Negating disjunctions and conjunctions

**Theorem 1.3.7.** Let $P$, $Q$ and $R$ be any logical statements and let “≡” stand for the phrase “have the same logical truth value”. Then we have

- $\neg(\neg P) \equiv P$ and $P \land P \equiv P$ and $P \lor P \equiv P$ (Idempotence)
- $\neg(P \land Q) \equiv (\neg P) \lor (\neg Q)$ and $\neg(P \lor Q) \equiv (\neg P) \land (\neg Q)$ (De Morgan’s Laws)
- $P \land (Q \lor R) \equiv (P \land V) \lor (P \land R)$ and $P \lor (Q \land R) \equiv (P \lor V) \land (P \lor R)$ (Distribution)

**Proof.** Some of these parts are pretty obvious (e.g.. idempotence), and some we’ll leave as exercises. But we’ll prove the second De Morgan’s law. First: this statement should make sense intuitively.

Suppose I tell you that it’s not the case that you’ll get soup or salad with dinner. You can conclude that you’ll not get soup, and you’ll not get salad. Or suppose you tell me that you’ll get a B or C in the class, and I tell you that you’re wrong. I mean that you wont get a B and you wont get a C.

Here’s the formulaic proof:

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<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \lor Q$</th>
<th>$\neg(P \lor Q)$</th>
<th>$\neg P$</th>
<th>$\neg Q$</th>
<th>$\neg(P) \land (\neg Q)$</th>
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</thead>
<tbody>
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Since the fourth column, $\neg(P \lor Q)$, and the last column, $\neg(P) \land (\neg Q)$ are identical, the two formulas are logically equivalent.

**Converse, Contrapositive and If-and-only-if**

**Definition 1.3.8.** Let $P$ and $Q$ be any statements.

1. The two statements $P \implies Q$ and $Q \implies P$ are **converses** of each other.
2. The two statements $P \implies Q$ and $\neg Q \implies \neg P$ are **contrapositives** of each other.
3. The statement $P \iff Q$ means $(P \implies Q) \land (Q \implies P)$. In other words, $P \iff Q$ is true when $P \implies Q$ and $Q \implies P$ are both true. Verbally we read “$P \iff Q$” as “$P$ if and only if $Q$”.

**Proposition 1.3.9.** Let $P$ and $Q$ be two logical statements. The following are equivalent:

1. $P \implies Q$
2. $\neg Q \implies \neg P$
3. $(\neg P) \lor Q$

**Proof.** We give a truth table for the values of “if $P$, then $Q$” and “if (not $Q$), then (not $P$)” and $(\neg P) \lor Q$

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<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \implies Q$</th>
<th>$\neg Q$</th>
<th>$\neg P$</th>
<th>$\neg P \iff \neg Q$</th>
<th>$(\neg P) \lor Q$</th>
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Since the third column and the last two columns are the same, we see that the third statement and the last two statements are logically equivalent.

**Example 3.** Consider the following open sentences where $x$ is allowed to be any real number.

$P$: If $x = 4$, then $x^2 = 16$
$Q$: If $x^2 = 16$, then $x = 4$
$R$: If $x^2 \neq 16$, then $x \neq 4$
$S$: If $x \neq 4$, then $x^2 \neq 16$
Find the truth value for each of $P$, $Q$, $R$, and $S$.

**Solution:** Statement $P$ is true. Let $x = 4$. Then $x^2 = 4^2 = 16$.

Statement $Q$ is false. Let $x = -4$. Then $x^2 = 16$, so the hypothesis is true, but $x \neq 4$, so the conclusion is false. Since $T \implies F$ is false, we are done.

Statement $R$ is true. Note that $R$ is the contrapositive of $P$. Therefore $R$ has the same truth value as $P$.

Statement $S$ is false. Note that $S$ is the contrapositive of $Q$. Therefore $S$ has the same truth value as $Q$.

**Example 4.** Each open sentence below allows $x$ and $y$ to be any real number. Prove each of the following true or find a counter example to show that it is false.

(a) If $x = 0$ or $y = 1$ then $x^2 + y^2 = 1$.
(b) If $x = 0$ and $y = 1$ then $x^2 + y^2 = 1$.
(c) If $x^2 = -1$ and $y = 1$ then $x^2 + y^2 = 1$.
(d) If $x^2 + y^2 = 1$ then $x = 1$ and $y = 1$.
(e) If $x^2 + y^2 = 1$ then $x = 1$ or $y = 1$.
(f) If $x^2 + y^2 = 1$ then $|x| \leq 1$ and $|y| \leq 1$.
(g) If $x^2 + y^2 > 1$ then $|x| > 1$ or $|y| > 1$.
(h) If $x^2 + y^2 > 1$ then $|x| > 1/\sqrt{2}$ or $|y| > 1/\sqrt{2}$.

**Solution:** Note: Below we only show the proofs, but of course the reader may want to look at pictures to figure out whether the given statement is true or not.

(a) If $x = 0$ or $y = 1$ then $x^2 + y^2 = 1$.

False. Let $x = 1$ and $y = 1$. Then "$y = 1$" is true and so "$x = 0$ or $y = 1$" is true. But, $x^2 + y^2 = 1 + 1 = 2$ and so "$x^2 + y^2 = 1$" is false. Since $T \implies F$ is false, the original statement is false.