5.4 The Interpretations of the Definite Integral

Fact. If we apply an integral to a rate of change of any type of quantity we get the following:

\[ \int f(t) \, dt = \text{sum of terms like } f(t) \times \Delta t \]

so \[ \int \text{rate of change} \, dt = \text{sum of rate of change } \times \Delta t \]

\[ = \text{sum of changes} \quad \text{(because rate of change } \times \Delta t = \text{change)} \]

\[ = \text{total change} \]

In other words

\[ \int_a^b \text{rate of change} \, dt = \text{total change from } t = a \text{ to } t = b \]

Note the following:

- The units are found as follows:
  
  units of the total change = units of rate of change \times units of time

  These will be the units of whatever quantity we were measuring the rate of change of.

- “Total change” means “net change”. In other words, it’s the increase minus the decrease of whatever quantity we were measuring the rate of change of.

Example 1. The table below shows the rate of flow \( F \) of water out of a huge water tank, in gallons per minute.

\[
\begin{array}{c|cccccccc}
 t & 0 & 5 & 10 & 15 & 20 & 25 & 30 \\
 F & 1000 & 633 & 414 & 264 & 155 & 69 & 0 \\
\end{array}
\]

(a) What does \( \int_0^{30} F(t) \, dt \) mean? What are its units?

(b) Estimate \( \int_0^{30} F(t) \, dt \).

Solution. (a)

\( F \times \Delta t = \text{rate of change of water in tank } \frac{\text{gallons}}{\text{minute}} \times \text{minutes} \)

\[ = \Delta \text{ amount of water in tank in gallons} \]

\[ \int_0^{30} F \, dt = \text{total change in amount of water in tank in gallons} \]

\[ = \text{total amount of water that flowed out of tank} \]

(b)

\[ \int_0^{30} F(t) \, dt \approx \text{sum of things like } F(t) \times \Delta t \]

\[ = 1000 \times 5 + 633 \times 5 + 414 \times 5 + 264 \times 5 + 155 \times 5 + 69 \times 5 \]

\[ = 12675 \text{ gal} \]
Example 2. A certain population $P$ of bacteria has growth rate given by the following formula:

$$\frac{dP}{dt} = \frac{1576e^{-0.7944t}}{(1 + 31e^{-0.7944t})^2} \quad (t = \text{days})$$

Set up an integral, that represents the total change in bacteria from $t = 0$ to $t = 15$, and then use your calculator/computer to numerically evaluate this integral.

Solution.

<table>
<thead>
<tr>
<th>Total change of population from $t = 0$ to $t = 15$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_0^{15} \text{rate of change of population } dt$</td>
</tr>
<tr>
<td>$= \int_0^{15} \frac{1576e^{-0.7944t}}{(1 + 31e^{-0.7944t})^2} , dt$</td>
</tr>
<tr>
<td>$= 61.9 , #\text{bacteria}$</td>
</tr>
</tbody>
</table>

In other words, there are 62 more bacteria at the end of 15 days than we started with.