5.3 The Definite Integral as Area

Example 1. The function \( f(x) = \frac{7}{2}\sqrt{-x^2 + 8x - 12} \) does a reasonably good job modeling the shape of one McDonald’s Golden Arch.

Find the area under one arch by using the function \( f(x) \), an integral, and your calculator.
Example 2. The chart below shows (estimates) for cumulative percentages of population, and cumulative percentages of income for Census Tract 1901 (Franklin Square) in Baltimore, as of calendar year 2017:

<table>
<thead>
<tr>
<th>population %</th>
<th>0.000</th>
<th>0.298</th>
<th>0.393</th>
<th>0.500</th>
<th>0.548</th>
<th>0.647</th>
<th>0.701</th>
<th>0.746</th>
<th>0.790</th>
<th>0.863</th>
<th>0.920</th>
<th>0.969</th>
<th>0.975</th>
<th>0.982</th>
<th>1.000</th>
</tr>
</thead>
<tbody>
<tr>
<td>income %</td>
<td>0.000</td>
<td>0.065</td>
<td>0.097</td>
<td>0.146</td>
<td>0.174</td>
<td>0.243</td>
<td>0.288</td>
<td>0.330</td>
<td>0.377</td>
<td>0.399</td>
<td>0.476</td>
<td>0.578</td>
<td>0.695</td>
<td>0.714</td>
<td>0.750</td>
</tr>
</tbody>
</table>

For instance, this means that roughly the bottom 30% of population (i.e. 30% in terms of number of people, “bottom” in terms of income) earns 6.5% of the total income.

If it helps to visualize this, we can connect the dots with a series of straight lines, and get something that looks like a curve:

![Figure 5.1: Lorentz curve for Baltimore Census Tract 1901 in year 2017](image)

This is called the Lorentz curve. There are other kinds of Lorentz curves: for wealth instead of income, for educational access, for insurance claims, etc. These curves also change over time and location: for instance from the beginning of the 20th century to the end, or from Baltimore to the US as a whole, etc.

Estimate the area under this Lorentz curve.
Example 3. The graph of \( f(x) \) consists of straight lines and a semicircle, shown below.

(a) Find \( \int_0^2 f(x) \, dx \) exactly. 
(b) Find \( \int_2^6 f(x) \, dx \) exactly. 
(c) Find \( \int_0^7 f(x) \, dx \) exactly. 
(d) Find \( \int_0^{10} f(x) \, dx \) exactly.