4.2 Intervals of Increase, Decrease, Concavity, Inflection points

Definition. For a function \( f(x) \), an inflection point is a number \( x = c \) such that \( f(x) \) changes concavity at \( x = c \).

We find inflection points the same way we find local max/mins: (1) take the second derivative, (2) set it equal to 0, (3) solve this equation, (4) confirm your answers by looking at the graph.

Example 1. For the function \( f(x) = x^4 - x^3 - x^2 \) do the following:

(a) Find the critical points and the possible inflection points algebraically.

(b) Using your answers from part (a), and by looking at a graph, find the intervals of increase and decrease, and the intervals of concavity, and the location of the local max/mins.

Solution. (a)

\[
\begin{align*}
f(x) &= x^4 - x^3 - x^2 \\
f'(x) &= 4x^3 - 3x^2 - 2x \\
f''(x) &= 0 \\
4x^3 - 3x^2 - 2x &= 0 \\
x(4x^2 - 3x - 2) &= 0 \\
x &= \frac{3 \pm \sqrt{9 - 4(4)(-2)}}{8} \\
&= \frac{3 \pm \sqrt{41}}{8} \\
x &= 0 \quad \text{or} \quad x \approx -0.43, 1.18
\end{align*}
\]

So there are three critical points

C.P. = \(-0.43, 0, 1.18\).

\[
\begin{align*}
f''(x) &= 12x^2 - 6x - 2 \\
f'''(x) &= 0 \\
12x^2 - 6x - 2 &= 0 \\
x &= \frac{6 \pm \sqrt{36 - 4(12)(-2)}}{24} \\
x &= \frac{6 \pm \sqrt{132}}{24} \\
x &\approx -0.23, 0.73
\end{align*}
\]

So there are two possible inflection points (“possible” because we don’t know yet if the concavity changes there, but it should be emphasized these are all the possibilities, if there are any inflection points, they must be one of these):

I.P. (possible) = \(-0.23, 0.73\)
(b) Now, if we try to graph \( f(x) \), the standard window shows this:

That’s not a great view of the graph and from part (a) we know that there’s more going on, for instance we should be able to see three critical points.

The following shows the whole graph:

This graph shows us most of the interesting information, although it’s hard to read exactly. In particular, I don’t think we could use the graph alone to see exactly where the inflection points are, but since we found the possible inflection points algebraically, we can use the graph to confirm that they are inflection points.

Based on the graph, combined with the algebra we did earlier, we can identify the interesting features:

\[
\begin{align*}
  f \uparrow: & \quad (-0.43, 0) \cup (1.18, \infty) \\
  f \downarrow: & \quad (-\infty, -0.43) \cup (0, 1.18) \\
  \text{l. mins:} & \quad x = -0.43 \text{ and } x = 1.18 \\
  \text{l. max:} & \quad x = 0, \\
  f\text{C.U.} : & \quad (-\infty, -0.28) \cup (0.73, \infty) \\
  f\text{C.D.} : & \quad (-0.28, 0.73)
\end{align*}
\]
Infl. pts: $x = -0.28, 0.73$.

**Example 2.** Let $f(x)$ be defined by the graph below

(a) Over which intervals does it appear that $f(x)$ is increasing? Decreasing?
(b) Over which intervals does it appear that $f(x)$ is concave up? Concave down?
(c) Where are the local max/mins (ignore endpoints)? Where does the concavity change?

**Solution.**
(a) Based on where the graph is going up and where it's going down, we can say that it's increasing from $-1.3$ to $1.3$, and again from $2.2$ to $2.8$ (remember: we record just the $x$-values, not the $y$-values).
   It's decreasing from from $-1.6$ to $-1.3$ and from $1.3$ to $2.2$, and from $2.8$ to $3.1$.
   Here's how we can describe this some information more compactly:

\[
\begin{align*}
  f \uparrow & \quad -1.3 < x < 1.3 \text{ and } 2.2 < x < 2.8 \\
  f \downarrow & \quad -1.6 \leq x \leq -1.3 \text{ and } 1.3 \leq x \leq 2.2 \text{ and } 2.8 \leq x \leq 3.1
\end{align*}
\]

Here's an even more compact notation:

\[
\begin{align*}
  f \uparrow & \quad (-1.3, 1.3) \cup (2.2, 2.8) \\
  f \downarrow & \quad (-1.6, -1.3) \cup (1.3, 2.2) \cup (2.8, 3.1)
\end{align*}
\]

(b) Concave up: from $-1.6$ to $-0.9$ and from $0$ to $0.9$, and from $1.8$ to $2.5$.
   Concave down: from $-0.9$ to $0$, from $0.9$ to $1.8$, and from $2.5$ to $3.1$.
   Here's more compact notation:

\[
\begin{align*}
  f \text{ C.U.} & : \quad -1.6 \leq x \leq -0.9 \text{ and } 0 \leq x \leq 0.9 \text{ and } 1.8 \leq x \leq 2.5 \\
  f \text{ C.D.} & : \quad -0.9 \leq x \leq 0 \text{ and } 0.9 \leq x \leq 1.8 \text{ and } 2.5 \leq x \leq 3.1
\end{align*}
\]

Here's the most compact notation:

\[
\begin{align*}
  f \text{ C.U.} & : \quad (-1.6, -0.9) \cup (0, 0.9) \cup (1.8, 2.5)
\end{align*}
\]
\[ f \text{ C.D. : } (-0.9, 0) \cup (0.9, 1.8) \cup (2.5, 3.1) \]

(c) It appears that \( x = -1.3 \) and \( x = 2.2 \) are local mins.
The local maxs are \( x = 1.3 \) and \( x = 2.8 \).
The concavity changed at \( x = -0.9, 0, 0.9, 1.8, 2.5 \).