1. In class, and in the notes, we showed that it is possible to find, through Origami Operations, one intersection of a circle and a line. Now show that when the line intersects the circle twice it is possible to find both intersections.

Solution

We are given a point $A$, a point $B$ and a point $B'$. We need to construct another point $B''$ that is also on the line $\ell$ and on the circle with center $A$.

Apply OO4 to the point $A$ and the line $\ell$. This creates a fold through $A$ that is perpendicular to $\ell$. Let $B''$ be point that $B'$ moves two when we apply this fold. Since the fold goes through $A$, we have that $A$ doesn’t move with the fold, and so $AB' = AB''$, which shows that $B''$ is on the circle. Since the fold is perpendicular to the line, we have that all points on the line $\ell$ are taken to points also on the line by the fold. Thus $B''$ is on the line $\ell$. Note that $B' \neq B''$ (unless $B'$ is on Fold 2, in which case there is only one intersection).

(Problems 2–5 come from College Geometry by Howard Eves, 0.3#9) As an example of axiomatic reasoning, consider the following set-up. A certain school has a set $S$ of students. Certain subsets of $S$ are called clubs. Instead of saying “a student is an element of a certain subset” we use the more familiar phrase “a student is a member of a club”. We will not attempt to define students or clubs, but we will assume the following properties (“Club Axioms”) about them:

CA0 $S$ is nonempty and finite, and every club is nonempty.

CA1 Every student in $S$ is a member of at least one club.

CA2 For every pair of students there exists a unique club to which they both belong.

CA3 For every club there exists a unique club that has no students in common with the first.

Prove the following propositions about these clubs. Justify every logical assertion by one of the “Club Axioms”. (Be extremely rigorous, make no assumptions aside from what’s given above. I would recommend drawing pictures to represent students and clubs. Test your understanding of the problems by drawing different pictures and asking true and false questions about your pictures and the problems you’re trying to prove.)

2. Every student in $S$ is a member of at least two clubs.

Solution
Let $x$ be any student. Then CA1 implies that $x$ is a member of a club, $X$. By CA3 there is another club $Y$ that has no members in common with $X$. Since $Y$ is nonempty (by definition of a club), there is at least one member $y \in Y$. By CA2 there is a club $Z$ containing both $x$ and $y$. Then $x$ is in two clubs: $X$ and $Z$ (note that $Z \neq X$ since $y \notin X$).

3. Every club contains at least two members.

**Solution**

Let $X$ be any club. Since $X$ is nonempty (by definition of a club), there exists at least one member $x \in X$. By CA3 there exists a club $Y$ that does not intersect $X$. Since $Y$ is nonempty there exists a member $y \in Y$. By CA2 there exists a club $Z$ that contains $x$ and $y$. By CA3 there exists a club $W$ that does not intersect $Z$. By the “unique” part of CA3, $Y$ is the only club that does not intersect $X$. Therefore, $W$ does intersect $X$. Let $w \in X \cap W$. Then $X$ has two students $x$ and $w$.

4. The set $S$ contains at least four students.

**Solution**

Since $S$ is nonempty, there exists at least one student. Applying CA1 there exists at least one club, $X$. Applying CA3, there exists a second club $Y$ that has no students in common with $X$. Then by part (b), each of $X$ and $Y$ have at least two students, so there are at least four students in $X$ and $Y$ together.

5. There are at least 6 different clubs.

**Solution**


FALSE Proof: By problem 3 there are at least four distinct students: \(x, y, z, w\). These four students form 6 distinct pairs: \(x\&y, x\&z, x\&w, y\&z, y\&w, z\&w\). By CA2 each distinct pair defines a distinct club. Since we had 6 distinct pairs, this produces 6 distinct clubs.

Comments: This is false proof because we don’t know that the clubs produced by different pairs are distinct. Suppose all the students are in one club. Then this satisfies CA1 and CA2. So CA2 does not produce 6 different clubs. I think any proof of problem 4 has to use not just the conclusions of problems 1–3, but some of the steps in the proofs of them (or repeat some of the steps).

Real proof: Let \(x\) be any student. As in the proof of problems 1 and 2, we know that there exists students \(y\) and \(w\), and clubs \(X, Y, Z\) and \(W\) such that \(x \in X, y \in Y, w \in W, Y \cap X = \emptyset, W \cap Z = \emptyset, X \cap Z = x, X \cap W = w\). Since \(W \cap Z = \emptyset\), we know that \(Z\) is the unique club that does not intersect \(W\). Then \(W\) must intersect \(Y\). Let \(W \cap Y = u\). Then by CA2 there is a club \(U\) that contains \(x\) and \(u\), and a club \(V\) that contains \(y\) and \(w\).

It is easy to believe that all the clubs are distinct, but here the details. We already knew that \(x, y, w, X, Y, Z, W\) were distinct (as in the proofs of problems 1 and 2). Now note that \(u \notin X\), since \(u \in Y\) and \(x \notin Y\). Note that \(u \neq y\) since \(u \in W\) and \(y \notin Y\). Note that \(u \neq w\) since \(u \in Y\) and \(w \notin Y\). Since \(u \neq y\) we have \(U \neq Y, Z\). Since \(u \neq w\) we have \(U \neq X, W\). Since \(w \neq x\) we have that \(V \neq X, Z\). Since \(w \neq u\) we have that \(V \neq Y, W\).

Challenge. Eves goes on to say the following: “The persevering student may care to try to establish the following much more difficult theorem.” For a challenge, for extra credit, see if you can prove the following:

Theorem. No club contains more than two members.