10.3 Polar Coordinates

Definition. For any point in the plane, with Cartesian coordinates \((x, y)\), we define \textbf{polar coordinates} of this point as follows:

\[
(x, y)_{\text{cartesian}} = (r, \theta)_{\text{polar}} \text{ where } r \text{ and } \theta \text{ satisfy}
\]
\[
x = r \cos(\theta) \text{ and } \\
y = r \sin(\theta)
\]

Note that \(r\) and \(\theta\) are not unique.

We can picture \((r, \theta)\). Shown below is an example where \((x, y)\) is in Quadrant II and \(r\) and \(\theta\) are chosen in a natural fashion:

\[
(x, y) \quad r \quad \theta
\]

Fact. To convert from Cartesian to Polar coordinates, the following formulas can be used:

\[
r = \sqrt{x^2 + y^2}
\]

\[
\theta = \begin{cases} 
\tan^{-1}(y/x) & \text{if } (x, y) \text{ is in QI or QIV} \\
\tan^{-1}(y/x) + \pi & \text{if } (x, y) \text{ is in QII or QIII} \\
\pm \pi/2 & \text{as appropriate, if } (x, y) \text{ is on the } y\text{-axis}
\end{cases}
\]

Definition. A \textbf{polar equation} is an equation relating \(r\) and \(\theta\), where \((r, \theta)\) is interpreted in terms of polar coordinates. Usually a polar equation is given in the form \(r = f(\theta)\). A \textbf{polar curve} is the set of all points that satisfy a polar equation.

Fact. Given a polar equation \(r = f(\theta)\), the Cartesian derivative equals

\[
\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta) \sin(\theta) + f(\theta) \cos(\theta)}{f'(\theta) \cos(\theta) - f(\theta) \sin(\theta)}
\]

To understand this, recall that

\[
\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta \theta}
\]

and that

\[
x = r \cos(\theta) = f(\theta) \cos(\theta) \quad \text{and} \quad y = r \sin(\theta) = f(\theta) \sin(\theta)
\]