Example 1. Imagine we want to find the volume of hard boiled egg. We could put the egg in a measuring cup and measure how much water it displaces. But we suppose we want to do it more mathematically, using functions and formulas. We will imagine cutting the egg into slices, and then measuring the volume of each slice as a cylinder.

(a) Use the ellipse

\[ \frac{x^2}{1.155^2} + \frac{y^2}{0.8^2} = 1 \]

to model the outline of an egg with length 2.31 in and diameter 1.6 in. Draw the ellipse and the shape it makes when rotated around the x-axis.

(b) Imagine cutting the shape from (a) into slices, and draw the result.

(c) Imagine replacing each slice from (b) with a cylinder and draw the result.

(d) Figure out how to find the volume of each cylinder-slice in (c) and write down a formula for the sum of the volumes.

(e) Translate the formula from (d) into an integral, and solve the integral.
CHAPTER 6. INTEGRAL APPLICATIONS

Now each slice has a slight curve on the top and bottom. This makes it hard to find the exact volume of the slice, but we can approximate the volume by using a cylinder of the same size:

$$\pi \left( \frac{1}{m} - \frac{1}{n} \right) \frac{l}{t} \Delta x$$

To find the radius of the curve from the x-axis, this distance is what we usually call the y-value. Now we find the y-values from the ellipse:

$$\frac{x^2}{m^2} \pm \sqrt{\frac{y^2}{l^2} \mp \frac{x^2}{m^2}}$$

Combining this formula with the above sum of volumes we get:

$$\pi \left( \frac{1}{m} - \frac{1}{n} \right) \frac{l}{t} \Delta x$$

This last line is the most important conclusion of this example, translating a volume by rotation into an integral. Of course, it's also a part of the integral:

$$\pi \left( \frac{1}{m} - \frac{1}{n} \right) \frac{l}{t} \Delta x$$
Example 2. Find the volume generated by rotating \( y = -x^2 + 2 \) around the \( x \)-axis, between \( x = -\sqrt{2} \) and \( x = \sqrt{2} \).

Example 3. Find the volume generated by rotating \( y = x^2 \) around the \( y \)-axis, between \( y = 0 \) and \( y = 2 \).
Example 4. Set up an integral, and use your calculator to find it, for the volume of the napkin ring made by rotating the region bounded by $y = \sqrt{1 - x^2} + 2$, and $y = 2$ around the $x$-axis.
The volume of the slice is given by the area of the face $\Delta h$ times.

We start with the two-dimensional shape involving $r$ and $R$.

Area $\pi R^2 - \pi r^2$.

Volume $\pi R^2 - \pi r^2 \Delta x$.

So to finish we just need to figure out formulas for $R$ and $r$ in the shape that we have.

Thus

$R = \sqrt{m - x^2}$

and

$r = \sqrt{m - x^2}$

Putting all together we have the integral that we want.

$$\int 1 - \frac{1}{\pi} \sqrt{m - x^2} \, dx$$
Example 5. Find the volume of the napkin ring made by rotating the region bounded by \( y = -\frac{4}{9}x^2 + 2 \), and \( y = \frac{1}{4}x^2 + \frac{7}{16} \) (shown below) around the line \( y = -1 \).
The larger radius is defined by the top curve \( y - \frac{p}{u}x^2 \) and is the distance between this curve and the horizontal axis \( y \).

The smaller radius is defined by the bottom curve \( y - \frac{m}{p}x^2 \) and is the distance between this curve and the horizontal axis \( y \).

Now we integrate, skipping some of the messy steps:

\[
V = \pi \left( \left[ \left( \frac{3}{2} \right)^{3/2} - \left( \frac{3}{2} \right)^{3/2} \right] \right) - \pi \left( \left[ \left( \frac{m}{p} \right)^{3/2} - \left( \frac{m}{p} \right)^{3/2} \right] \right) dx
\]

(Steps of foiling skipped.)
Example 6. Find the volume of the region between the curves \( y = g(x) = x^3 \) and \( y = f(x) = \sqrt{x} \) rotated around the line \( x = 1 \) (shown below).
Extra examples

Example 7. Derive the formula for the volume of a sphere of radius \( r \).

Challenge. 
- Can you figure out how to find the volume of a shape rotated around the line \( y = x \)? What about other lines?
- Can you figure out how to apply washers when \( f(x) \) and \( g(x) \) switch places with respect to which one is farther from \( c \)? What about if they switch places also with respect to which side of \( c \) they fall on?