7.3 Trigonometric Substitution

Example 1. Find the area under $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = 1/2$. 

Solution. This is the area under part of a circle, as shown, 

But this area is not one quarter of the half circle. There is no elementary way to find this area.

The area we want is given by an integral

$$\int_{x=0}^{x=1/2} \sqrt{1 - x^2} \, dx.$$ 

We do a kind of backwards substitution: instead of letting $u = g(x)$, we will let $x = g(\theta)$ where $g$ is some trig function. We will soon give the rules of thumb for which trig function to use, but for now, as an act of faith, we do the correct substitution.

Let $x = \sin(\theta)$ 

$dx = \cos(\theta) \, d\theta$.

Plugging these into the above integral we obtain

$$\int_{x=0}^{x=1/2} \sin^2(\theta) \cos(\theta) \, d\theta.$$ 

We need to do two things to this integral: change the endpoints from $x$-values to $\theta$-values, and simplify the formula on the inside. Here are the calculations,

$x = 0 \implies \sin(\theta) = 0 \implies \theta = 0$

$x = 1/2 \implies \sin(\theta) = 1/2 \implies \theta = \pi/6$

$$\sin^2(\theta) = \frac{1}{2} \cos(\theta).$$ 

Note where we put "\(*\)": this last equation is true as long as $\cos(\theta)$ is positive.

If $\cos(\theta)$ is negative, then $\sqrt{\cos^2(\theta)} = |\cos(\theta)|$. The integral now becomes

$$\int_0^{\pi/6} \cos(\theta) \cos(\theta) \, d\theta = \int_0^{\pi/6} \cos^2(\theta) \, d\theta.$$ 

We look up this integral from section 7.2 and get

$$\int_0^{\pi/6} \cos^2(\theta) \, d\theta = \frac{\pi}{2} - \frac{1}{2} \left(1 + \cos(2\theta)\right).$$
Example 2. Find $\int \sqrt{4 - x^2} \, dx$. 
Example 3. Find \( \int x \sqrt{1 - x^2} \, dx \).

Example 4. Find \( \int x^3 \sqrt{9 + x^2} \, dx \).
Example 5. Find \( \int \frac{1}{\sqrt{x^2 - 2}} \, dx \)
Extra examples

Example 6. Find \( \int_{2/3}^{\sqrt{2}/3} \frac{1}{\sqrt{9x^2 - 1}} \, dx \).