11.6 Absolute Convergence and the Ratio and Root Tests

Example 1. Can we apply any of the tests we’ve learned so far to the series
\[ \sum_{n=1}^{\infty} \frac{\cos(n)}{n^2} \]?

Solution. If we write out the first five terms we get
\[
\cos 1, \cos 2, \cos 3, \cos 4, \cos 5, \ldots
\]
where we have indicated the sign of each term with a + or − below the term.
Since some of the terms are negative, we cannot apply the Comparison Tests or the Integral Test. Since the terms do not alternate perfectly, we cannot apply the alternating series test.

Example 2. [Continued] Show that \[ \sum_{n=1}^{\infty} \frac{\cos(n)}{n^2} \] is absolutely convergent.

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\[ \sum_{n=1}^{\infty} \frac{\cos(n)}{n^2} \] convergest
Thereforer by Theorem
\[ \sum_{n=1}^{\infty} \frac{\cos(n)}{n^2} \] converges toot In other words, \[ \sum_{n=1}^{\infty} \frac{\cos(n)}{n^2} \] is absolutely convergent.
Example 3. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

\[ \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}. \]

Example 4. Apply the Ratio Test to \( \sum_{n=1}^{\infty} (-1)^n \frac{10^n}{n!} \).
Example 5. Apply the Ratio Test to \( \sum_{n=1}^{\infty} (-1)^n \frac{n^{10}}{1.5^n} \).

Example 6. What happens when you apply the Ratio test to the following series:

\[
\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}
\]
Example 7. Determine if the following series converges:

\[
\sum_{n=1}^{\infty} \left( \frac{-2n^3 + n}{3n^3 + n^2 + 1} \right)^n
\]

Make sure you see that the absolute values make the final limit \( \frac{2}{3} \) here, not \( -\frac{2}{3} \). In this example, the difference doesn’t matter, but it would matter a lot if we had an example with \( L = | -3/2 | \).