1. Hyperbolic AAA congruence. Let $\triangle ABC$ and $\triangle DEF$ be two triangles with $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$. Prove that the triangles are congruent.

Hint: WLOG we can prove that $\overline{AB} = \overline{DE}$ (why?)

Proceed by contradiction. Justify the construction in the following picture, then apply SAS, then show the angles in some quadrilateral add to 360. What is the contradiction?

Solution

Suppose, for contradiction, that $\overline{AB} \neq \overline{DE}$. Without loss of generality, suppose $\overline{AB} > \overline{DE}$. Let $G \in \overline{AB}$ with $\overline{AG} = \overline{DE}$ and let $H \in \overline{AC}$ such that $\overline{AH} = \overline{DF}$.

We apply SAS with $\overline{AG} = \overline{DE}$, $\angle A = \angle D$, $\overline{AH} = \overline{DF}$ to conclude that $\triangle AGH \cong \triangle DEF$. Combining this with the given angles equality we have

$\angle 1 = \angle 7 = \angle 5,$
$\angle 3 = \angle 8 = \angle 6.$

This shows that $GH$ and $BC$ are parallel. Therefore the second picture is impossible and also $\square BCHG$ is a simple quadrilateral. We also know that

$\angle 1 + \angle 2 = 180^\circ$ and $\angle 3 + \angle 4 = 180^\circ$

so

$\angle 1 + \angle 2 + \angle 3 + \angle 4 = 360^\circ$.

Replacing $\angle 1$ and $\angle 3$ with $\angle 5$ and $\angle 6$ we get

$\angle 5 + \angle 2 + \angle 6 + \angle 4 = 360.$

Thus, the angles inside the quadrilateral $\square BCHG$ add to 360, a contradiction.

2. Distance in the Poincare Disc.
Let $A$ and $B$ be points in the Poincaré disc model of Hyperbolic Geometry. We define the hyperbolic distance between them as $d(A, B)$ where

$$d(A, B) = \left| \ln \left( \frac{AY \cdot BX}{AX \cdot BY} \right) \right|$$

where $X$ and $Y$ are the intersection of the hyperbolic line $\overline{AB}$ with the boundary of the Poincaré disc, and $AY$, $BX$, $AX$, and $BY$ are Euclidean distances, i.e. the length of Euclidean line segments.

Note that the definition of $d(A, B)$ appears to require that we put $X$ and $Y$ in a certain order with respect to $A$ and $B$, but part (b) (or the same argument as used in part (b)) shows that it doesn’t matter.

With $A$, $B$ any points as described above, show that

(a) $d(A, B) \geq 0$,

(b) $d(A, B) = d(B, A)$ (please assume for this problem that $X$ and $Y$ are fixed even when we switch $A$ and $B$).

**Solution**

(a) Absolute value of any quantity will be $\geq 0$.

(b) We leave $X$ and $Y$ in the same order for both $A$ and $B$ so we need the absolute values. Then

$$d(A, B) = \left| \ln \left( \frac{AY \cdot BX}{AX \cdot BY} \right) \right| \text{ definition}$$

$$= -\ln \left( \frac{AX \cdot BY}{AY \cdot BX} \right) \text{ taking reciprocal and using properties of log}$$

$$= \ln \left( \frac{AX \cdot BY}{AY \cdot BX} \right) \text{ absolute values cancel negative}$$

$$= d(B, A)$$
3. Fill in the gaps in proof posted online that every right triangle in Hyperbolic Geometry contains a defective sub-triangle.

4. Extra Credit: Here is what you should have had for HW 5#1.

**Lemma**

If $C_1$ is a circle with radius $a$, $C_2$ a circle with radius $b$, and if $c$ is the distance between the centers then

$$C_1 \cap C_2 = \begin{cases} 
0 \text{ points} & \text{if } c < |a-b| \text{ or } c > a+b \\
1 \text{ point} & \text{if } 0 < c = |a-b| \text{ or } c = a+b \\
2 \text{ points} & \text{if } |a-b| < c < a+b 
\end{cases}$$

To see these six conditions represented look at Figure 1. Most people missed some of these conditions, and no one stated it exactly as I have here, which is fine, because I've tried to make this statement as elegant as possible. But we can make it even more compact, although perhaps a little less clear. Find an elementary formula $D$ (i.e. one that does not directly involve if-thens or piecewise definitions) such that

$$C_1 \cap C_2 = \begin{cases} 
0 \text{ points} & \text{if } D \text{ (fill in) } \\
1 \text{ point} & \text{if } D \text{ (fill in) } \\
2 \text{ points} & \text{if } D \text{ (fill in) }
\end{cases}$$

**Solution**

The conditions can be given in terms of a single number/formula. Let

$$D = (a - |b - c|)(b + c - a).$$

Then

$$C_1 \cap C_2 = \begin{cases} 
0 \text{ points} & \text{if } D < 0 \\
1 \text{ point} & \text{if } D = 0 \\
2 \text{ points} & \text{if } D > 0 
\end{cases}$$

Figure 1: Six cases of circles intersecting
Note that the equivalence of these conditions with the original ones given above requires $b + c > 0$. 