Homework 9
Due April 15, 2020

1. Do ONE (and only one) of the problems (a) or (b) below. In both cases we assume that $X, A, B,$ and $Y$ are points as needed in the definition of $d(A,B)$ in the Poincare Disc, in the order $X — A — B — Y$ (i.e. $X$ is on the boundary of the circle; $A$ is between $X$ and $B$; $B$ is between $A$ and $Y$; and $Y$ is on the boundary of the circle).

(a) With $A, B$ any points as described above, show that $d(A,B) = 0$ if and only if $A = B$

**Solution**

Note that $X — A — B — Y$ (where “between” means along the circular arc). We claim that $\overline{AX} \leq \overline{BX}$, with equality $\iff A = B$.

Clearly, if $A = B$ then equality holds. Conversely, if $A \neq B$, then the Euclidean circle $C$ with center at $X$ and radius $\overline{XA}$ intersects the arc $XABY$ in one place, namely at $A$. Since $B$ is on the opposite side of $A$ than $X$, we have that $B$ is outside of $C$. Therefore, $\overline{AX} < \overline{BX}$.

Now we combine inequalities:

- $\overline{AX} \leq \overline{BX}$, (above)
- $\overline{BY} \leq \overline{AY}$, (similar to previous line)
- $\overline{AX} \cdot \overline{BY} \leq \overline{BX} \cdot \overline{AY}$, (multiply previous lines)
- $\frac{\overline{AX}}{\overline{AY}} \leq \frac{\overline{BX}}{\overline{BY}}$ (divide previous line by $\overline{AY}$ and $\overline{BY}$)

Furthermore, equality holds in the first line if and only if $A = B$ if and only if equality holds in all the lines.

Now we calculate the distance formula:

$$d(A,B) = 0$$

$$\iff \left| \ln \left( \frac{\overline{AY} \cdot \overline{BX}}{\overline{AX} \cdot \overline{BY}} \right) \right| = 0$$

$$\iff \ln \left( \frac{\overline{AY} \cdot \overline{BX}}{\overline{AX} \cdot \overline{BY}} \right) = 0$$

$$\iff \frac{\overline{AY} \cdot \overline{BX}}{\overline{AX} \cdot \overline{BY}} = 1$$

$$\iff \frac{\overline{AY}}{\overline{AX}} = \frac{\overline{BY}}{\overline{BX}}$$

As above, we saw that this is equivalent to $A = B$.

(b) With $A$ and $B$ any points as described above, let $C$ be any point on the circular arc between $A$ and $B$. Show that $d(A,B) = d(A,C) + d(C,B)$

**Solution**
Theorem (SAS Congruence) Given two biangles, if one pair of corresponding angles are equal, and the bases are equal, then the biangles are congruent.

Proof. We assume biangles ... and are given with ... and ... as pictured.

Then we only need to prove ....

Suppose, for contradiction, that ... and ... are not equal. WLOG assume that $\angle B > \angle D$. We copy angle ... to the point $E$ so that $\angle \ldots E = \angle D$. Note that Point $E$ is in the interior of $\angle \ldots$ as pictured because ....

In case we need more points labeled, let's all use the same letters:

Since $\angle ABE$ is less than the $\angle ABY$, and since $BY$ is the asymptotic parallel to $AX$, the line $BE$ intersects $AX$. Let $F$ be the intersection. Let $G$ be the point on $CW$ with $CG = AF$. 

Now we apply SAS to the two triangles $\triangle ABF$ and $\triangle CDG$:

... $= ...$ because ...

... $= ...$ because ...

... $= ...$ because ...

Therefore $\triangle ABF \cong \triangle CDG$. Therefore ........... So $\angle CDG = \angle CDZ$. This implies that the line $DG$ equals the line $DZ$. This is contradiction because DG is ... and DZ is not ....

**Solution**

We assume biangles ... and are given with $\angle A = \angle C$ and $\overline{AB} = \overline{CD}$ as pictured.

Then we only need to prove $\angle B = \angle D$.

Suppose, for contradiction, that $\angle B \neq \angle D$. WLOG assume that $\angle B > \angle D$. We copy angle $\angle D$ to $\angle B$ to create the point $E$ so that $\angle ABE = \angle D$. Note that Point $E$ is in the interior of $\angle B$ as pictured because $\angle ABE < \angle B$.

Let $X$, $Y$, $W$ and $Z$ be points labeled as shown.

Since $\angle ABE$ is less than the $\angle ABY$, and since $BY$ is the asymptotic parallel to $AX$, the line $BE$ intersects $AX$. Let $F$ be the intersection. Let $G$ be the point on $\overline{CW}$ with $\overline{CG} = \overline{AF}$.

Now we apply SAS to the two triangles $\triangle ABF$ and $\triangle CDG$:

$\overline{AB} = \overline{CD}$

$\angle XAB = \angle WCD$

$\overline{AF} = \overline{CG}$

The “S” in our given SAS

The “A” in our given SAS

definition of $G$
Therefore $\triangle ABF \cong \triangle CDG$. Therefore

\[
\angle CDG \cong \angle ABF \quad \text{congruent triangles}
\]
\[
= \angle ABE \quad \text{because } \overline{BE} = \overline{BF}
\]
\[
\cong \angle CDZ \quad E \text{ was constructed by copying } \angle D
\]

Since $\angle CDG = \angle CDZ$, the line $DG$ equals the line $DZ$, which is a contradiction since $DG$ is not parallel to $CW$ and $DZ$ is.