7.8 Improper Integrals

The word “improper” here just means that \( \int_a^b f(x) \, dx \) has one (or more) of the following:

- \( a = \infty \), or
- \( b = -\infty \), or
- \( f(x) \) has a vertical asymptote (VA) in the interval \([a, b] \) (i.e. we have \( y \)-values approaching \( \pm \infty \)).

Perhaps it’s best to start with a quick review of vertical and horizontal asymptotes.

\[
\lim_{x \to \infty} f(x) = L \quad \text{means that } f(x) \text{ has a right side horizontal asymptote of } y = L \]

\[
\lim_{x \to -\infty} f(x) = L \quad \text{means that } f(x) \text{ has a left side horizontal asymptote of } y = L \]

\[
\lim_{x \to a^+} f(x) = \pm \infty \quad \text{means that } f(x) \text{ has a vertical asymptote on the right of } x = a \]

\[
\lim_{x \to a^-} f(x) = \pm \infty \quad \text{means that } f(x) \text{ has a vertical asymptote on the left of } x = a \]

Pretty much all the basic functions that you know that have vertical asymptotes are shown below:

- \( \lim_{x \to 0^+} \frac{1}{x} = \infty \), i.e. \( \frac{1}{x} \) has a V.A. at \( x = 0 \). Notational shortcut: “\( \frac{1}{0^+} = \infty \).”
- \( \lim_{x \to c} \frac{f(x)}{g(x)} = \pm \infty \) if \( g(c) = 0 \) and the top does not equal 0. I.e. \( \frac{f(x)}{g(x)} \) has a V.A. if we have \( \div \) by 0 on bottom but not on top. Notational shortcut: “\( \frac{f(x)}{g(x)} (\neq 0) \) = \( \pm \infty \).”
- \( \lim_{x \to 0^+} \ln(x) = \infty \), i.e. \( \ln(x) \) has a V.A. at \( x = 0 \). Notational shortcut: “\( \ln(0) = \ln(0^+) = \infty \).”
- \( \lim_{x \to \pi/2^-} \tan(x) = \infty \) and \( \lim_{x \to \pi/2^+} \tan(x) = -\infty \). I.e. \( \tan(x) \) has a V.A. at \( x = \pm \pi/2 \) (this is actually \( \div \) by 0 since \( \cos(\pm \pi/2) = 0 \)). Notational shortcut “\( \tan(\pm \pi/2) = \pm \infty \).”

Pretty much all the basic functions that you know that have horizontal asymptotes are shown below (as well as two functions that don’t have them).

- \( \lim_{x \to \pm \infty} \frac{1}{x} = 0 \) for any real number \( p > 0 \). Notational shortcut: “\( \frac{1}{\pm \infty} = 0 \).”
- \( \lim_{x \to -\infty} e^x = 0 \). Notational shortcut: “\( e^{-\infty} = 0 \).”
- \( \lim_{x \to \infty} \tan^{-1}(x) = \pi/2 \), \( \lim_{x \to -\infty} \tan^{-1}(x) = -\pi/2 \). Notational shortcut: \( \tan^{-1}(\pm \infty) = \pm \pi/2 \).
- \( \lim_{x \to \pm \infty} \ln(x) = \pm \infty \). Notational shortcut: \( \ln(\pm \infty) = \pm \infty \).
- \( \lim_{x \to \pm \infty} e^x = \pm \infty \). Notational shortcut: \( e^{\pm \infty} = \pm \infty \).
- \( \lim_{x \to \pm \infty} \sqrt{x} = \infty \). Notational shortcut: \( \sqrt{\pm \infty} = \pm \infty \).

Now, returning to integrals, if an integral involves infinity either with the \( x \)-values or \( y \)-values, then you use limits, specifically, you take the limit \( \lim_{x \to ?} \).
equals $\infty$ or the location of a vertical asymptote. In all of the following definitions, we assume that $F(x)$ is the anti-derivative of $f(x)$.

- $\int_{-\infty}^{b} f(x) \, dx = F(b) - \lim_{x \to -\infty} F(x)$.
- $\int_{a}^{b} f(x) \, dx = \lim_{x \to \infty} F(x) - F(a)$.

- $\int_{-\infty}^{b} f(x) \, dx = \int_{-\infty}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx$ where $c$ is any point between $-\infty$ and $\infty$, and the two integrals on the right are defined as in the previous steps.

- If $x = b$ is a VA then $\int_{a}^{b} f(x) \, dx = \lim_{x \to b} F(x) - F(a)$.

- If $x = a$ is a VA then $\int_{a}^{b} f(x) \, dx = F(b) - \lim_{x \to a} F(x)$.

- If $x = c$ is a VA and $c$ and $a < c < b$, then $\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx$ where the two integrals on the right are defined as in the previous steps.

- If one of the previous limits does not exist (this includes the case where the limit equals $\infty$ or $-\infty$), then the corresponding integral also does not exist. If the integral exists, we say that it is **convergent**, otherwise **divergent**.

- In the two cases where we split one integral into the sum of two new integrals, we require both of the new integrals to exist. If this is not the case, then we declare that the original integral does not exist.