7.4 Integration of Rational Functions by Partial Fractions

A rational function is one of the form
\[
\frac{\text{polynomial}}{\text{polynomial}}
\]

In this section we will cover three techniques to integrate them: polynomial division, partial fractions, and completing the square.

Here are the integrals of the simplest rational functions. The first four we know already, the last two we’ll find in examples.

\[
\int \frac{1}{x} \, dx = \ln |x|
\]
\[
\int \frac{1}{ax ± b} \, dx = \frac{1}{a} \ln |ax ± b|
\]
\[
\int \frac{1}{x^2 + 1} \, dx = \tan^{-1}(x)
\]
\[
\int \frac{x}{x^2 ± a} \, dx = \frac{1}{2} \ln |x^2 ± a|
\]
\[
\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right)
\]

Strategy for integrating rational functions:

Given an integral of the form \( \int \frac{\text{polynomial}}{\text{polynomial}} \),

- If you can split the fraction up at + and/or − in the numerator and get a basic anti-derivative, then do this.
- If the degree on top is ≥ the degree on the bottom, then do polynomial division.
- If you can factor the bottom then do partial fractions.
- If you cannot factor the bottom, and the bottom is a quadratic, then complete the square.

The degree of a polynomial is the biggest power of \( x \) that appears (or would appear if you multiplied it all out). For rational functions, the top and bottom of the fractions are polynomials, and so the top and bottom have degrees.

For instance

- \( \frac{4}{x^2 - x + 1} \), degree of top = \( \_\_\_\_ \), degree of bottom = \( \_\_\_\_ \).
- \( \frac{x^3 - x + 1}{x^2(x + 1)^3} \), degree of top = \( \_\_\_\_ \), degree of bottom = \( \_\_\_\_ \) (if you multiply everything out).