Chapter 6

Antiderivatives

6.2 Antiderivatives

Definition. An anti-derivative of \( f(x) \) is a function \( F(x) \) such that \( F'(x) = f(x) \).

In other words, an anti-derivative is a function that correctly fills in \( \frac{d}{dx} \)?

Example 1. Guess an anti-derivative of each of the following functions, and then verify your guess:

(a) \( f(x) = 3x^2 \)
(b) \( f(x) = x^5 \)
(c) \( f(x) = x^{17} \)

Solution. (a) We are trying to fill in the following: \( \frac{d}{dx} \)\( ? \) = \( x^3 \). Since taking the derivative decreases the power, our guess for the anti-derivative should be to increase the power. So we guess \( F(x) = x^3 \). Verify: \( \frac{d}{dx} x^3 = 3x^2 \checkmark \).

(b) We are trying to fill in the following: \( \frac{d}{dx} \)\( ? \) = \( x^6 \). With the same reasoning as in part (a) we guess \( F(x) = x^6 \). When we verify this doesn’t quite work: \( \frac{d}{dx} x^6 = 6x^5 \neq x^5 \). We need the coefficient of 6 to disappear when we take the derivative, so we put an extra constant in before we take the derivative that will end up canceling. Guess \( F(x) = \frac{1}{6}x^6 \). Check: \( \frac{d}{dx} \frac{1}{6}x^6 = \frac{1}{6} \cdot 6x^5 = x^5 \checkmark \).

(c) With the same reasoning as in part (b) to cancel we guess \( F(x) = \frac{1}{18}x^{18} \).

Check \( \frac{d}{dx} \frac{1}{18}x^{18} = \frac{1}{18} \cdot 18x^{17} = x^{17} \checkmark \).

Rule. An anti-derivative of \( x^n \) is given by \( \frac{1}{n+1}x^{n+1} \).

Definition. Given a function \( f(x) \), we define the symbol “\( \int f(x) \, dx \)” to be an anti-derivative of \( f(x) \). In other words

1. \( \int f(x) \, dx \) is a function,
2. and \( \frac{d}{dx} \left( \int f(x) \, dx \right) = f(x) \) by definition of the symbol “\( \int f(x) \, dx \)”
Rule. \[ \int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \]

**Theorem 1** (Fundamental Theorem of Calculus II). Let \( f(x) \) be a continuous function defined on \([a, b]\). Then
\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]
where \( F(x) \) is any anti-derivative of \( f(x) \).

**Example 2.** Let \( v(t) = \frac{\sqrt{t}}{16} \) be a velocity function, in miles per hour. Use the Fundamental Theorem of Calculus to find the total distance traveled from \( t = 0 \) to \( t = 60 \). (Compare this with Example 1 in Section 5.2.)

**Solution.** By our work on interpreting definite integrals, we know that \( \int_0^{60} v(t) \, dt \) is the distance. From the fundamental theorem of calculus we know that \( \int_0^{60} v(t) \, dt = P(60) - P(0) \) where \( P(t) \) is an anti-derivative of \( v(t) \). So mostly, we need to find the anti-derivative. Using the power rule we have
\[
P(t) = \int \frac{\sqrt{t}}{16} \, dx
= \int \frac{1}{16} t^{1/2} \, dx
= \frac{1}{16} \frac{t^{3/2}}{3/2} + C
= \frac{1}{24} t^{3/2} + C
\]

Maybe you want to double check that we have the right anti-derivative:
\[
\frac{d}{dt} \left( \frac{1}{24} t^{3/2} \right) = \frac{1}{24} \frac{3}{2} t^{1/2} = \frac{1}{16} \sqrt{t}
\]
In other words let \( P(t) = \frac{2}{3} \frac{t^{3/2}}{16} \).

\[
\text{total distance} = \int_0^{60} \frac{\sqrt{t}}{16} \, dt
= P(60) - P(0)
= \frac{1}{24} (60)^{3/2} - \frac{1}{24} 0^{3/2}
= \frac{1}{24} (60)^{3/2}
\]
This agrees very well with our answer from before.

Here is a list of two basic facts and 5 basic anti-derivative formulas that we should memorize:

**Table 6.1: Basic anti-derivatives and facts**

\[
\begin{align*}
\int C f(x) \, dx &= C \int f(x) \, dx \\
\int x^n \, dx &= \frac{x^{n+1}}{n+1} + C \text{ if } n \neq -1 \\
\int e^x \, dx &= e^x + C \\
\int f(x) \pm g(x) \, dx &= \int f(x) \, dx \pm \int g(x) \, dx \\
\int \frac{1}{x} \, dx &= \ln |x| + C \\
\int a^x \, dx &= \frac{1}{\ln(a)} a^x + C \\
\int e^{kx} \, dx &= \frac{1}{k} e^{kx} + C
\end{align*}
\]

**Example 3.** Find \( \int 5e^x - 10\sqrt{x} + \frac{\pi}{x} - \frac{13.5}{x^2} \, dx \).

**Solution.** To find \( F(x) \) we use the basic facts about anti-derivatives shown in Table 6.1. In this example we break it down into more steps than many readers will need.

Before we start, it’s worth remembering the right way to look at three terms of this function:

\[
\begin{align*}
10\sqrt{x} &= 10x^{1/2} \\
\frac{\pi}{x} &= \pi \cdot \frac{1}{x} \\
\frac{13.5}{x^2} &= 13.5 \cdot x^{-2}
\end{align*}
\]

Now we can break our integral up into pieces:

\[
\begin{align*}
\int 5e^x - 10\sqrt{x} + \frac{\pi}{x} - \frac{13.5}{x^2} \, dx &= 5 \int e^x \, dx - 10 \int x^{1/2} \, dx + \pi \int \frac{1}{x} \, dx - 13.5 \int x^{-2} \, dx \\
&= 5e^x - 10 \frac{x^{3/2}}{3/2} + \pi \ln |x| - 13.5(-x^{-1}) + C \\
&= 5e^x - \frac{20}{3} x^{3/2} + \pi \ln |x| + \frac{1}{x} + C
\end{align*}
\]

**Comments.** Notation: we usually write \( F(x) \bigg|_a^b \) as an abbreviation for \( F(b) - F(a) \).

**Example 4.** The function \( C'(t) = 0.716e^{0.0192t} \) pretty accurately models the rate of change of CO\(_2\) in the atmosphere (where \( t \) is years since 1950 and the units of \( C'(t) \) are ppm per year). In 1950 the CO\(_2\) level was about 310 ppm (parts per million).
(a) Use your calculator/computer to find the amount of atmospheric carbon predicted in calendar year 2020.
(b) Find a formula for $C(t)$ and graph it for the years 1950–2020.

**Solution. (a)**

Carbon in 2020 = $C(70)$

\[
C(0) + \int_{0}^{70} C'(t) dt
\]

\[= 310 + \int_{0}^{70} 0.716e^{0.0192t} \, dt \]

\[= 310 + \left. \frac{0.716e^{0.0192t}}{0.0192} \right|_{0}^{70} \]

\[= 310 + \left( \frac{0.716e^{0.0192 	imes 70}}{0.0192} - \frac{0.716e^{0}}{0.0192} \right) \]

\[= 310 + (37.29e^{0.0192t} - 37.29) \]

\[= 272.7 + 37.292e^{0.0192t} \]

Shown below is the graph of $C(t)$, together with the actual data points of measured carbon in the atmosphere.

**Example 5.** Find $\int_{5}^{10} \frac{3}{x} - \frac{11}{x^4} \, dx$. 
Solution.

\[
\int_5^{10} \frac{3}{x} - \frac{11}{x^4} \, dx = \int_5^{10} \frac{1}{x} - 11x^{-4} \, dx \\
= 3 \ln(x) - 11 \left. \frac{x^{-3}}{-3} \right|_5^{10} \\
= 3 \ln(x) + \frac{11}{3} \left. \frac{1}{x^3} \right|_5^{10} \\
= 3 \ln(10) + \frac{11}{3} \frac{1}{10^3} - \left( 3 \ln(5) + \frac{11}{3} \frac{1}{5^3} \right)
\]