Example 1. (a) Find the Maclaurin series for $f(x) = e^x$.
(b) Write the series using $\sum$ notation.
(c) Find the radius of convergence.
(d) Use the series to approximate $\sqrt{e}$ to the 4th decimal place.
Example 2. Find the Taylor series of $f(x) = \ln(x)$ at $x = 1$, write the series using $\sum$ notation, and find its radius of convergence, and see if you recognize what you get using this series to calculate $\ln(2)$.

Example 3. Find the Maclaurin series for $f(x) = \sin(x)$, write the series using $\sum$ notation, and find its radius of convergence.
Thus we have
\[\sin ax = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.\]

We have to work harder to write this in notation mostly because we need a formula that just gives the odd powers of \(x\) and the odd factorials. In other words, we need to figure out a formula for the following outputs:

<table>
<thead>
<tr>
<th>Value of (n)</th>
<th>Resulting power of (x) and factorial</th>
<th>( s_{ijklmn} ) resulting power of (x) and factorial</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
</tbody>
</table>

Although we know how to write these formulas for the powers of \(x\) and the factorials, we thus have:

\[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.\]

Now we find the radius of convergence. We start with the Ratio Test and multiply by the reciprocal of \(x^{2n+1}\):

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \right| = |x^2|.
\]

To see how to simplify the factorials, think about what sorts of numbers we have: \(s_{ijklmn}\) is an odd number and \(d_{ijklmn}\) is the next odd number. So we have a fraction with a factorial on top and the next odd number on the bottom. For example, the fraction could be something like:

\[
\frac{p}{q} \cdot |x| \cdot \frac{r}{s}.
\]

Now we find the radius of convergence. We start with the Ratio Test and multiply by the reciprocal of \(x^{2n+1}\):

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \right| = |x^2|.
\]

To see how to simplify the factorials, think about what sorts of numbers we have: \(s_{ijklmn}\) is an odd number and \(d_{ijklmn}\) is the next odd number. So we have a fraction with a factorial on top and the next odd number on the bottom. For example, the fraction could be something like:

\[
\frac{p}{q} \cdot |x| \cdot \frac{r}{s}.
\]
Example 4. [The Binomial Series] Let \( k \) be any real number. Find the Maclaurin Series for \( f(x) = (1 + x)^k \), write your answer in \( \sum \) notation.
Extra Examples

Example 5. Find the Taylor series of $f(x) = \sin(x)$ centered at $a = \pi/4$. You do not have to write this using $\sum$ notation, but you can try if you like.
Example 6. Find the Taylor series of \( f(x) = x^{-2} \) centered at \( a = 2 \), write the series using \( \sum \) notation, and find its radius of convergence.