Homework 2  
Due January 29, 2020

1. Look up and prepare a complete solution for constructing a regular pentagon. Do it using real ruler and compass steps, with real ruler and compass drawings on your paper. Explain every step.

As far as justification that your construction is correct, I think I'll need to be very generous on how well you justify your answer. It would be nice if you at least try to explain why the 5 sides are equal, and I won't expect more than an attempt at this. But for textbook perfection one would need more than just 5 equal sides, you would need to show that the angles inside are correct, i.e. that the interior angles of the pentagon are 108°. Equivalently you can prove that one side of the pentagon makes a central angle with the center of the circle of 72°. The way I did this was to show that \( \cos(\alpha) = \frac{\sqrt{5} - 1}{4} \) where \( \alpha \) is the central angle I've constructed. But even this takes some work. So, like I said, at least try to explain why the 5 sides are equal and I'll let it go at that.
Goal: Construct a regular pentagon using only straight edge and compass construction.

We follow the Richmond solution. The main step is to construct one side of the pentagon along the circumference of a circle: this segment is then copied around the circle. So, assuming that the angle made from the center of the circle to this segment is $72^\circ$ (which we prove below), then the rest of the construction will be automatic.

The first side is constructed using steps pictured and explained below:

Distilled version of steps: Draw any circle and a diameter in that circle, and let point $A$ be an endpoint of the diameter; point $B$ is the midpoint of the perpendicular radius; point $C$ is determined by bisecting the angle made at $B$; point $D$ is determined by erecting a perpendicular at $C$.

On the next few pages I show hand drawn steps of this solution, including the helper lines.
A circle and its diameter

Intersecting arcs with centers at endpoints, so we can erect a perpendicular

Connect the midpoint of the top radius to the end of the diameter

To prepare for angle bisection, mark two equal distances on diameters

From the points marked in the last step, draw two intersecting arcs

Using the arcs just drawn, draw the line that bisects the angle, until it hits the horizontal diameter

To prepare for erecting a perpendicular, mark two equal distances from last point (one is right end of diameter)
From the points marked in the last step, draw two intersecting arcs.

Using the arcs just drawn, erect a perpendicular.

Connect the point where the perpendicular hits the circle, with the right end of the diameter.

Copy the length of the last line segment and mark it off all around the circle.

Connect all the lengths we have marked off.

Erase most intermediate steps.
It remains to prove that this solution is correct, which is to say that all the interior angles are equal. One way to do this is to show that the angle from the center of the circle to the constructed line segment is 72° (i.e. 360°/5). For convenience assume that the circle has unit radius. Let α and β be angles, and h a side as labeled below:

![Diagram](image)

It suffices to show that α = 72°, which means we can show that cos(α) = cos(72°).

Applying the Pythagorean Theorem, it’s easy to show that $h = \sqrt{5}/2$. Therefore, sin(β) and cos(β) are known. Applying a half-angle identity we have

$$\tan(\beta/2) = \frac{1 - \cos(\beta)}{\sin(\beta)}$$

$$= \frac{1 - \frac{1/2}{\sqrt{5}/2}}{\frac{1}{\sqrt{5}/2}}$$

$$= \frac{\sqrt{5} - 1}{2}.$$

On the other hand, since we bisected angle β, we know β/2 is in a right triangle and has opposite side equal to cos(α), and so $\tan(\beta/2) = \frac{\cos(\alpha)}{1/2} = 2\cos(\alpha)$. Therefore

$$2\cos(\alpha) = \frac{\sqrt{5} - 1}{2}$$

and so

$$\cos(\alpha) = \frac{\sqrt{5} - 1}{4}.$$

According to Wikipedia, $\cos(72°) = \frac{\sqrt{5} - 1}{4}$, and so we are done.
However, the interested reader might want to know why \( \cos(72^\circ) = \frac{\sqrt{5} - 1}{4} \). Here's one derivation. Start with the following trig identity:

\[
\sin(5\theta) = 5 \sin(\theta) - 20 \sin^3(\theta) + 16 \sin^5(\theta).
\]

With \( \theta = 36 \) we get

\[5 \sin(36) - 20 \sin^3(36) + 16 \sin^5(36) = 0.\]

Factoring out \( \sin(36) \), and setting \( x = \sin(36) \) we get a hidden quadratic in \( x^2 \):

\[16x^4 - 20x^2 + 5 = 0.\]

The quadratic formula gives

\[
x^2 = \frac{1}{8}(5 \pm \sqrt{5})
\]

\[
x = \pm \sqrt{\frac{1}{8}(5 \pm \sqrt{5})}
\]

Since \( \sin(36) \) is positive the first “±” will be the positive root. Since \( 36 < 45 \) we must have \( \sin(36) < \sin(45) = 1/\sqrt{2} \). The solution that satisfies this condition has the second “±” given by “−” and so

\[
\sin(36) = \sqrt{\frac{5 - \sqrt{5}}{8}} = \frac{1}{4} \sqrt{10 - 2\sqrt{5}}
\]

From this get

\[
\cos(36) = \frac{1}{4}(-1 + \sqrt{5})
\]

and then use a double angle identity to get

\[
\cos(72) = \frac{1}{4}(\sqrt{5} - 1).
\]
2. Prove that the string construction we did in class actually creates an ellipse.

To be more specific, suppose we have points \((-1, 0)\) and \((1, 0)\) for the foci, and we use a string of length 4 with the ends of the string fixed on the foci. Suppose we catch a pencil against the string and then pull the pencil so that the string gets tight, and let \((x, y)\) be the coordinates of the pencil tip.

Prove that \((x, y)\) satisfy the following equation of an ellipse:

\[
\frac{x^2}{4} + \frac{y^2}{3} = 1.
\]

Hint: The two segments of string represent two distances between points, and the sum of these distances equals 4.

Solution

Let \(A\) and \(B\) be the foci and let \(C\) be the point \((x, y)\). Since the string has length 4 and since it is pulled tight we have

\[
\sqrt{(x + 1)^2 + y^2} + \sqrt{(x - 1)^2 + y^2} = 4,
\]

\[
\left(\sqrt{(x + 1)^2 + y^2}\right)^2 = \left(4 - \sqrt{(x - 1)^2 + y^2}\right)^2
\]

\[
(x + 1)^2 + y^2 = 16 - 8\sqrt{(x - 1)^2 + y^2} + (x - 1)^2 + y^2,
\]

\[
x^2 + 2x + 1 + y^2 - x^2 + 2x - 1 - y^2 - 16 = 8\sqrt{(x - 1)^2 + y^2},
\]

\[
4x - 16 = 8\sqrt{(x - 1)^2 + y^2},
\]

\[
(x - 4)^2 = \left(2\sqrt{(x - 1)^2 + y^2}\right)^2
\]

\[
x^2 - 8x + 16 = 4x^2 - 8x + 4 + 4y^2,
\]

\[
12 = 3x^2 + 4y^2,
\]

\[
1 = \frac{x^2}{4} + \frac{y^2}{3}.
\]

3. Show that string construction we did in class actually creates a parabola.

To be more specific, suppose the horizontal line \(\ell\) we used was the \(x\)-axis and the focus was the point \(P = (0, 1)\).

(a) Let \((x, y)\) be any point that is equal distant from \(\ell\) and \(P\). Show that this point is on the parabola \(y = \frac{1}{2}x^2 + \frac{1}{2}\).

(b) We have a right angle tool that is exactly 4 high, and we have a length of string that is also exactly 4 high. We fix one end of the string on the focus, and one end at the tip of the vertical edge of a right angle tool. We put the horizontal edge of the right angle tool on \(\ell\). Suppose we catch a pencil against the string and then pull the pencil so that the string gets tight, and at the same time the pencil is against the vertical edge of the right angle tool. Two different positions are shown below.
Let \((x, y)\) be the coordinates of the pencil tip. Show that this point \((x, y)\) is on the parabola 
\[ y = \frac{1}{2}x^2 + \frac{1}{2}. \]
Hint: turn this problem back into part (a).

**Solution**

(a) Let \(P\) be the focus and let \(A\) be the point \((x, y)\).

\[
\begin{align*}
\overline{AP} &= \text{dist}(A, \ell) \\
\sqrt{x^2 + (y - 1)^2} &= y \\
x^2 + (y - 1)^2 &= y^2 \\
x^2 + y^2 - 2y + 1 &= y^2 \\
2y &= x^2 + 1 \\
y &= \frac{1}{2}x^2 + \frac{1}{2}
\end{align*}
\]

(b) Note that

\[
\overline{AP} = \begin{cases} 
\text{amount of string pulled} \\
\text{off the } L\text{-shape} 
\end{cases} \\
= 4 - \begin{cases} 
\text{amount of string still} \\
\text{touching } L\text{-shape} 
\end{cases} \\
= \text{dist}(A, \ell).
\]

Now apply part (a).

4. Assume that we have a segment of unit length, and segments of length \(a\) and \(b\). Find an S&C construction for \(a/b\) and prove that it is correct.

**Solution**

Start with \(BC\) of length \(b\). Using Prop I.11, erect a perpendicular through \(C\). Using SC3, draw a circle with center \(C\) and radius 1. Using SC4, let \(A\) be the intersection of this circle and the perpendicular. Using SC1, draw \(AB\).

\[
\text{Now we let } EF\text{ be a line segment with length } a. \text{ Using Prop I.11, erect a perpendicular through } F. \text{ Using Prop I.23, construct an angle with vertex } E \text{ and one side given by } EF, \text{ such that the angle is congruent to } \angle ABC. \text{ Using SC2, let } D \text{ be the intersection of the other side of the angle with the perpendicular through } F. \text{ Let } x \text{ be the length of the line segment } DF.
\]
Now the two triangles $\triangle ABC$ and $\triangle DEF$ have all the same angles, and are therefore similar. Thus, the ratios of corresponding sides are equal. So we have

\[
\frac{a}{b} = \frac{x}{1}.
\]

Thus, we have constructed a segment $\overline{DF}$ whose length equals the product $a/b$.

5. This problem attempts to track and estimate the inaccuracy in our ruler and compass construction of a square root.

Let’s assume that every time I make a mark on a piece of paper, or use my ruler to measure a length on a piece of paper, or set my compass to a distance on a piece of paper, that I’m potentially off by 0.05 inches. For instance, suppose I want to calculate $\sqrt{3}$ with the ruler and compass steps from class. Here’s how it would start:

(a) Let’s take a line of length 1 as given, and add to this a line length 3. The total should have length 4 but actual numbers will vary from 3.95 to 4.05 (I’m only using one setting on the compass to move one line: if you move them both the results would be between 3.9 and 4.1).

(b) Find the midpoint of the total line: location is 2 from left end, but actual midpoint will vary from 1.925 to 2.075. Why? The line we’ve drawn may be actually 3.95, in which case the midpoint is 1.975, but we are off by 0.05 and so mark it at 1.925. Similarly, the line we’ve drawn may be actually 4.05, in which case the midpoint is 2.025, but we are off by 0.05 and so mark it at 2.075.

(c) Set compass to radius of circle: radius may be 1.875 to 2.125 (above measurements ±0.05).

(d) The next step is to read the length of intersection of perpendicular with semicircle: the circle we draw may be off in where the center location is and it may be off in the radius, and the location of the perpendicular may be off. In each case there are two numbers that mark the upper and lower bound.

That gives rise to a total of $2 \times 2 \times 2 = 8$ combinations of inaccuracy should get plugged into an ideal circle to read off the length. The ideal circle would have form $y = \sqrt{R^2 - (X - C)^2}$ where $R$ is the radius, $X$ is where the original line segment and the unit line segment come together, and $C$ is the center of the circle. In our case, we could have something like $h = \sqrt{1.875^2 - (3.05 - 2.125)^2}$, (I’ve used a circle of radius 1.875, with center at 2.125, and plugged in the $x$-value of 3.05 to calculate the $y$-value).

(e) Now we read $h$ as the length just constructed. It should be one of the 8 numbers we just calculated, but when we read it on the ruler we may again be off by 0.05.

Applying these ideas, find upper and lower bounds for the calculation we will get starting with a line segment of length $r = 3.5$. Explain your work.
Solution

We start with $r = 3.5$ and track the inaccuracy of our construction of $\sqrt{3.5}$. The main steps are this: (a) form a line segment $DF$ with $DE$ having length $r$ and $EF$ having length 1, (b) find the midpoint of the segment just created, (c) set compass on midpoint and endpoint and draw half circle, (d) find the height of the circle at point $E$, i.e. plug $X \approx 3.5$ into equation of a circle $y = \sqrt{R^2 - (X - C)^2}$ where $R \approx 1.75$ and $C \approx 1.75$.

(a) We should have $1 + 3.5 = 4.5$, with upper and lower bounds as shown

\[4.45 \leq DF = 4.5 \leq 4.55\]

(b) Midpoint should be a distance of 2.25, upper and lower bounds as shown

\[4.45/2 - 0.05 \leq 2.25 \leq 4.55/2 + 0.05, \quad 2.175 \leq 2.25 \leq 2.325\]

(c) Radius should be 2.25, upper and lower bounds as shown

\[2.175 - 0.05 \leq 2.25 \leq 2.325 + 0.05, \quad 2.125 \leq 2.25 \leq 2.375\]

(d) We take the formula

\[y = \sqrt{R^2 - (X - C)^2}\]

and plug in a range of values as shown

\[3.45 \leq X \leq 3.55, \quad 2.175 \leq C \leq 2.325, \quad 2.125 \leq R \leq 2.375\]

We can do this “blindly”, just plugging in all the possible options:

\[\sqrt{(2.125)^2 - (3.450 - 2.175)^2} \approx 1.700, \quad \sqrt{(2.375)^2 - (3.450 - 2.175)^2} \approx 2.004,\]
\[\sqrt{(2.125)^2 - (3.450 - 2.325)^2} \approx 1.803, \quad \sqrt{(2.375)^2 - (3.450 - 2.325)^2} \approx 2.092,\]
\[\sqrt{(2.125)^2 - (3.550 - 2.175)^2} \approx 1.620, \quad \sqrt{(2.375)^2 - (3.550 - 2.175)^2} \approx 1.936,\]
\[\sqrt{(2.125)^2 - (3.550 - 2.325)^2} \approx 1.736, \quad \sqrt{(2.375)^2 - (3.550 - 2.325)^2} \approx 2.035\]

But we can also be more clever and figure out how to find the maximum value of $y$ above and how to find the minimum. The maximum should come from making $R$ as big as possible, and $(X - C)$ as small as possible, so $R = 2.375, X = 3.450, C = 2.325$. The minimum should come from making $R$ as small as possible, and $(X - C)$ as big as possible, so $R = 2.125, X = 3.550, C = 2.175$.

(e) We should have $h$ somewhere between 1.620 and 2.092. But, when we read the value on our ruler, we can add or subtract an additional 0.05:

\[1.620 - 0.05 \leq h \leq 2.092 + 0.05, \quad 1.570 \leq h \leq 2.142\]

Note that the ideal, correct value, would be $h = \sqrt{3.5} \approx 1.871$. 