1. In class I claimed that if two lines
\[ ax + by = e, \]
\[ cx + dy = f \]
intersect, then their intersection is given by
\[ x = \frac{1}{ad-bc} (de-bf), \]
\[ y = \frac{1}{ad-bc} (af-ce). \]

(a) Find the general solution for intersecting the following line and circle:
\[ ax + by = e, \]
\[ (x-c)^2 + (y-d)^2 = f^2. \]

Justify your steps as appropriate.

Solution

Solving the first equation for \( y \) we get
\[ y = -\frac{a}{b}x + e. \]

Substituting this into the second equation we get
\[ (x-c)^2 + (-\frac{a}{b}x + e - d)^2 = f^2, \]
\[ x^2 - 2cx + c^2 + \frac{a^2x^2}{b^2} + \frac{2adx}{b} + d^2 - \frac{2ae}{b} + e^2 = f^2. \]

Clearing denominators and gather coefficients we get
\[ (a^2 + b^2)x^2 + (2abd - 2b^2c - 2abe)x + b^2c^2 + e^2 - 2bde + b^2d^2 - b^2f^2 = 0. \]

This is a quadratic, and for convenience we relabel all the coefficients:
\[ A = a^2 + b^2, \]
\[ B = 2abd - 2b^2c - 2abe, \]
\[ C = b^2c^2 + e^2 - 2bde + b^2d^2 - b^2f^2, \]
\[ Ax^2 + Bx + C = 0. \]

Then we have a solution
\[ x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \]
\[ = \frac{-abd + b^2c + ae \pm \sqrt{a^2b^2c^2 + a^2b^2f^2 - 2ab^3cd - b^4d^2 + b^4f^2 + 2ab^2ce + 2b^3de - b^2e^2}}{a^2 + b^2}. \]

With work, we can make this solution seem more elegant
\[ x = \frac{ae - b(ad - bc) \pm |b| \sqrt{(a^2 + b^2)f^2 - (ac + bd - e)^2}}{a^2 + b^2}. \]

Once we calculate \( x \), then \( y \) is given as above, \( y = -\frac{a}{b}x + e. \)
(b) Find the general solution for intersecting the following two circles:

\[(x - a)^2 + (y - b)^2 = e^2,\]
\[(x - c)^2 + (y - d)^2 = f^2.\]

Justify your steps as appropriate.

**Solution**

If we subtract the second equation from the first, and expand, we get a linear equation:

\[2(c - a)x + 2(d - b)y = c^2 + d^2 + e^2 - a^2 - b^2 - f^2.\]

Any \((x, y)\) point that satisfied both the original equations also satisfies this one. Therefore, the points of intersection of the original circles satisfy this linear equation. Therefore, this equation is the equation of the line through the two points. Therefore, all we have to do is find the intersection of this line with either of the original circles. But this is what we did in part (a).

It’s a simple matter of notation now:

\[\tilde{a} = 2(c - a),\]
\[\tilde{b} = 2(d - b),\]
\[\tilde{e} = c^2 + d^2 + e^2 - a^2 - b^2 - f^2,\]
\[\tilde{a}x + \tilde{b}y = \tilde{e},\]
\[(x - c)^2 + (y - d)^2 = f^2,\]
\[x = \frac{\tilde{a} - \tilde{b}(\tilde{a}d - \tilde{b}c) \pm |\tilde{b}| \sqrt{(\tilde{a}^2 + \tilde{b}^2)f^2 - (\tilde{a}c + \tilde{b}d - \tilde{e})^2}}{\tilde{a}^2 + \tilde{b}^2},\]
\[y = -\frac{\tilde{a}}{\tilde{b}}x + \tilde{e}.\]

2. Use Gauss’s Theorem to list all constructible regular \(n\)-gons, with \(3 \leq n \leq 100\). Show enough work to justify your answer. Also, think about organizing your work so that if I ask, e.g., for you to list a different range of constructible \(n\) you have a nice simple approach for creating the list.

**Solution**

Let \(3 \leq n \leq 100\). By Gauss’s Theorem, and the list of Fermat primes, we have that the regular \(n\)-gon is constructible if and only if \(n\) can be written as a

\[n = 2^k 3^a 5^b 17^c\]

where \(k = 0, 1, 2, \ldots\), and each of \(a, b\) and \(c\) can be either 0 or 1.

To put it differently, we can take any product of an element listed below on the left, with an element listed on the right, as long as the result \(n\) satisfies \(3 \leq n \leq 100\).

\[
\begin{align*}
&\{1, 2, 4, 8, 16\} \times \{1, 3, 5, 15\} \\
&\{32, 64\} \times \{17, 51, 85\}
\end{align*}
\]

There are two obvious ways to organize your calculations: start with one thing on the left, and
combine it with everything on the right, or vice versa. We do both:

1 × stuff on right : \( n = 3, 5, 15, 17, 51, 85, \)
2 × stuff on right : \( n = 6, 10, 30, 34, \)
4 × stuff on right : \( n = 4, 12, 20, 60, 68, \)
8 × stuff on right : \( n = 8, 24, 40, \)
16 × stuff on right : \( n = 16, 48, 80, \)
32 × stuff on right : \( n = 32, 96, \)
64 × stuff on right : \( n = 64. \)

or

stuff on left ⇨ 1: \( n = 4, 8, 16, 32, 64, \)
stuff on left ⇨ 3: \( n = 3, 6, 12, 24, 48, 96, \)
stuff on left ⇨ 5: \( n = 5, 10, 20, 40, 80, \)
stuff on left ⇨ 15 : \( n = 15, 30, 60, \)
stuff on left ⇨ 17 : \( n = 17, 34, 68, \)
stuff on left ⇨ 51 : \( n = 51, \)
stuff on left ⇨ 85 : \( n = 85. \)

Sorting them in order we have, 24 total:

3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96.

3. Use Origami operations to trisect the angle on the following page. (Actually print the page out, cut out the square, fold it up, and mark the final results in pencil.)

4. Prove that the operations used in our origami construction of a trisection actually work, i.e. that the triangles \( \triangle BD'F', \triangle BB'F' \) and \( \triangle BB'J \) are all congruent.

(Hint: What I did was to prove that \( \triangle BD'F' \cong \triangle BB'F' \) using SAS, and that \( \triangle BB'F' \cong \triangle BB'J \) using SSS.)

**Solution**

The fold in step 3 takes the relation \( DF = BF \) to \( D'F' = B'F' \). The same fold takes the the relation \( FH \perp BD \) to \( F'H \perp B'D' \). Finally, we have \( BF = B'F' \). Thus, SAS shows \( \triangle BD'F' \cong \triangle BB'F' \).

Now, \( BF = BJ \) since both of these are perpendicular segments between the parallels \( FG \) and \( BC \). Applying the fold in step 3 shows \( B'F' = BF = BJ \). Now, \( BB' = BB' \). We apply the Pythagorean Theorem to the two right triangles \( \triangle BB'F' \) and \( \triangle BB'J \) to conclude that \( BF = BJ \). Then SSS implies that \( \triangle BB'F' \cong \triangle BB'J \).

Since the three triangles \( \triangle BD'F', \triangle BB'F' \) and \( \triangle BB'J \) are congruent, the angle \( \angle ABC \) has been divided into three equal angles (namely \( \angle DB'F', \angle F'BB' \) and \( \angle B'BJ \)).