Chapter 1: Functions and Applications

Basic Functions

1. Identify each of the graphs below as one of the following functions:
Function notation, solving function equations

2. (This problem as a whole is obviously too long for the final: but I could have a problem with a couple of parts like this.)

(a) Let \( f(x) = 3x + 5 \). Solve \( f(x) = 3 \).

\[ \text{Solution:} \]
\[ 3x + 5 = 3 \]
\[ 3x = -2 \]
\[ x = -\frac{2}{3} \]

(b) Let \( f(x) = x^2 \). Solve \( f(x) = 5 \).

\[ \text{Solution:} \]
\[ x^2 = 5 \]
\[ x = \pm \sqrt{5} \]
(c) Let $f(x) = 2x^3 - 5$. Solve $f(x) = -10$.

Solution:

$$2x^3 - 5 = -10$$
$$2x^3 = -5$$
$$x^3 = -\frac{5}{3}$$
$$x = \sqrt[3]{-\frac{5}{3}}$$
$$= -\sqrt[3]{\frac{5}{3}}$$

(d) Let $f(x) = x^2 - 2x - 15$. Solve $f(x) = 0$.

Solution:

$$x^2 - 2x - 15 = 0$$
$$(x - 5)(x + 3) = 0$$
$$(x - 5) = 0 \text{ or } (x + 3) = 0$$
$$x = 5 \text{ or } -3$$

(e) Let $f(x) = x^2 - 2x - 14$. Solve $f(x) = 0$.

Solution:

$$x^2 - 2x - 14 = 0$$
$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-14)}}{2(1)}$$
$$= \frac{2 \pm \sqrt{4 + 56}}{2}$$
$$= \frac{2 \pm \sqrt{60}}{2}$$

(f) Let $f(x) = e^x - 2x^2 e^x$. Solve $f(x) = 0$.

Solution:

$$e^x - 2x^2 e^x = 0$$
$$e^x(1 - 2x^2) = 0$$
$$e^x = 0 \text{ or } (1 - 2x^2) = 0$$
$$e^x = 0 \text{ is impossible}$$
$$1 - 2x^2 = 0$$
$$2x^2 = 1$$
$$x^2 = 1/2$$
$$x = \pm \sqrt{1/2}$$
(g) Let \( f(x) = \frac{1}{x} + 1 \). Solve \( f(x) = 0 \).

Solution:

\[
\frac{1}{x} + 1 = 0
\]
\[
\frac{1}{x} = -1
\]
\[
\frac{1}{x} = \frac{-1}{1}
\]
\[
x = -1 \quad \text{(cross multiply)}
\]

(h) Let \( f(x) = \frac{2}{x^2} - 5 \). Solve \( f(x) = 0 \).

Solution:

\[
\frac{2}{x^2} - 5 = 0
\]
\[
\frac{2}{x^2} = 5
\]
\[
\frac{2}{x^2} = \frac{5}{1}
\]
\[
2 = 5x^2 \quad \text{(cross multiply)}
\]
\[
x^2 = \frac{2}{5}
\]
\[
x = \pm \sqrt{\frac{2}{5}}
\]

(i) Let \( f(x) = 3e^x - 5 \). Solve \( f(x) = 0 \).

Solution:

\[
3e^x - 5 = 0
\]
\[
3e^x = 5
\]
\[
e^x = \frac{5}{3}
\]
\[
\ln(e^x) = \ln(5/3)
\]
\[
x = \ln(5/3)
\]

(j) Let \( f(x) = 4e^{2x} - 3 \). Solve \( f(x) = 0 \).

Solution:

\[
4e^{2x} - 3 = 0
\]
\[
4e^{2x} = 3
\]
\[
e^{2x} = \frac{3}{4}
\]
\[
\ln(e^{2x}) = \ln(3/4)
\]
\[ 2x = \ln(3/4) \]
\[ x = \frac{1}{2} \ln(3/4) \]

(k) Let \( f(x) = \ln(x) + 5 \). Solve \( f(x) = 0 \).

**Solution:**

\[
\ln(x) + 5 = 0 \\
\ln(x) = -5 \\
e^{\ln(x)} = e^{-5} \\
x = e^{-5}
\]

(I) Let \( f(x) = 2 \ln(x) - 5 \). Solve \( f(x) = 0 \).

**Solution:**

\[
2 \ln(x) - 5 = 0 \\
2 \ln(x) = 5 \\
\ln(x) = 5/2 \\
e^{\ln(x)} = e^{5/2} \\
x = e^{5/2}
\]

(m) Let \( f(x) = 3 \ln(x + 1) + 7 \). Solve \( f(x) = 0 \).

**Solution:**

\[
3 \ln(x + 1) + 7 = 0 \\
3 \ln(x + 1) = -7 \\
\ln(x + 1) = -7/3 \\
e^{\ln(x+1)} = e^{-7/3} \\
(x + 1) = e^{-7/3} \\
x = e^{-7/3} - 1
\]

3. Let \( f(x) = x^2 \) and \( g(x) = \frac{1}{x} + 1 \).

(a) Find \( f(x) + g(x) \).

**Solution:**

\[ x^2 + \frac{1}{x} + 1. \]

(b) Find \( f(x)g(x) \).

**Solution:**

\[ x^2 \left( \frac{1}{x} + 1 \right). \]
(c) Find \( \frac{f(x)}{g(x)} \).

Solution:
\[
\frac{x^2}{\frac{1}{x} + 1}.
\]

(d) Find \( f(g(x)) \).

Solution:
\[
f \left( \frac{1}{x} + 1 \right) = \left( \frac{1}{x} + 1 \right)^2.
\]

(e) Find \( g(f(x)) \).

Solution:
\[
g(x^2) = \frac{1}{x^2} + 1.
\]

Linear, Power Function, and Exponential Modelling

4. The two sections of Applied Calculus that I taught during Fall 2017 started with 61 students. 15 weeks later I had 56 students.

(a) Assuming that the students continued dropped the class at a linear rate, write an equation for \( S \), the number of students, as a function of \( t \), the number of weeks since the beginning of the semester.

(b) According to your function, how many students did I have in week 5?

Solution: (a)

line through (0, 61) and (15, 56)
\[
y = m(x - x_0) + y_0
\]
\[
x_0 = 0
\]
\[
y_0 = 61
\]
\[
m = \frac{56 - 61}{15 - 0} = -\frac{5}{15} = -\frac{1}{3}
\]
\[
y = -\frac{1}{3}(x - 0) + 61
\]
\[
= -\frac{1}{3}x + 61
\]

(b)
\[
y = \frac{1}{3}(5) + 61
\]
\[
= -\frac{5}{3} + 61
\]
\[
61 - \left(1 + \frac{2}{3}\right) = 59.333
\]

5. The arctic ice caps appears to be shrinking at a rate of 4\% per decade\(^1\).

(a) Assuming that the rate of decrease of ice remains the same, write an equation for \(I\), the percent of the current ice that will be left, as a function of \(t\), the number of decades from now.

(b) According to your equation, what percent of the current amount ice will be left in 50 years?

Solution: (a)

\[
I = Ca^t \quad C = \% \text{ of current ice}, \quad t = \text{decades}, \quad a = 1 + r, \quad r = \% \text{ change}
\]

\[
= 1 \cdot (1 - 0.04)^t
\]

\[
I = (0.96)^t
\]

(b)

\[
I = (0.96)^5
\]

\[
= 0.815
\]

\[
= 81.5\% \text{ of current ice}
\]

**Functions in Economics**

6. (Hughes-Hallet, 4e, #9) A company that makes Adirondack chairs has fixed costs of $5000 and variable costs of $30 per chair. The company sells the chairs for $50 each.

(a) Find formulas for the cost and revenue functions.

(b) Find the marginal cost and marginal revenue.

(c) Graph the cost and revenue functions on the same axes.

(d) Find the break-even point.

Solution:

(a)

\[
C(q) = 5000 + 30q
\]

\[
R(q) = 50q
\]

(b)

\[
MC = 30
\]

\[
MR = 50
\]

\(^1\)From [http://www.theguardian.com/environment/2013/sep/18/how-fast-is-arctic-sea-ice-melting](http://www.theguardian.com/environment/2013/sep/18/how-fast-is-arctic-sea-ice-melting)
7. (Hughes-Hallett, 4e, #28) The demand curve for a product is given by \( q = 120,000 - 500p \) and the supply curve is given by \( q = 1000p \) for \( 0 \leq q \leq 120,000 \), where price is in dollars.

(a) At a price of $100, what quantity are consumers willing to buy and what quantity are producers willing to supply? Will the market push prices up or down?

(b) Find the equilibrium price of quantity. Does your answer to part (a) support the observation that market forces tend to push prices closer to the equilibrium price?

**Solution:**

(a) 

\[
\text{consumers willing to buy: } q = 120,000 - 500p \\
= 120,000 - 500(100) \\
= 70,000
\]

\[
\text{suppliers willing to make: } q = 1000p \\
= 1000(100) \\
= 100,000
\]

Market forces will move prices down, because there are too many items being made.

(b) 

\[
\text{supply = demand} \\
1000p = 120000 - 500p
\]
This agrees with (a) because from \( p = 100 \) the price will move down to \( p = 80 \).

Chapter 2: The derivative

Approximating the Derivative

8. Let \( f(x) = e^{-x^2} \). Find an approximation of \( f'(2) \) by filling in the following table using the same rules as on the quiz\(^1\).

Solution:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( \frac{f(2) - f(x)}{2 - x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.9</td>
<td>0.02705185</td>
<td>-0.08736208</td>
</tr>
<tr>
<td>1.99</td>
<td>0.01906121</td>
<td>-0.07455692</td>
</tr>
<tr>
<td>1.999</td>
<td>0.01838903</td>
<td>-0.07339089</td>
</tr>
<tr>
<td>2</td>
<td>0.01831564</td>
<td>ERR</td>
</tr>
<tr>
<td>2.1</td>
<td>0.01215518</td>
<td>-0.06160461</td>
</tr>
<tr>
<td>2.01</td>
<td>0.01759571</td>
<td>-0.07199261</td>
</tr>
<tr>
<td>2.001</td>
<td>0.01824520</td>
<td>-0.07313447</td>
</tr>
</tbody>
</table>

The numbers we calculated for \( x = 1.999 \) and 2.001 are our best approximations. From them we can say

\[ f'(2) \approx -0.073 \]

Graphing Derivatives

9. Shown below is the graph of a cubic function, i.e. a function of the form \( y = ax^3 + bx^2 + cx + d \). Sketch a graph of the derivative; make sure that your sketch is positive/negative/zero in the right places, and that it has the right shape.

Solution:

\(^1\)You should plug numbers in for \( x \) that are close to 2. They should be close enough that you get two approximations of \( f'(2) \) that are the same to the second decimal place. You should always use at least 8 digits in every step of the calculation.
To see how to make the second graph, start with the points where $f$ is flat:

graph of $f(x)$: \( m = 0 \) at $x = -1.5$ and $x = 1.5$
Therefore the graph of \( f'(x) : y = 0 \) at \( x = -1.5 \) and \( x = 1.5 \).

Similarly, see where \( f \) is going up:

\[
\text{graph of } f(x) : m \approx 2 \text{ at } x = -2 \text{ and } x = 2
\]

Therefore the graph of \( f'(x) : y = 2 \) at \( x = -2 \) and \( x = 2 \).

Finally, see where \( f \) is going down:

\[
\text{graph of } f(x) : m \approx -3 \text{ at } x = 0
\]

Therefore the graph of \( f'(x) : y = -3 \) at \( x = 0 \).

**Marginal Cost and Revenue**

10. The graph below shows a total cost function. Estimate the marginal cost at \( q = 700 \).

*Solution:* We start by graphing a tangent line (simply move a straight edge towards the graph of \( C(q) \) until it just barely touches the graph at \( q = 700 \)).

Now we estimate the slope of the straight line. The best way to do this is to use points that are as far apart as possible, or to see if the line intersects exactly at one of the grid points.

I'll use points far apart:

\[
(0, 210) \text{ and } (1000, 810)
\]

appear to be on the line. Thus

\[
MC = m \approx \frac{810 - 210}{1000 - 0} = \frac{600}{1000} = \frac{3}{5}
\]

**Chapter 3: Rules for Derivatives**
Basic derivatives

11. Find the following derivatives

(a) \( \frac{d}{dx} 3x^6 \)
   
   Solution:
   
   \( 18x^5 \)

(b) \( \frac{d}{dx} 5\sqrt{x} \)
   
   Solution:
   
   \( \frac{5}{3} x^{-2/3} \) (note that \( \sqrt[3]{x} = x^{1/3} \))

(c) \( \frac{d}{dx} \frac{10}{x^5} \)
   
   Solution:
   
   \( -50x^{-6} \) (note that \( \frac{10}{x^5} = 10x^{-5} \))

(d) \( \frac{d}{dx} 17e^x \)
   
   Solution:
   
   \( 17e^x \)

(e) \( \frac{d}{dx} 17 \ln(x) \)
   
   Solution:
   
   \( 17 \cdot \frac{1}{x} \)

Combinations of functions

12. Find the following derivatives

(a) \( \frac{d}{dx} 3(2x - 5)^6 \)
   
   Solution:
   
   \[
   \frac{d}{dx} 3 \left( 2x - 5 \right)^6 = 18 \left( 2x - 5 \right)^5 \cdot 2x - 5' \\
   = 18(2x - 5)^5 \cdot 2
   \]

(b) \( \frac{d}{dx} 5\sqrt{-7x + 11} \)
   
   Solution:
   
   \[
   \frac{d}{dx} 5\sqrt{-7x + 11} = \frac{5}{3} \left( -7x + 11 \right)^{-2/3} \cdot (-7x + 11)' \\
   = \frac{5}{3} (-7x + 11)^{-2/3} \cdot (-7)
   \]
(c) \[ \frac{d}{dx} \frac{10}{(4x - 5)^5} \]

Solution:
\[ \frac{d}{dx} \frac{10}{(4x - 5)^5} = -50 \left( \frac{10}{4x - 5} \right)^6 \frac{d}{dx} (4x - 5) \]
\[ = -50 (4x - 5)^{-6} \cdot 4 \]

(d) \[ \frac{d}{dx} 17e^{-x+10} \]

Solution:
\[ \frac{d}{dx} 17e^{-x+10} = 17e^{-x+10} \cdot \frac{d}{dx} (-x + 10) \]
\[ = 17e^{-x+10} \cdot (-1) \]

(e) \[ \frac{d}{dx} 17 \ln(3x + 5) \]

Solution:
\[ \frac{d}{dx} 17 \ln(3x + 5) = 17 \cdot \frac{1}{3x + 5} (3x + 5)' \]

13. Find the following derivatives

(a) \[ \frac{d}{dx} (3x^6 + 5\sqrt{x})\left(\frac{10}{x^5} - 17e^x\right) \]

Solution:
\[ \frac{d}{dx} \left(3x^6 + 5\sqrt{x}\right) \left(\frac{10}{x^5} - 17e^x\right) \]
\[ f = 3x^6 + 5\sqrt{x} \]
\[ g = \left(\frac{10}{x^5} - 17e^x\right) \]
\[ f' = 18x^5 + \frac{5}{3} x^{-2/3} \]
\[ g' = -50x^{-6} - 17e^x \]
\[ f'g + fg' = \left(18x^5 + \frac{5}{3} x^{-2/3}\right) \left(\frac{10}{x^5} - 17e^x\right) + (3x^6 + 5\sqrt{x}) \cdot (-50x^{-6} - 17e^x) \]

(b) \[ \frac{d}{dx} \frac{2x^2 - x}{e^x + x^2} \]

Solution:
\[ \frac{2x^2 - x}{e^x + x^2} \leftarrow f \]
\[ e^x + x^2 \leftarrow g \]
\[ f' = 4x - 1 \]
\[ g' = e^x + 2x \]
\[ \frac{f'g - fg'}{g^2} = \frac{(4x - 1)(e^x + x^2) - (2x^2 - x)(e^x + 2x)}{(e^x + x^2)^2} \]
Derivatives and Tangent Lines

14. (a) Find the Equation of tangent line at $x = 5$ for $f(x) = \ln(x)$.

Solution:

$$y_m(x - x_0) + y_0$$

$x_0 = 5$

$y_0 = f(5) = \ln(5)$

$f'(x) = \frac{1}{x}$

$m = f'(5) = \frac{1}{5}$

$y = \frac{1}{5}(x - 5) + \ln(5)$

(b) Find the Equation of tangent line at $x = 9$ for $f(x) = 5\sqrt{x}$.

Solution:

$$y_m(x - x_0) + y_0$$

$x_0 = 9$

$y_0 = f(9) = 5\sqrt{9} = 15$

$f'(x) = \frac{5}{2}x^{-1/2}$

$m = f'(9) = \frac{5}{2}(9)^{-1/2}$

$$= \frac{5}{2} \cdot \frac{1}{\sqrt{9}}$$

$$= \frac{5}{2} \cdot \frac{1}{3}$$

$$= \frac{5}{6}$$

$y = \frac{5}{6}(x - 9) + 15$

Chapter 4: Using the Derivative

Local Max/Mins

15. Let $f(x) = 3x^4 - 4x^3$.

(a) Find the critical points algebraically.

Solution:
\[ f'(x) = 12x^3 - 12x^2 \]
\[ f'(x) = 0 \]
\[ 12x^3 - 12x^2 = 0 \]
\[ 12x^2(x - 1) = 0 \]
\[ x^2 = 0 \text{ or } (x - 1) = 0 \]
\[ x = 0, 1 \]

(b) There should be two critical points. Use your calculator to calculate \( f' \) at three numbers: one to the left of both critical points, one in the middle, and one to the right of both critical points.

Solution:
\[ f'(-1) = -24 \]
\[ f'(0.5) = -4.5 \]
\[ f(2) = 48 \]

(c) Based on your answers from part (b), identify each critical point as a local max/min/Neither.

Solution: Since \( f'(x) \) is negative to both the left and right of \( x = 0 \), we see that this point is neither.
Since \( f'(x) \) is negative to the left of \( x = 1 \) and positive to the right, we see that \( x = 1 \) is a local min.

(d) Using Desmos (or something similar), graph \( f(x) \). Use a window that shows all the “interesting” features (in particular it should show the local max/mins), and verify your answers from (a)–(c).

Solution:

Note: your window has to be small enough that I can see two things: (1) that the point at \((0,0)\) is neither a max nor min, (2) that the point \((1,-1)\) is a local min.
Solution:

critical points: $x = 0, 1$

$x = 1$ is l. min at $x = 1$.

$x = 0$ is neither

$f \uparrow$ to right of $x = 1$.

$f \downarrow$ to left of $x = 1$.

(e) Summarize your work in a “1D# table” (1st Derivative Number Line Table\(^1\)) table that shows the first derivative test and the conclusions that it gives you.

Solution:

<table>
<thead>
<tr>
<th></th>
<th>neither $x = 0$</th>
<th>l.min $x = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'$</td>
<td>$f' &lt; 0$</td>
<td>$f' &lt; 0$</td>
</tr>
<tr>
<td>$f$</td>
<td>$f \downarrow$</td>
<td>$f \downarrow$</td>
</tr>
<tr>
<td>$f'$</td>
<td>$f' = 0$</td>
<td>$f' = 0$</td>
</tr>
<tr>
<td>$f$</td>
<td>$f \uparrow$</td>
<td>$f \uparrow$</td>
</tr>
</tbody>
</table>

16. Repeat #16 with $f(x) = x^2e^x$, and/or $f(x) = \frac{1}{2}x^2 - 3x + 2 \ln(x)$ and/or $f(x) = xe^{-x^2}$.

Solution:

First function:

$$f(x) = x^2e^x$$
$$f'(x) = 2xe^x + x^2e^x$$
$$2xe^x + x^2e^x = 0$$
$$xe^x(x + 2) = 0$$

$x = 0$ OR $e^x = 0$ OR $x + 2 = 0$

$x = 0, -2$

test points $f'(-3) = 0.15$ [POSITIVE]

$f'(-1) = -0.37$ [NEGATIVE]

$f'(1) = 8.15$ [POSITIVE]

conclusions $x = -2$ is l.max, $x = 0$ is l.min

---

\(^1\)This should be a number line with the following information: you should label on the number line each critical point. Above each critical point you should indicate whether that point is a local max/min/neither. On top of the number line and between the critical points you should indicate whether $f$ is increasing or decreasing. On the bottom of the number line, between the critical points and at each critical point, you should indicated whether $f'$ is $+$, $-$ or $0$. 

l.max \quad l.min
x = -2 \quad x = 0

\begin{array}{c|c|c|c|c}
\hline
& f \nearrow & f \searrow & f \nearrow & f \nearrow \\
f' & > 0 & < 0 & = 0 & > 0 \\
\hline
\end{array}

Second function:

\[ f(x) = \frac{1}{2}x^2 - 3x + 2 \ln(x) \]
\[ f'(x) = x - 3 + \frac{2}{x} \]
\[ x - 3 + \frac{2}{x} = 0 \quad \text{(mult. eqn. by } x) \]
\[ x^2 - 3x + 2 = 0 \]
\[ (x - 2)(x - 1) = 0 \]
\[ x = 1, 2 \]

Test points:
\[ f'(0.5) = 1.5 \quad \text{POSITIVE} \]
\[ f'(1.5) = -0.17 \quad \text{NEGATIVE} \]
\[ f'(2.5) = 0.3 \quad \text{POSITIVE} \]

Conclusions:
\[ x = 1 \text{ is l.max, } x = 2 \text{ is l.min} \]
Third function:

\[ f(x) = xe^{-x^2} \]
\[ f'(x) = e^{-x^2} - 2x^2e^{-x^2} \]
\[ e^{-x^2} - 2x^2e^{-x^2} = 0 \]
\[ e^{-x^2}(1 - 2x^2) = 0 \]
\[ e^{-x^2} = 0 \quad \text{OR} \quad 1 - 2x^2 = 0 \]

no solution \quad \text{OR} \quad x^2 = 1/2

\[ x = \pm 1/\sqrt{2} \approx \pm 0.707 \]
\[ f'(-1) = -0.37 \quad \text{NEGATIVE} \]
\[ f'(0) = 1 \quad \text{POSITIVE} \]
\[ f'(1) = -0.37 \quad \text{NEGATIVE} \]

conclusions \( x = -1/\sqrt{2} \) is l.min, \( x = 1/\sqrt{2} \) is l.max
Global Max/Mins

17. Let \( f(x) = x^{10} - 10x \).

(a) Find the critical points of \( f(x) \).

\[ f'(x) = 10x^9 - 10 \]
\[ f'(x) = 0 \]
\[ 10x^9 - 10 = 0 \]
\[ 10(x^9 - 1) = 0 \]
\[ x^9 - 1 = 0 \]
\[ x^9 = 1 \]
\[ x = \sqrt[9]{1} \]
\[ x = 1 \]

(b) Apply the Global Max/Min test to find the absolute max/min on the interval \( 0 \leq x \leq 2 \).
Solution:

Solution:

\[
\begin{array}{c|c}
  x & y \\
  1 & -9 \\
  0 & 0 \\
  2 & 1004 \\
\end{array}
\]

G. max: \( x = 2, \ y = 1004 \)
G. min: \( x = 1, \ y = -9 \)

18. Repeat #18 with \( f(x) = x^2e^x \) on the interval \([−3, 2]\) and/or \( f(x) = \frac{1}{2}x^2 − 3x + 2\ln(x) \) on the interval \([0.25, 2]\) and/or \( f(x) = xe^{-x^2} \) on the interval \([0, 2]\).

Solution:

First Function: From #17 the critical numbers are \( x = -2, 0 \). We calculate the \( y \)-values at these points and the endpoints.

\[
\begin{array}{c|c}
  x & y = xe^x \\
  -3 & 0.45 \\
  -2 & 0.54 \\
  0 & 0 \\
  2 & 29.6 \\
\end{array}
\]

G. max: \( x = 2, \ y = 29.6 \)
G. min: \( x = 0, \ y = 0 \)

Second Function: From #17 the critical numbers are \( x = 1, 2 \). We calculate the \( y \)-values at these points and the endpoints.

\[
\begin{array}{c|c}
  x & y = \frac{1}{2}x^2 − 3x + 2\ln(x) \\
  0.25 & -3.5 \\
  1 & -2.5 \\
  2 & -2.6 \\
\end{array}
\]

G. max: \( x = 1, \ y = -2.5 \)
G. min: \( x = 0.25, \ y = -3.5 \)

Third Function: From #17 the critical numbers are \( x = \pm 1/\sqrt{2} \approx \pm 0.707 \). We ignore \(-0.707 \) since it is not in the interval \([0, 2]\). We calculate the \( y \)-values at 0.707 and the endpoints.

\[
\begin{array}{c|c}
  x & y = xe^{-x^2} \\
  0 & 0 \\
  1/\sqrt{2} & 0.43 \\
  2 & 0.04 \\
\end{array}
\]

G. max: \( x = 1/\sqrt{2}, \ y = 0.43 \)
G. min: \( x = 0, \ y = 0 \)
Optimizing Cost and Revenue

19. Shown below is a graph of cost and revenue for a certain company:

(a) If the production level is $q = 400$, should the company make a small increase in production? Why or why not?

Solution:
Here’s what the graphs look like with a small part of the tangent lines shown:

It should be clear that the slope of $C$ is less than the slope of $R$. In other words:

\[ MR > MC \Rightarrow \text{profit increases with } q \]
so they should increase production.

(b) Find two critical points of profit. Which one has maximum profit? Why?

Solution:
We find two points where the slopes are the same for $C$ and $R$: these are at $q = 126$ and $q = 730$:

The one that is maximum profit is $q = 730$. The best way to tell this is just to the left of 730 we have $MR > MC$ and just to the right we have $MR < MC$.

20. Repeat parts (a) and (b) from the previous problem, using the table below

<table>
<thead>
<tr>
<th>$q$</th>
<th>0</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>500</th>
<th>600</th>
<th>700</th>
<th>800</th>
<th>900</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$MR$</td>
<td>2.00</td>
<td>1.25</td>
<td>1.05</td>
<td>0.95</td>
<td>0.89</td>
<td>0.84</td>
<td>0.80</td>
<td>0.77</td>
<td>0.75</td>
<td>0.72</td>
<td>0.70</td>
</tr>
<tr>
<td>$MC$</td>
<td>2.10</td>
<td>1.35</td>
<td>0.78</td>
<td>0.39</td>
<td>0.18</td>
<td>0.15</td>
<td>0.30</td>
<td>0.63</td>
<td>1.14</td>
<td>1.83</td>
<td>2.70</td>
</tr>
</tbody>
</table>

Solution: (a) At $q = 400$ we have $MR > MC$ so profit is increasing at that point. They should increase production at least a little.

(b) Critical points are where $MR = MC$. From the table this appears to happen between $q = 100$ and $q = 200$, say $q = 125$, and again between $q = 700$ and $q = 800$, say $q = 725$. The second of these is where the maximum profit is, because $MR > MC$ at $q = 700$ and $MR < MC$ at $q = 800$.

21. A company has a cost function given by $C(q) = 10q + 1$ and revenue function given by $R(q) = 15\sqrt{q}$.

(a) Find the critical point of profit.

Solution:

\[
MC = 10
\]
\[
MR = \frac{15}{2}q^{-1/2}
\]
\[
MC = MR
\]
\begin{align*}
10 &= \frac{15}{2} q^{-1/2} \\
10 &= \frac{15}{2} \cdot \frac{1}{\sqrt{q}} \\
10 &= \frac{15}{2 \sqrt{q}} \\
20 \sqrt{q} &= 15 \text{ (cross multiply)} \\
\sqrt{q} &= \frac{15}{20} \\
\sqrt{q} &= \frac{3}{4} \\
q &= (3/4)^2 \\
q &= 9/16 \approx 0.562
\end{align*}

(b) Is the critical point a maximum for profit? Why?

Solution:
The critical point is a maximum for profit. The best way to see this is to note that just to the left we have \( MR > MC \) and just to the right we have \( MR < MC \):

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graph.png}
\caption{Graph of \( R(q) \) and \( C(q) \).}
\end{figure}

**Average Cost**

22. A company is making simple things with a cost function given by \( C(q) = 5q^2 + 1 \).

(a) Find the average cost at \( q = 3 \).

Solution:

\begin{align*}
\alpha(q) &= \frac{C(q)}{q} \\
&= \frac{5q^2 + 1}{q}
\end{align*}
\[ a(3) = \frac{5(3)^2 + 1}{3} = \frac{46}{3} \approx 15.333 \]

(b) Is the average cost increasing or decreasing at \( q = 3 \)?

Solution:
\[ MC = 10q \]
\[ MC(3) = 30 \]

The average cost is increasing because \( MC(3) > a(3) \).

(c) Repeat the previous two parts graphically, using the graph of \( 5q^2 + 1 \):

Solution:
The average cost equals the slope of the straight line shown below:

\[
a(3) = \text{slope of line} = \frac{\text{rise}}{\text{run}} = \frac{45}{3}
\]

If we look at \( q = 3 \) where the straight line intersects the curve, it's clear that the curve is steeper at that point. Thus, \( MC(3) > a(3) \) and so the average cost is increasing.
23. Repeat parts (a) and (b) from the previous problem using the table below

<table>
<thead>
<tr>
<th>q</th>
<th>0.0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>C(q)</td>
<td>1.00</td>
<td>2.25</td>
<td>6.00</td>
<td>12.25</td>
<td>21.00</td>
<td>32.25</td>
<td>46.00</td>
</tr>
<tr>
<td>MC</td>
<td>0.00</td>
<td>5.00</td>
<td>10.00</td>
<td>15.00</td>
<td>20.00</td>
<td>25.00</td>
<td>30.00</td>
</tr>
</tbody>
</table>

Solution: (a)

\[ a = \frac{C}{q} = \frac{46}{3} = 15.333 \]

(b) Since \( a = 15.3 \) and \( MC = 30 \), we have \( a < MC \) and therefore increasing \( q \) will increase the average cost.

**Elasticity of Demand**

24. You are given a demand function \( q = 1000 - 10p^2 \).

(a) Using the derivative definition, find the elasticity at \( p = 5 \).

Solution:

\[
E = \left| \frac{p \cdot \frac{dq}{dp}}{q \cdot \frac{dq}{dp}} \right|
\]

\( p = 5 \)

\[
q = 1000 - 10(5)^2
\]

\[
= 1000 - 250
\]

\[
= 750
\]

\[
\frac{dq}{dp} = -20p
\]

\[
\frac{dq}{dp} \bigg|_{p=5} = -20(5)
\]

\[
= -100
\]

\[
E = \left| \frac{5 \cdot (-100)}{750} \right|
\]

\[
= \frac{-500}{750}
\]

\[
= \frac{-2}{3}
\]

\[
= \frac{2}{3} \approx 0.666
\]
(b) If we increase price, do we expect $R$ to increase or decrease? Why?

Solution:
We expect $R$ to increase because $E < 1$.

(c) If we increase price 7%, how much of a percentage change do we expect from demand?

Solution:

\[
\%\text{-change of } q = -E \cdot \%\text{-change of } p \\
= -0.6666(0.07) \\
= -0.047 \\
= 4.7\% \text{ decrease}
\]

25. Repeat parts (a)–(c) from the previous problem, using the table below

<table>
<thead>
<tr>
<th>$p$</th>
<th>0.0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
<th>4.5</th>
<th>5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>1000</td>
<td>998</td>
<td>990</td>
<td>978</td>
<td>960</td>
<td>938</td>
<td>910</td>
<td>878</td>
<td>840</td>
<td>798</td>
<td>750</td>
</tr>
</tbody>
</table>

Solution: (a)

\[
E \approx -\frac{\Delta q}{\Delta p} = -\frac{798}{750 - 798} \\
= -\frac{798}{-4.5} \\
= 177.33 \\
= 0.54 \\
\]

(b) We expect $R$ to decrease, because $E < 1$.

(c)

\[
\%\text{-change of } q = -E \cdot \%\text{-change of } p \\
= -0.54(0.07) \\
= -0.038 \\
= 3.8\% \text{ decrease}
\]

Note: the answers between this problem and the previous one are a little different, due to the fact that here we are approximating $E$ with discrete steps of $\Delta p = 0.5$, and in the previous problem we calculated $E$ exactly using derivatives.

Chapter 5: Definite Integrals
Interpreting Integrals as Velocity and Area

26. Set up an integral to find the area below \( y = e^{-x^2} \), above the \( x \)-axis, and between 0 and 1.

Solution:

\[ \int_0^1 e^{-x^2} \, dx \]

27. Set up an integral to find the net distance travelled by a falling object with velocity given by \( v(t) = -9.8t + 12 \), from \( t = 1 \) to \( t = 5 \).

Solution:

\[ \int_1^5 -9.8t + 12 \, dt \]

28. The graph below consists of straight lines and parts of perfect circles. Calculate the indicated integrals

\begin{center}
\begin{tikzpicture}
\begin{axis}[
axis lines=left,\]
\end{axis}
\end{tikzpicture}
\end{center}

(a) \( \int_0^2 f(x) \, dx \)

Solution: \( \int_0^2 f(x) \, dx \) is the area of the first quarter circle, so \( \frac{1}{4} \pi r^2 = \frac{1}{4} \pi (2^2) = \pi \).

(b) \( \int_0^4 f(x) \, dx \)

Solution: \( \int_0^4 f(x) \, dx \) is the area of the first half circle, so \( \frac{1}{2} \pi r^2 = \frac{1}{2} \pi (2^2) = 2\pi \).

(c) \( \int_0^6 f(x) \, dx \)

Solution: \( \int_0^6 f(x) \, dx \) is the area from part (b), plus the area of the first triangle (i.e. the triangle from 4 to 6). This gives \( 2\pi + \frac{1}{2}bh = 2\pi + \frac{1}{2}(2)(2) = 2\pi + 2 \).

(d) \( \int_0^{12} f(x) \, dx \)

Solution: For \( \int_0^{12} f(x) \, dx \) the area of the half circle above the \( x \)-axis cancels with the area of the half circle below the \( x \)-axis. This leaves the area of the double triangle (i.e. the triangle from 4 to 8.) This gives \( \frac{1}{2}bh = \frac{1}{2}(4)(2) = 4 \).
Calculating using tables, graphs and calculators

29. Calculate \( \int_1^{10} x^2 \, dx \) with a Left Hand Riemann Sum and \( n = 9 \).

**Solution:**
We start with the number line

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]

Now we divide into 9 equal pieces:

\[
\Delta x
\]

So the \( x \)-values are 1, 2, 3, \ldots, 10.

For the left hand rule we use 1, 2, \ldots, 9:

\[
\int_1^{10} x^2 \, dx = f(1) \times 1 + f(2) \times 1 + f(3) \times 1 + f(4) \times 1 + f(5) \times 1 \\
+ f(6) \times 1 + f(7) \times 1 + f(8) \times 1 + f(9) \times 1 \\
= (1)^2 \times 1 + (2)^2 \times 1 + (3)^2 \times 1 + (4)^2 \times 1 + (5)^2 \times 1 \\
+ (6)^2 \times 1 + (7)^2 \times 1 + (8)^2 \times 1 + (9)^2 \times 1 \\
= 285
\]

For the right hand rule we use 2, 3, \ldots, 10:

\[
\int_1^{10} x^2 \, dx = f(2) \times 1 + f(3) \times 1 + f(4) \times 1 + f(5) \times 1 + f(6) \times 1 \\
+ f(7) \times 1 + f(8) \times 1 + f(9) \times 1 + f(10) \times 1 \\
= (2)^2 \times 1 + (3)^2 \times 1 + (4)^2 \times 1 + (5)^2 \times 1 + (6)^2 \times 1 \\
+ (7)^2 \times 1 + (8)^2 \times 1 + (9)^2 \times 1 + (10)^2 \times 1 \\
= 384
\]

Note: it is *required* that you first write out the sum in at least one of the ways I’ve done (i.e. either with “\( f(\)” or with “\( (\)\)\(^2\)”) before you calculate the total.

30. (Based on Hughes-Hallet, #10) The marginal cost function for a company is given by

\[
C'(q) = q^2 - 16q + 70 \text{ dollars/unit,}
\]

where \( q \) is the quantity produced.

(a) If \( C(0) = 500 \), set up an integral formula to find the total cost of producing 20 units. Which part of the formula represents the fixed cost and which part represents the total variable cost for this quantity?
Solution:

\[ C(20) = C(0) + \int_0^{20} q^2 - 16q + 70 \, dq \]

\( C(0) \) = fixed cost

\[ \int_0^{20} q^2 - 16q + 70 \, dq = \text{total variable cost} \]

(b) Use your calculator or Desmos to calculate the fixed cost, the total variable cost, and \( C(20) \).

Solution:

\[ \int_0^{20} q^2 - 16q + 70 \, dq \approx 866.67 \]

\( C(0) = 500 \) (given)

\( C(20) = 500 + 866.67 \)
\[ = 1366.67 \]