined via \( E \), to another line \( L' \).

![Diagram of cross ratio and central projection](image)

Then \([A, B, C, D] = [AB] \div [BD] = [\triangle AEC] \div [\triangle AED] \div [\triangle BEC] \div [\triangle BDE] \) (because the triangles have the same height measured from \( E \) to \( L \)).

This equals \([AE] \cdot |CE| \sin(\angle AEC) \div [AE] \cdot |DE| \sin(\angle AED) \div [BE] \cdot |CE| \sin(\angle BEC) \div [BE] \cdot |DE| \sin(\angle BDE) \) Now these angles are all unchanged by replacing \( A \) with \( A' \), etc.

**Example 1.** In the Euclidean model \( \mathbb{R}^2 \), let \( A, B, C \) and \( D \) be points on the \( x \)-axis with \( A = (1, 0), B = (3, 0), C = (6, 0), D = (10, 0) \). Let \( \pi \) be the projection from the \( x \)-axis to the \( y \)-axis via the point \( E = (-1, 1) \), as defined in homework, i.e. \( \pi(x, 0) = \frac{x}{x+1} \). Let \( A' = \pi(A) \), etc.

Find the cross ratios \([A, B, C, D]\) and \([A', B', C', D']\).
\[ [A, B, C, D] = \frac{5}{3} \cdot \frac{7}{2} = \frac{35}{6} \quad D' = \frac{10}{11} \]
\[ [A', B', C', D'] = \frac{1}{2} - \frac{5}{11} \cdot \frac{3}{4} - \frac{10}{22} = \frac{5}{14} - \frac{7}{22} - \frac{7}{11} - \frac{28}{33} = \frac{35}{27} \]

**Example 2.** Suppose we have collinear points \( A', B', C', D' \), and define \( a', b', c' \) and \( d' \) as shown

\[
\begin{align*}
  a' &= |AC| \\
  b' &= |AD| \\
  c' &= |BD| \\
  d' &= |BC|
\end{align*}
\]

Suppose \( a' = 12.35 \), \( b' = 13.05 \), \( c' = 1 \), \( d' = 0.3 \). Suppose we know that these points are the projection of \( A, B, C \) and \( D \) with \( d = 4.55 \) and \( c = 14.55 \). Find \( b \).

We have

\[
\frac{a}{b} \cdot \frac{14.55}{4.55} = \frac{12.35}{13.05} \cdot \frac{1}{0.3}
\]

If we “solve” for \( b \) then this has \( a \) in it. Can we eliminate \( a \)? Note that \( a = b - (c - d) \).

Thus, we can eliminate \( a \)

\[
\frac{b - 10}{b} \cdot \frac{14.55}{4.55} = \frac{12.35}{13.05} \cdot \frac{1}{0.3}
\]

\[
\frac{b - 10}{b} = 0.9865
\]

\[
0.01353b = 10
\]

\[
b = 739.06
\]
Exercises

1. Find the cross ratio of the following lengths

![Diagram with points A, B, C, D]

where $|AB| = 2$, $|BC| = 3$, $|CD| = 4$.

2. Let $\pi$ be the projection described in Section ??.

(a) Find the image of $\pi$ applied to the following points: $A = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 \\ 10 \\ 0 \end{bmatrix}$.

(b) Find the cross ratio of $\pi(A)$, $\pi(B)$, $\pi(C)$, $\pi(D)$.

3. Repeat problem 1 where $|CD| = \infty$.

4. Repeat problem 2 where $D = \begin{bmatrix} 1 \\ \infty \\ 0 \end{bmatrix}$.

5. Suppose the following image is of a street and that we know that the house is 1 mile away from the bottom of the picture. Find the distance between the house and the rocket.

Hint: Use the cross ratio with one point equal to $\infty$. I am assuming that you will need a ruler.
7.4 The real projective plane

Let \( \mathbb{R}^2 \) be the usual Euclidean plane, endowed with coordinates.

**Definition 7.4.1.** 1. For each line \( \ell \) in \( \mathbb{R}^2 \) with slope \( m \) we create a symbol \( \infty_m \).
   (We note the following technical points: for vertical lines we create the symbol \( \infty_\infty \); also, by construction the symbol \( \infty_m \) is not equal to any element or subset of \( \mathbb{R}^2 \); finally, for two or more different lines with the same slope, we just create one symbol. Also, note that the symbol is just meant to be some element of some set that is not already in \( \mathbb{R}^2 \). For instance, it could be \( mi \) where \( i \) is the complex square root of \(-1\).) We call each such symbol \( \infty_m \) an **ideal point**.

2. For each line \( \ell \) with slope \( m \) in \( \mathbb{R}^2 \) we define the *extended* line \( \ell^* \) to equal \( \ell \), together with the element \( \infty_m \) (on a technical, set-theoretical, level \( \ell^* \) equals the union \( \ell \cup \{\infty_m\} \)).

3. We let \( L_\infty \) be set of all such symbols \( \infty_m \).

4. The **real projective plane** \( \mathbb{RP}^2 \) consists of the following:
   - **Points** in \( \mathbb{RP}^2 \): elements in \( \mathbb{R}^2 \) together with each element of the form \( \infty_m \), as described above.
   - **Lines** in \( \mathbb{RP}^2 \): extended lines \( \ell^* \), as described above, together with the set \( L_\infty \) of all possible elements \( \infty_m \).

Note that \( \mathbb{RP}^2 \) contains \( \mathbb{R}^2 \) within it. Points in \( \mathbb{RP}^2 \) that are not ideal points are points in \( \mathbb{R}^2 \); we call these **ordinary points**. Furthermore, all of Euclidean geometry can be viewed as special cases in Projective geometry; specifically, Euclidean geometry deals with those points not of the form \( \infty_m \) (consequently it deals with part of each projective line). Therefore, every Euclidean axiom applies to part of \( \mathbb{RP}^2 \); any axiom which refers to parallel lines applies to the ordinary points on an extended line; any axiom which refers to lengths or size of angles applies again only to ordinary points.

**Definition 7.4.2.** For two points \( A \) and \( B \) in \( \mathbb{RP}^2 \) we define \( |AB| \) as follows: \( |AB| \) equals the Euclidean distance \( |AB| \) if \( A \) and \( B \) are both ordinary points; \( |AB| = \infty \) if either \( A \) or \( B \) or both are ideal points.

We define \( \infty_\infty = 1 \) and extend the cross ratio to \( \mathbb{RP}^2 \). Now we extend this definition to \( \mathbb{RP}^2 \). Let \( L \) be a line in \( \mathbb{RP}^2 \) with \( L \neq L_\infty \) and fix a direction on \( L \). If \( X \) or \( Y \) is infinite then we define \( XY \) to equal \( \infty \) or \( -\infty \) according to the direction of \( L \). We define \( \infty_\infty = 1 \), and use the usual rules of signs to extend this definition to \( \infty_\infty \), \( \infty_\infty \), etc. In this manner we can define \( [A, B, C, D] \) when one point is at infinity. We cannot extend this definition to the case when more than one point is infinite. For then the line containing these four points in \( L_\infty \). We do not claim to be able to put distances on all infinite points in a meaningful way!

**Example 3.** Suppose we have collinear points \( A', B', C', D' \), and define \( a', b', c', d' \) as shown

\[
\begin{align*}
a' &= |AC| \pm \\
b' &= |AD| \pm \\
c' &= |BD| \pm \\
d' &= |BC| \pm 
\end{align*}
\]

Suppose \( a' = 12.35 \), \( b' = 13.05 \), \( c' = 1 \), \( d' = 0.3 \). Suppose we know that these points are the projection of \( A, B, C \) and \( D \) with some of the coordinates of \( A, B, C \) and \( D \) known, as shown

\[
A = 0, B = 6, D = \infty
\]
CHAPTER 7. PROJECTIVE GEOMETRY

Find $C$.

We have

\[
\frac{a}{\infty} \cdot \frac{\infty}{d} = \frac{12.35}{13.05} \cdot \frac{1}{0.3}
\]

If we “solve” for $a$ then this has $d$ in it, and if we solve for $d$ this has $a$ in it. As before, we can eliminate $d$ (or $a$), but not using $a = b - (c - d)$ (can you see why?). Rather, $a - d = 6$, since $A = 0$ and $B = 6$.

\[
a = d3.1545
\]

\[
a = (a - 6)3.1545
\]

\[
a = 8.785
\]

Example 4. Consider Figure 7.9, which shows the picture of one of the pyramids in Egypt (taken from Wikipedia) If we assume that the man near the close corner is 6 feet tall, how big is the pyramid?

We start by making a drawing of crucial parts of the pyramid, shown in Figure 7.10. The application of the cross ratio will be along the bottom line that we have drawn, but we have used the other lines to help determine the point at infinity. We extend the sides of the pyramid, and some of the lines along the steps, so that we can determine the point at infinity, as shown in

Now we take out our rulers and measure, right on the paper, the apparent positions of $B'$, $C'$ and $D'$. Here’s what I get (with $A' = 0$)

\[
A' = 0, \ B' = 0.65, \ C' = 12.8, \ D' = 16.6
\]

Now, we’ve assumed that the man is about 6 feet tall. The length of the first block that is next to him is pretty close to this, maybe just a bit more. Let’s say that it’s 7 feet tall.

This means that we have the following information about the original points:

\[
A = 0, \ B = 7, \ C = ?, \ D = \infty.
\]

We solve this for $C$ as before $C = a = d \cdot \frac{a'}{b'} \cdot \frac{b'}{c'}$ with $a' = 12.8, b' = 16.6, c' = 15.95, d' = 12.15$ and $d = a - 7$

\[
C = a = 578.6 \text{ feet}
\]

Figure 7.9: The Great Pyramid of Kheops
Figure 7.10: Outline of pyramid

Figure 7.11: Pyramid outline extended
That’s a pretty good estimate, considering how crude all of our guesses and techniques were. The real length is about 755 feet.

These examples are wonderful, but we have not proven anything yet! Let’s do so now. Recall that points in \( \mathbb{RP}^2 \) are points in \( \mathbb{R}^2 \) (ordinary points) together with ideal points (i.e. points at infinity). Lines in \( \mathbb{RP}^2 \) are of the form \( \ell^* \) (where \( \ell \) is a euclidean line and \( \ell^* \) is \( \ell \) together with the point at infinity), or \( L_\infty \) (the line consisting of all the ideal points).

**Proposition 7.4.3.** For any two distinct points in \( \mathbb{RP}^2 \) there exists a unique line containing them.

**Proof.** Let \( A \) and \( B \) be two points in \( \mathbb{RP}^2 \). There are three cases: both are ordinary points, one is an ordinary point and one is ideal, and both are ideal points.

Let both be ordinary points. Then there is a unique euclidean line \( \ell \) containing \( A \) and \( B \). Then \( L = \ell^* \) is a projective line and contains \( A \) and \( B \) as well. Furthermore, if \( M \) is another projective line containing \( A \) and \( B \), then \( M = m^* \) for some euclidean line \( m \), and \( A, B \in m \), so \( m = \ell \) since uniqueness holds in euclidean geometry, and then \( M = L \).

Let \( A \) be ordinary and \( B \) be ideal. Then \( B = \infty_m \) for some slope \( m \). Let \( \ell \) be the ordinary line through \( A \) with slope \( m \). Then \( L = \ell^* \) contains \( A \) and \( \infty_m \), i.e. \( A, B \in L \). Uniqueness of \( L \) follows from the uniqueness of the line through \( A \) with slope \( m \).

Let \( A \) and \( B \) be ideal. Then \( L_\infty \) contains \( A \) and \( B \). Furthermore, no other line in \( \mathbb{RP}^2 \) contains both, since each other line contains exactly one ideal point.

**Proposition 7.4.4.** For any two distinct lines in \( \mathbb{RP}^2 \) there exists a unique point in their intersection.

**Proof.** Let \( L \) and \( M \) be two distinct lines in \( \mathbb{RP}^2 \). There are two cases: both are extended ordinary lines, or one is the line \( L_\infty \).

Suppose that both are extended ordinary lines: \( L = \ell^* \) and \( M = m^* \). If \( \ell \) and \( m \) are not parallel, then they intersect in a point. If they are parallel, then they have the same slope, \( r \), and then \( \ell^* \) and \( m^* \) are both defined as their ordinary line together with \( \infty_r \), whence both contain the same ideal point. Now suppose that one of these lines, say \( m \), is \( L_\infty \). Then \( L = \ell^* = \ell \cup \infty_r \) for some slope \( r \). Then \( \infty_r \in L_\infty \), and so \( \infty_r \) is in the intersection of \( L \) and \( M \).

The preceding paragraph shows that \( L \) and \( M \) have at least one point in their intersection. But they cannot have more than one point in their intersection, since if they did they would be the same line by the previous proposition.

Next we re-prove Theorem 7.3.2.

**Definition 7.4.5.** Let \( E \) be any point in \( \mathbb{RP}^2 \) and let \( \ell \) be any line. The central projection determined by \( E \) and \( \ell \) is the function \( \pi \) defined as follows: given \( A \in \mathbb{RP}^2 \), we let \( \pi(A) \) equal the intersection of the line \( AE \) with \( \ell \). Note that \( \pi \) is defined on \( \mathbb{RP}^2 \) minus \( E \) (unless \( E \) is on \( \ell \), in which case \( \pi \) is defined on \( \mathbb{RP}^2 \) minus \( \ell \)).

**Theorem 7.4.6.** Let \( \pi \) be any central projection defined on a subset of \( \mathbb{RP}^2 \), and projecting to a line \( \ell \) where \( \ell \) is not the line \( L_\infty \). Then \( \pi \) preserves the cross ratio. In other words, if \( A, B, C, D \) are four distinct collinear points, at most one of which is an ideal point, then
\[
[A, B, C, D] = [\pi(A), \pi(B), \pi(C), \pi(D)]
\]

**Proof.** There are three cases: All points are ordinary, in which case we use Theorem 7.3.2; \( E \) is an ideal point (i.e. \( E = \infty \)); or one of the other points, say, \( D \), is ideal (i.e. \( D = \infty \)).

When \( E = \infty \) we have four parallel lines: \( AA', BB', CC', DD' \). For details, see the notes from class.
When \( D = \infty \) we have \( ED' \) is parallel to \( AB \). For details, see the notes from class.

As in Euclidean geometry, there are some facts about \( \mathbb{R}P^2 \) which are so obvious that we might be too lazy to state officially (like the fact that there are at least 3 distinct points; or at least 4 distinct non-collinear points, etc.) however there are two ultra-important axioms which you should verify:

A1 For every two distinct points there is a unique line which contains them.

A2 For every two distinct lines there is a unique point contained in their intersection.

These two axioms are dual: if you start with A1, then switch the words “point” and “line” and replace “contains” with “contained in”, then you obtain A2. Similar replacements will turn A2 into A1. This suggests the following fact.

Fact (Duality). Every true statement made about \( \mathbb{R}P^2 \) remains true if you switch “point” and “line” as well as “contains” and “contained in” (as well as all synonyms).

Be careful though: I used the word “suggests” because it takes more work to prove the fact; in particular, although A1 and A2 are dual, I have referred to other axioms that \( \mathbb{R}P^2 \) satisfies, which we have not even stated, and these axioms would also have to be scrutinized to verify that their dual versions are also true.

\(^1\)Technically, I’m misusing the term “axiom” here, but for a reason. Usually axioms are assumed and not verified. However, we have been given the model of \( \mathbb{R}P^2 \) and we can prove that the properties stated in the axioms hold. I continue to call them axioms because they are often used as axioms (i.e. assumed), in which case one proves things with them without referring to specific facts or knowledge about \( \mathbb{R}P^2 \).
Exercises

1. Show the following principle of duality applies to trigonometry. Given an identity involving a single angle, the equation obtained by replacing each trigonometric function with its cofunction is also an identity.

2. Find the equation of a plane which contains the points \( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \), \( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \) and \( \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \).

3. In each part find a description of the line through the two points in \( \mathbb{RP}^2 \):
   (a) \( (1, 2) \) and \( (10, 15) \).
   (b) \( (1, 2) \) and \( \infty_{17} \).
   (c) \( \infty_{-3} \) and \( \infty_{22} \).

4. In each part find the point in the intersection of the two lines
   (a) \( y = 3x + 2 \) and \( 5x + y = 2 \)
   (b) \( 3x + 2y = 5 \) and \( y = -3/2x + 5 \)
   (c) \( 3x + 2y = 5 \) and \( L_{\infty} \).

5. Prove that axioms 1 and 2 hold in \( \mathbb{RP}^2 \).

6. From [10, p37] Each of the following descriptions applies to points and lines in a projective plane. In each case give (1) a written version of the dual description and (2) draw a picture of the original and the dual.
   You might want to use \( A, B, C \), etc. for points and \( a, b, c \), etc. for lines.
   (a) Three non-collinear points and the three lines determined by them.
   (b) Four points, no three of which are collinear, and the six lines determined by them.
   (c) Three concurrent lines \( a, b, c \), through a point \( O \), and their points of intersection with a fourth line \( d \) not through \( O \).
   (d) The line \( AB \) intersecting the line \( AD \) in the point \( E \).

7. Define a dual version of the cross ratio, i.e. a quantity that applies to four concurrent lines.

8. State and prove a dual version of Theorem 7.4.6.
7.5 Axioms for Projective Geometry

The real projective plane \( \mathbb{P} \) is a set of points, and a nonempty collection of points called lines, with the following properties:

1. There exists at least one line.
2. For each line there exists a point not on the line.
3. Every pair of distinct points are contained in a unique line.
4. Every pair of distinct lines intersect in a unique point.
5. For every line \( L \), there is a bijection between the real numbers, and \( L \) minus one point.

7.6 Real projective space

For the purpose of proving the Projective Desargue's Theorem, we will use \( \mathbb{RP}^3 \), real projective space. (Actually, there are a number of times where it's convenient to have real projective space. For instance, projections. These operate on linear subsets, so if we want the right context for applying them to a two dimensional problem, we should work in three dimensional space.) This will be defined in a manner as similar as possible to \( \mathbb{RP}^2 \). The main difference is that for lines in \( \mathbb{R}^3 \) we do not have the concept of slope. More completely, in \( \mathbb{R}^3 \) we can't measure the slope of a line with just one number; we must use more than one number, such as the slope in the \((x,y)\)-plane, the slope in the \((x,z)\)-plane, and the slope in the \((y,z)\)-plane.

For each Euclidean line \( \ell \) in \( \mathbb{R}^3 \) define its slope triple as follows: fix two points on \( \ell \), let \( \Delta x \), \( \Delta y \) and \( \Delta z \) be the change in \( x \)-coordinates, \( y \)-coordinates and \( z \)-coordinates respectively. Then the slope triple is the triple of numbers \( \vec{m} \) defined as

\[
\text{slope triple of } \ell = \vec{m} = \left( \frac{\Delta y}{\Delta x}, \frac{\Delta x}{\Delta z}, \frac{\Delta z}{\Delta y} \right).
\]

If one of these fractions involves division by 0, we set it equal to \( \infty \). For example, the \( x \)-axis has slope triple given by \( (0, \infty, \infty) \).

Example 5. Let \( \ell \) be a line through the two vectors \( v = \begin{pmatrix} 2 \\ 5 \\ -7 \end{pmatrix} \) and \( u = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} \). Then the slope triple of \( \ell \) is

\[
\vec{m} = (-7/3, 10/3, -10/7).
\]

For each Euclidean line \( \ell \) in \( \mathbb{R}^3 \) with slope triple \( \vec{m} \) we create a symbol \( \infty_{\vec{m}} \). For each line \( \ell \) in \( \mathbb{R}^3 \) we define the extended line \( \ell^* \) to equal \( \ell \), together with the element \( \infty_{\vec{m}} \).

For each Euclidean plane \( W \) in \( \mathbb{R}^3 \) we define the projective plane \( W^* \) by adding to \( W \) all the symbols \( \infty_{\vec{m}} \), where \( \vec{m} \) is the slope triple of any possible line \( \ell \) in \( W \). Each \( W^* \) has also a line at infinity consisting of all the elements \( \infty_{\vec{m}} \) which are part of \( W^* \). Note that in this notation \( \mathbb{RP}^2 \) equals \( (\mathbb{R}^2)^* \) (where we identify \( \mathbb{R}^2 \) is the \((x,y)\)-plane in \( \mathbb{R}^3 \)). In a similar way, we can view each \( W^* \) as a copy of \( \mathbb{RP}^2 \).

We now define real projective space \( \mathbb{RP}^3 \).

Points in \( \mathbb{RP}^3 \): elements in \( \mathbb{R}^3 \) together with each \( \infty_{\vec{m}} \) as described above.

Lines in \( \mathbb{RP}^3 \): extended lines \( \ell^* \), in some \( W^* \) as described above; also, each line at infinity which is part of some \( W^* \).

Planes in \( \mathbb{RP}^3 \): each \( W^* \) as described above, also the set \( W_{\infty} \) consisting of all elements \( \infty_{\vec{m}} \) for all possible Euclidean lines in \( \mathbb{R}^3 \).

Again, we will state some (but not all) of the axioms for \( \mathbb{RP}^3 \).

A1 For every two distinct points there is a unique line which contains them.
A2 For every two distinct coplanar lines there is a unique point contained in their intersection.

A3 For every two distinct planes there is a unique line contained in their intersection.

A4 For every plane and every line not contained in the plane, the intersection there is a unique point contained in their intersection.

Let me mention one crucial step in verifying these axioms: you would have to prove that parallel Euclidean lines produce the same point at infinity, and that parallel Euclidean planes produce the same line at infinity.
Exercises

1. For each pair of planes find the line in the intersection
   (a) $2x + 3x + 4z = 1$ and $x - 3y + 2z = 0$.
   (b) $2x + 3y + 4z = 0$ and $2x + 3y + 4z = 10$.

2. Find the intersection of the line $(1 - t) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ and the plane $2x + 3x + 4z = 0$.

3. Find a plane, a line which is parallel to the plane, and the intersection of the line and the plane.

4. Prove that axioms 1, 2, 3, 4 hold in $\mathbb{R}P^3$. 
7.7 Desargue’s Theorem

Recall that three or more lines are **concurrent** if they all intersect at a common point.

We say that two triangles \(\triangle ABC\) and \(\triangle A'B'C'\) are in **perspective** from a point \(O\) if the lines \(AA'\), \(BB'\) and \(CC'\) are all concurrent at \(O\).

**Theorem 7.7.1** (Desargue’s Theorem). If triangles \(\triangle ABC\) and \(\triangle A'B'C'\) are in perspective from the point \(O\) then the intersections

\[
\begin{align*}
AB \cap A'B', & \quad AC \cap A'C', & \quad BC \cap B'C'
\end{align*}
\]

are collinear.

**Proof.** Case 1: the triangles are not in the same plane.

Case 2: the triangles are in the same plane.
Exercises

1. Draw a statement of Desargue’s theorem in two stages: First showing the hypotheses of the theorem, and second showing the conclusions.

2. Let $\triangle ABC$ be a triangle in $\mathbb{R}P^2$. List the possible Euclidean descriptions of $\triangle ABC$.

3. Suppose $\triangle ABC$ and $\triangle A’B’C’$ are in perspective from a point $O$. List the possible Euclidean descriptions of $O$, $\triangle ABC$ and $\triangle A’B’C’$.

4. State the dual version of Desargue’s Theorem.

5. State the converse of Desargue’s Theorem.

6. (From [6, p125].) The following construction shows how to draw three lines that intersect at a common point which is not necessarily on the paper. Prove that the construction works.

   Given two lines $L$ and $L’$ and a point $X$ we will draw a third line $L''$ which goes through $X$ and intersects $L$ and $L’$ at some other common point.

   Draw three concurrent lines $M_1$, $M_2$, $M_3$ with $X \in M_1$. Let

   $A = M_1 \cap L$  \hspace{1cm}  $A’ = M_1 \cap L’$
   $B = M_2 \cap L$  \hspace{1cm}  $B’ = M_2 \cap L’$
   $C = M_3 \cap L$  \hspace{1cm}  $C’ = M_3 \cap L’$
   $P = AB’ \cap A’B$  \hspace{1cm}  $Q = BC’ \cap B’C$
   $Y = XP \cap M_2$  \hspace{1cm}  $Z = YQ \cap M_3$

   Now set $L'' = XZ$ (see Figure 7.12).

   (Note: in practice it might be easiest to draw the lines $M_1$, $M_2$ and $M_3$ as parallel, or almost parallel.)

7.8 Axiomatizing projective geometry

Before a person starts studying Euclidean geometry as a mathematical subject, he or she has a firm geometrical intuition than influences how they view of the subject. In this context, writing down the axioms for Euclidean geometry does not serve so much to guide the student, but rather to make geometric arguments depend on simple premises, and logical deduction rather than assertions like “see, the triangles are similar because they look the same.” Ultimately, it is a surprising triumph of the axiomatic method that everything in Euclidean geometry can actually be derived from so few axioms (about 20 if we use the complete list due to Hilbert).

I hope the previous sections will now allow us to do something similar. We have some intuition about projections and about $\mathbb{RP}^2$, now we will record the most important properties that hold in $\mathbb{RP}^2$.

A1 For every two distinct points there is a unique line which contains them.
A2 For every two distinct lines there is a unique point contained in their intersection.
A3 There exist four points, no three of which are in a line.
A4 Desargue’s Theorem holds.

Figure 7.17: Something to do with Desargue
Exercises

All of the following problems assume only a geometry which satisfies axioms (1)–(4); you might think about $\mathbb{RP}^2$ when you try to solve them, but your final solution should use only these four axioms.

1. Prove that there are at least four lines, no three of which are concurrent.
2. Prove that the smallest set of points and lines which satisfy axioms (1)–(3) has 7 points and 7 lines, and give a picture of this object.
3. Prove that every point is contained in at least three lines.
4. Prove that every line contains at least three points.
5. (From [10, p37]) Prove that if there are exactly $n$ points on one line, then there are exactly $n$ points on every line.
6. (From [10, p37]) Prove that if there are exactly $n$ points on one line, then there are exactly $n$ lines through every point.
7. (From [10, p37]) Prove that if there are exactly $n$ points on a line then the whole geometry contains exactly $n^2 - n + 1$ points.

(The number $n$ in this case is called the order of the geometry. A topic of much recent research is to classify projective geometries of various orders, see [7] for an easy article about the search for a projective plane of order 10, and see [8].)
Appendix A

Linear Algebra

Recall that \( \mathbb{R}^3 \) represents three dimensional space. We assign each point in \( \mathbb{R}^3 \) coordinates, \( x, y \) and \( z \). When we do linear algebra, we usually write these coordinates in a column vector like this

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]

in \( \mathbb{R}^3 \).

The reason for using the notation of column vectors is that it makes certain calculations a little bit easier, namely adding vectors and multiplying by scalars.

For example, if we pick two points

\[
\begin{pmatrix}
-2 \\
3.5 \\
11
\end{pmatrix}
\text{ and } \begin{pmatrix}
4 \\
7 \\
-0.5
\end{pmatrix}
\]

in \( \mathbb{R}^3 \), we can add them together

\[
\begin{pmatrix}
-2 \\
3.5 \\
11
\end{pmatrix} + \begin{pmatrix}
4 \\
7 \\
-0.5
\end{pmatrix} = \begin{pmatrix}
-2 + 4 \\
3.5 + 7 \\
11 - 0.5
\end{pmatrix} = \begin{pmatrix}
2 \\
10.5 \\
10.5
\end{pmatrix}.
\]

Note that writing these coordinates as column vectors made it a little bit easier to line up the coordinates and add them.

Similarly, if we pick one point and one scalar

\[
\begin{pmatrix}
-2 \\
3.5 \\
11
\end{pmatrix}
\text{ in } \mathbb{R}^3 \text{ and } 5 \text{ in } \mathbb{R}
\]

we can multiply them together

\[
5 \begin{pmatrix}
-2 \\
3.5 \\
11
\end{pmatrix} = \begin{pmatrix}
-10 \\
17.5 \\
55
\end{pmatrix}.
\]

Adding vectors, and multiplying by scalars can be used to describe a lot of geometrical concepts. For instance, if \( \vec{u} \) and \( \vec{v} \) are two vectors, then the points \( \vec{0}, \vec{u}, \vec{v} \) and \( \vec{u} + \vec{v} \) form the four corners of a parallelogram. Using these concepts, we can also describe the line through the vectors \( \vec{u} \) and \( \vec{v} \).

In particular, the point \( \frac{1}{2} \vec{u} + \frac{1}{2} \vec{v} \) is the midpoint between \( \vec{u} \) and \( \vec{v} \). The point \( \frac{1}{3} \vec{u} + \frac{2}{3} \vec{v} \) is on the line segment connecting \( \vec{u} \) and \( \vec{v} \), two thirds of the way towards \( \vec{v} \). In a similar fashion, the line through \( \vec{u} \) and \( \vec{v} \) is given by all the points of the form

\[
(1 - t)\vec{u} + t\vec{v}
\]
as \( t \) ranges through all real numbers. This description of a line makes it easy to solve for the intersection of the line with some other geometric shape: you solve for the value of \( t \), and then plug this into the formula for the line.

**Example 1.** Find the intersection of the line through the points \( \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} \) and \( \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \) with the plane defined by \( y = 4 \).

*Solution:* The line equals all the points of the form

\[
(1 - t) \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.
\]

We solve for \( t \) such that the \( y \)-coordinate is 4:

\[
(1 - t)2 + t3 = 4 \quad \Rightarrow \quad 2 + t = 4 \quad \Rightarrow \quad t = 2.
\]

To find the point on the line corresponding to \( t = 2 \), we plug this \( t \) value back into the formula for the line

\[
(1 - 2) \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}.
\]
References


   This book is fun reading. In pages 196 and following, the author discusses Desargue’s work in some detail, in particular some of the terminology he introduced, and his use of infinity. There’s an interesting review of this book at the AMS website: http://www.ams.org/notices/200001/rev-phillips.pdf

Appendix A

Mason and Dixon Tangent Line

Dimensions assume having access to a 54” drywall square.
To make this set up draw the parts in black. Here’s the order I did it in: draw MP 8 + 7/8 inches from the right edge. Then the line straight up from that. At 53 + 1/8 inches along that line I measured 6 + 7/8 inches over and marked NC.
I can think of a few ways to do this problem. They all start the same way, and they all use a little bit of iteration, or if you prefer, trial and error followed by an improved trial with much smaller error.

Here’s how the iteration, or successive approximation works.

Suppose we have any candidate line $\ell$ and let $TL$ be the line we are trying to find. Drop a perpendicular from $NC$ to $\ell$. Note that even if $\ell$ is a little more than 12 inches from $NC$, we can get a pretty good approximation of the perpendicular. Since we will iterate the whole process, a good approximation is all we need.

Let $a$ be the distance from $\ell$ to $NC$ and let $b$ be the distance from $MP$ to the point where the perpendicular hits $\ell$. Then $a$ and $b$ are both directly measurable (even if the distance is larger than 12 inches, we have a line to move along so we can do it with multiple uses of the ruler).

Now we compare $a$ to 13, the radius of the circle and then we displace the line $\ell$ by the difference. For example, suppose $a = 14.5$. This means that $\ell$ is too far to the west and we need to move it to the east. Since $14.5 - 13 = 1.5$, the displacement should be 1.5 inches at the intersection, and we use adjust the displacement linearly everywhere else. For instance, if $b = 56$ then the point 12 inches up from $MP$ on $\ell$ should be moved by $\frac{12}{56} \times 1.5$ inches to the east. Once we have this first point, we can extend the line through it, or make the other points by calculating the displacements.

In this fashion, $\ell$ produces a second line $m$ which should be much more accurate. If $\ell$ was close to correct, then $m$ will be within measurable error of $TL$.

So the question really is how to get the first approximation $\ell$. Following are some different ways of finding the first approximation.

1. Eyeball it: we can see roughly where $\ell$ should go. Careful sighting from the point $MP$ towards the edge of the circle should produce a line $\ell$ that’s within 2 or so inches of $TL$. 

![Diagram of perpendicular and displacement]
2. Trigonometry: In the following picture we are “given” 6.9, 13 and 53 (not to scale)

Then we have

\[ c = \sqrt{6.9^2 + 53^2} \]
\[ \tan(\beta) = \frac{6.9}{53} \]
\[ \sin(\beta + \theta) = \frac{13}{c} \]
\[ \theta = \sin^{-1}\left( \frac{13}{\sqrt{6.9^2 + 53^2}} \right) - \tan^{-1}(\frac{6.9}{53}) \]
\[ \approx 6.66^\circ \]

Now that \( \theta \) is known, we can use our protractor to draw an approximation of the line \( \ell \).

3. Start with \( \ell \) given by the “meridian,” the line due north from MP. So the end of the line needs to move about 6.1 inches to the west. The point 12” north of MP needs to move \( \frac{12}{53} \times 6.1” \) to the west. Using this point, we can extend a line until it hits the circle. The resulting adjusted line might not be close enough to TL since our first iteration was pretty far off. It might be necessary to make a second iteration.
Appendix B

Hexagons for a Hyperbolic Plane
The activity calls for enough of the hexagons and triangles to make a 2 foot diameter “circle”. How many hexagons is this? Each one is 4in = 1/3 of a foot in diameter. So 3 per foot, and 6 per two feet. But this is linear. How many for the “circle”?

Well, I don't know the answer. But, I would start with the Euclidean amount, and then add some. Maybe I would add 1/6th, since that’s what is added to a single hexagon. But maybe there’s some compounding?

This is 26. Now, we need an equilateral triangle for each hexagon. So 26 equilateral triangles. That may be enough, but I'm not sure, so increase each by 1/6th? Make it of 30 each. For three groups make it 90. So 30 copies of the hexagon sheets and 6 of the triangles.
Bibliography
