We start by recalling long division of numbers.

\[
\frac{123}{9} \rightarrow 9 \overline{123} \rightarrow 9 \overline{1} \quad 2 \quad 3 \rightarrow \frac{123}{9} = \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_.
\]

At each step of long division, we put on a number on top. We choose the top so that when we multiply it by the number on the side, we can subtract as much as possible away from the numbers inside.

Now we do the same thing with polynomials.

\[
\frac{2x^2 + 17x + 2}{2x + 1} \rightarrow 2x + 1 \overline{2x^2 + 17x + 2} \rightarrow 2x + 1 \overline{2x^2 + 17x + 2} + \frac{-2x^2 + 17x + 2}{2x + 1} = \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_.
\]

At each step of long division, we put a monomial on top. We choose the top so that when we multiply it by the polynomial on the side, we can subtract as much as possible away from the polynomial inside.

Here’s a fairly general description of partial fractions.

1. Make sure that you have \( \frac{\text{poly}}{\text{poly}} \) with the degree top < degree bottom.
2. Factor the bottom into linear and quadratic factors.
3. Setting up the fractions: this needs a case-by-case description.

**Distinct linear factors** (each factor on the left appears as a term on the right)

\[
\frac{\ast}{(x+a)(x+b)\ldots} = \frac{A}{x+a} + \frac{B}{x+b} + \ldots \quad (a \neq b)
\]

could be more here could be more here

**Repeated linear factors** (with 4 as an example of how many times \((x+a)\) is repeated)

\[
\frac{\ast}{(x+a)^4(x+b)\ldots} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3} + \frac{D}{(x+a)^4} + \frac{E}{x+b} + \ldots \quad (a \neq b)
\]

the power here, 4, equals \ldots the number of terms here

**Distinct quadratic factors**

\[
\frac{\ast}{(x^2 + ax + b)(x^2 + cx + d)\ldots} = \frac{Ax + B}{x^2 + ax + b} + \frac{Cx + D}{x^2 + cx + d} + \ldots
\]

could be more here could be more here
Repeated quadratic factors (with 3 as the number of times \((x^2 + ax + b)\) is repeated)

\[
\frac{1}{(x^2 + ax + b)^3(x^2 + cx + d)} = \frac{Ax + B}{x^2 + ax + b} + \frac{Cx + B}{(x^2 + ax + b)^2} + \frac{Dx + E}{(x^2 + ax + b)^3} + \frac{Fx + G}{x^2 + cx + d}
\]

the power here, 3, equals \ldots the number of terms here

Mixed linear and quadratic factors

\[
\frac{1}{(x + a)(x + b)^2(x^2 + cx + d)(x^2 + ex + f)^2} = \frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{(x + b)^2} + \frac{Dx + E}{x^2 + cx + d} + \frac{Fx + G}{x^2 + ex + f} + \frac{Hx + I}{(x^2 + ex + f)^2}
\]

4. After you get the above equation set up, you multiply both sides by the denominator from the left, and cancel all denominators. To finish there are two possible steps:

(a) You can plug in \(x\)-values that make one of the terms on the right equal to 0. This might allow you to solve for the other constants \(A, B, \ldots\), but it might not (it won’t if there are repeated factors or quadratics without roots).

(b) If the previous step doesn’t finish the problem, then you multiply everything out on the right, gather together all the \(x\)-terms on the right, gather the \(x^2\)-terms, the \(x^3\)-terms etc. Set up a new system of equations as follows:

\[
\begin{align*}
\text{constant from left} &= \text{constant from right} \\
\text{\(x\)-coeff from left} &= \text{\(x\)-coeff from right} \\
\text{\(x^2\)-coeff from left} &= \text{\(x^2\)-coeff from right}
\end{align*}
\]

This gives linear equations in \(A, B, C, \ldots\) Solve these equations in the usual way (i.e. solve one equation for one of the letters \(A, B, C, \ldots\), substitute this into the other equations, and repeat: or use matrices and linear algebra).

Completing the square is a trick to turn something of the form \(x^2 + ax + b\) into \((x + c)^2 + d\). The following pattern works only when the coefficient of \(x^2\) is 1.

\[
x^2 + 8x + 5 = \frac{x^2 + 8x + 5}{2} = \left(\frac{x + 4}{2}\right)^2 + \frac{9}{4}
\]

In short: Take half of the \(x\)-coefficient, square this, add and subtract the result into the formula. Then the first three terms (the \(x^2\) term, the \(x\) term and the part that we added) equal \((x + c)^2\).