1. Let $C_1$ be a circle with radius $a$, $C_2$ a circle with radius $b$, and let the centers of $C_1$ and $C_2$ have a distance of $c$ apart.

Figure out and state a lemma that gives necessary and sufficient conditions in terms of $a$, $b$, and $c$, for when $C_1 \cap C_2$ equals 0 points, 1 point and 2 points. In other words you want to be able to say: The intersection $C_1$ and $C_2$ has 0 points if and only if ____________________________, it has exactly 1 point if and only if ____________________________, it has 2 points if and only if ____________________________.

**Solution**

$$C_1 \cap C_2 = \begin{cases} 
0 \text{ points} & \text{if } c < |a-b| \text{ or } c > a+b \\
1 \text{ point} & \text{if } c = |a-b| \text{ or } c = a+b \\
2 \text{ points} & \text{if } |a-b| < c < a+b 
\end{cases}$$

We offer a few words to show that our cases are well-defined. Note that $|a-b| \leq a+b$. This means that $|a-b|$ and $a+b$ break the positive real number line into three intervals

$$0 \leq c < |a-b|, \quad c = |a-b|, \quad |a-b| < c < a+b, \quad c = a+b, \quad c > a+b.$$ 

In other words: there are only 5 relations that $c$ can have with $|a-b|$ and $a+b$ and we have included them all.

To see why these five conditions matter we look at six pictures. Here are three where the center of one circle is contained in the interior of the other circle:

Here are three pictures where each center is in the exterior of the other circle

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1Conditions that you can use as one side of and if-and-only-if statement
EXTRA: let’s prove that the circles do indeed intersect as described.

We stated the result above in the most symmetric way possible, with $c$ compared either to $|a-b|$ or to $a+b$. But it seems as if the proof is easier to tackle by focusing on a variable $x$ that plays the role of $a$. So we rewrite the conditions above, and in any case it’s useful to see how they can be combined.

$$c < |a-b| \text{ or } c > a+b \iff c < a-b \text{ or } c < b-a \text{ or } c > a+b$$
$$\iff b+c < a \text{ or } a+c < b \text{ or } c > a+b$$
$$\iff b+c < a \text{ or } a < b-c \text{ or } a < c-b$$
$$\iff b+c < a \text{ or } a < |b-c|$$

Using basically the same calculations we have

$$c = |a-b| \text{ or } c = a+b \iff a = |b-c| \text{ or } a = b+c$$
$$c > |a-b| \text{ and } c < a+b \iff a > |b-c| \text{ and } a < b+c$$

We will prove below the conditions involving $a$ rather than $c$.

Finally, before we start in earnest, we note the following: we were asked to prove if-and-only-ifs, but we do not appear to do so below. The key is that our five conditions given on $c$, or on $a$, are comprehensive and mutually exclusive. In other words, every possible combination of $a$, $b$ and $c$ fits exactly one of our five conditions. Keeping this in mind, suppose we want to prove “If $C_1 \cap C_2 = \emptyset$ then $a < |b-c|$ or $a > b+c$”. We can prove this by contrapositive and get “If $a \geq |b-c|$ and $a \leq b+c$ then $C_1 \cap C_2$ has one or two points.” But we will prove this last if-then as part of our case-by-case conditions on $a$, $b$ and $c$. In other words, every proof we give will be of the form “if (conditions on $a$, $b$ and $c$) then (conditions on $C_1 \cap C_2$)” but this is logically equivalent under these conditions to also giving proofs of the form “If (conditions on $C_1 \cap C_2$) then (conditions on $a$, $b$ and $c$).”

We finally start our proof.

Let $O_1$ be the center of $C_1$ and $O_2$ the center of $C_2$. Let $P$ be any point in $C_2$ and let $x$ be the distance from $O_1$ to $P$. Then the points $O_1$, $O_2$, and $P$ either form a triangle or are collinear and form a line segment. Applying the triangle inequality gives three inequalities; addition of lengths of line segments gives three equalities:

<table>
<thead>
<tr>
<th>One triangle</th>
<th>line segments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; b+c$</td>
<td>$x = b+c$</td>
</tr>
<tr>
<td>$b &lt; x+c$</td>
<td>$b = x+c$</td>
</tr>
<tr>
<td>$c &lt; b+x$</td>
<td>$c = b+x$</td>
</tr>
</tbody>
</table>

Using similar calculations to above (when we switched the role of $c$ and $a$) we can combine these to get

$$|b-c| \leq x \leq b+c$$
Note that equality holds if and only if we have collinear points. As indicated above, can be more specific with how the points are arranged:

\[ x = |b - c| \iff P \in O_2 O_1 \]
\[ x = b + c \iff P \in \overrightarrow{O_1 O_2} \text{ with } O_2 \text{ in the middle of } O_1 \text{ and } P \]

Case 1: \( a < |b - c| \). Then \( a < |b - c| \leq x \), so \( a < x \). This shows that \( P \) is in the interior of \( \mathcal{G}_1 \).
Therefore \( \mathcal{G}_2 \subseteq \text{int}(\mathcal{G}_1) \), so \( \mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset \).

Case 2: \( a > b + c \). Then \( a > b + c \geq x \), so \( a > x \). This shows that \( P \) is in the exterior of \( \mathcal{G}_1 \).
Therefore \( \mathcal{G}_2 \subseteq \text{ext}(\mathcal{G}_1) \), so \( \mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset \).

Case 3: \( a = |b - c| \). Then \( a = |b - c| \leq x \). Furthermore, equality holds if and only if \( P \in \overrightarrow{O_2 O_1} \).
Therefore, there is a unique point \( P \) that satisfies \( x = a \) and so \( P \in \mathcal{G}_1 \cap \mathcal{G}_2 \). Therefore, \( \mathcal{G}_1 \cap \mathcal{G}_2 \) has exactly one point.

Case 4: \( a = b + c \). Then \( a = b + c \geq x \). Furthermore, equality holds if and only if \( P \in \overrightarrow{O_1 O_2} \) with \( P \) past \( O_2 \). Therefore, there is a unique point \( P \) that satisfies \( x = a \) and so \( P \in \mathcal{G}_1 \cap \mathcal{G}_2 \).
Therefore, \( \mathcal{G}_1 \cap \mathcal{G}_2 \) has exactly one point.

Case 5: \( |b - c| < a < b + c \). If \( P \in \overrightarrow{O_1 O_2} \) with \( P \) past \( O_2 \) then we have \( x = b + c \) and so \( x > a \) and \( P \in \text{ext}(\mathcal{G}_1) \).
If \( P \in \overrightarrow{O_2 O_1} \) then we have \( x = |b - c| \) and so \( x < a \) and \( P \in \text{int}(\mathcal{G}_1) \).
Therefore, \( \mathcal{G}_2 \) contains points both interior and exterior to \( \mathcal{G}_1 \). Therefore \( \mathcal{G}_1 \cap \mathcal{G}_2 \) has two points.

2. (Based on Kinsey, Moore and Prassidis, 2.6, p. 19.)

We saw earlier that Euclid had a hidden assumption in Proposition 1, the construction of an equilateral triangle, namely that the two circles do in fact intersect. The way to fill that hidden assumption is simply to make it a clearly stated, additional postulate, before the proof begins. On the one hand, this may feel like cheating: “I skipped a step and now I’ll just add that step as universal assumption.” But it’s also both honest and unavoidable: there has to be a list of starting assumptions that we can’t prove, and we should honestly state what those are, and work as hard as possible to make them as few as possible.

In any case, here’s another example: a hidden assumption of Euclid’s, and an honest statement of that property.

**Definition.** (a) Given two points \( A \) and \( B \), the line segment from \( A \) to \( B \) is denoted by \( \overline{AB} \).

(b) Let \( \mathbb{E} \) be the Euclidean plane, i.e. the set of all points, lines, circles, etc., in the geometric plane we are studying.

(c) A set \( S \subseteq \mathbb{E} \) is **convex** if the following holds: for all \( A, B \in S \) we have \( \overline{AB} \subseteq S \).

(d) For any set \( S \subseteq \mathbb{E} \) the complement \( S^c \) is is the set of all points in \( \mathbb{E} \) that are not in \( S \).

**Postulate** (Plane Separation Axiom). If \( \ell \) is any line in \( \mathbb{E} \) we have, then \( \ell^c \) can be written as \( \ell^c = S_1 \cup S_2 \) where \( S_1 \) and \( S_2 \) are nonempty, disjoint and convex. Furthermore, if \( A \in S_1 \) and \( B \in S_2 \) then \( \overline{AB} \) intersects \( \ell \).

**Prove.** Let \( \ell \) be a line, let \( S_1 \) and \( S_2 \) be as in the postulate, and let \( A, B \in \ell^c \) and assume \( A \in S_1 \). Show that \( B \in S_1 \) if and only if \( \overline{AB} \) is disjoint from \( \ell \).

For later use we define the following common use of language: We call \( S_1 \) and \( S_2 \) the **sides** of the line \( \ell \). If \( A \in S_1 \) we can call \( S_1 \) the **A-side of \( \ell \)**. If \( A, B \in S_1 \) we say they are on the same side of \( \ell \). If \( A \in S_1 \) and \( B \in S_2 \) we say they are on opposite sides of \( \ell \).

**Solution**

**Proof.** “\( \Rightarrow \)” We prove that if \( B \in S_1 \), then \( \overline{AB} \cap \ell = \emptyset \). Let \( B \in S_1 \). Then \( A, B \in S_1 \). By Postulate Plane Separation Axiom, \( S_1 \) is convex, so we have that \( \overline{AB} \subseteq S_1 \). Since \( S_1 \subseteq \ell^c \) this shows that \( \overline{AB} \subseteq \ell^c \). This means that \( \overline{AB} \cap \ell = \emptyset \).

“\( \Leftarrow \)” We prove that if \( \overline{AB} \cap \ell = \emptyset \) then \( B \in S_1 \). Suppose that \( \overline{AB} \cap \ell = \emptyset \). If \( B \in S_2 \), then by the Plane Separation Axiom, \( \overline{AB} \cap \ell \neq \emptyset \), so we conclude \( B \notin S_2 \). Therefore \( B \in S_1 \). \( \square \)

In grading this, I wanted to see two things very explicitly: (i) A clear structure of two differ-
3. (Based on Kinsey, Moore and Prassidis, 2.9, p. 19.)

Use the idea of plane separation to define interior and exterior of an angle. (Your definition has to capture what we all “know” interior and exterior mean, but it can use only the language of sides of a line.)

Solution

We define interior and exterior of an angle.

Let \( \angle ABC \) be given. Let \( S_1 \) be the side of the line \( AB \) that contains \( C \) and let \( S_2 \) be the side of the line \( BC \) that contains \( A \). The interior of \( \angle ABC \) is given by \( S_1 \cap S_2 \). The exterior of \( \angle ABC \) is given by those points in \( E \) not on \( BA \), or \( BC \) or in the interior. (Another way to define the exterior is to take the union of the other sides of the lines, i.e. the side of \( AB \) that does not contain \( C \) union with the side of \( BC \) that does not contain \( A \).)

4. Prove that in our proof of the Weak Exterior Angle Theorem, that point \( F \) was in the interior of \( \angle ABC \).

Solution

This should have been “interior of \( \angle ACD \)” but I’ll stick with the typo.

The main steps are: (1) Show that \( E \) is in the interior of the angle \( \angle ACD \), (2) Show that \( F \) is on the same side of \( AB \) as \( E \), and on the same side of \( BC \) as \( E \).

Actually, these two statements generalize very easily, and maybe the generalization helps reveal what is important. Claim 1: If \( X \in \overrightarrow{BA} \) and \( Y \in \overrightarrow{BC} \) and \( Z \in \overrightarrow{XY} \) with \( Z \neq X, Y \), then \( Z \) is in the interior of \( \angle ABC \). Claim 2: If \( X \in \text{int} \angle ABC \) and \( Y \in \overrightarrow{BX} \) with \( Y \neq B \), then \( Y \in \text{int} \angle ABC \).

We prove now that \( E \in \text{int} \angle ABC \). In particular, we claim that \( E \) is on the \( A \)-side of \( BC \) and on the the \( C \)-side of \( AB \). The two statements are mirror images of each other, so we omit the second one.

Since \( E \in \overrightarrow{AC} \), and \( E \neq A, C \) we have \( E \) is between \( A \) and \( C \). Therefore, \( C \) is not between \( A \) and \( E \). Since \( AE \cap BC = C \), and \( C \notin \overrightarrow{AE} \) we have \( \overrightarrow{AE} \cap BC = \emptyset \). Therefore \( E \) is on the \( A \)-side of \( BC \).

We prove now that \( F \) is on the same side of \( AB \) as \( E \), and on the same side of \( BC \) as \( E \). In particular, we claim that \( E \) is on the \( A \)-side of \( BC \) and on the the \( C \)-side of \( AB \). The two statements are mirror images of each other, so we omit the second one.

Since \( E \in \overrightarrow{BF} \), and \( E \neq B, F \) we have \( E \) is between \( B \) and \( F \). Therefore, \( B \) is not between \( F \) and \( E \). Since \( BF \cap BC = B \), and \( C \notin \overrightarrow{EF} \) we have \( \overrightarrow{EF} \cap BC = \emptyset \). Therefore \( F \) is on the same side of \( BC \) as \( E \).

5. (Based on Kinsey, Moore and Prassidis, 2.18, p. 24.) We define the interior of a triangle as the intersection of the interiors of the three angles. In other words

\[
\text{int} \triangle ABC = ( \text{int} \angle ABC ) \cap ( \text{int} \angle BCA ) \cap ( \text{int} \angle CAB ).
\]

Of course, you’ve defined \( \text{int} \angle ABC \), in problem 3.

Prove that the interior of any triangle is convex.
Solution

We show that every triangle has a convex interior.

Let $\triangle ABC$ be given, and let $D, E$ be interior to the triangle. Let $S_1$ be the side of $BC$ that contains $A$, let $S_2$ be the side of $AC$ that contains $B$ and let $S_3$ be the side of $AB$ that contains $C$. Then the interior is $S_1 \cap S_2 \cap S_3$. Then for each $i \in \{1, 2, 3\}$ we have $D, E \in S_i$, and so $DE \subseteq S_i$, since $S_i$ is convex. Thus, $DE \subseteq S_1 \cap S_2 \cap S_3$. 