3.4 Product and Quotient Rules

Example 1. In Fall 2018, the undergraduate enrollment at Loyola University Maryland was 3886 and the tuition was $47520 per year (information taken from the 2018–2019 catalogue). Hypothetically, suppose that the school is thinking of increasing it’s tuition by $125. Suppose that this would cause the enrollment to decrease by 3 students.

What would be the change in revenue?

Solution. Let \( \Delta R \) be the change in revenue, \( \Delta p = 125 \) the change in tuition, and \( \Delta q = -3 \) the change in number of students. The first thing to note is that

\[
\Delta R \neq \Delta q \ast \Delta p
\]

In fact, if you did this calculation you would find that \( \Delta R \) is negative, which is wrong.

The problem is that \( \Delta p \) needs to be combined with all the students, not just the 3 students who don’t come. After all, most of the students will be paying more tuition, and so this will increase the revenue. Similarly \( \Delta q \) needs to be combined with the entire tuition, not the change in tuition.

Here is the right way to find \( \Delta R \):

\[
\Delta R = \text{new revenue} - \text{old revenue}
= (p + \Delta p) \cdot (q + \Delta q) - p \cdot q
= p \cdot q + p \cdot \Delta q + \Delta p \cdot q + \Delta p \cdot \Delta q - p \cdot q
= p \cdot \Delta q + \Delta p \cdot q + \Delta p \cdot \Delta q
= 47520(-3) + 125(3886) + 125(-3)
= 343190 - 375
= 343015
\]

There is one more thing to note here:

\[
\Delta R \approx 343190 = p \cdot \Delta q + \Delta p \cdot q.
\]

This approximation is pretty accurate because \( \Delta p \) and \( \Delta q \) are both relatively small, and it would be even more accurate if \( \Delta p \) and \( \Delta q \) were smaller. If we imagine what happens as \( \Delta p \) and \( \Delta q \) both become so small that they approach 0, then we get the product rule:

\[
R' = p \cdot q' + p \cdot q'.
\]

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**Product rule**

- Function Notation: \((f \cdot g)' = f' \cdot g + f \cdot g'\)
- Leibniz Notation: \[
\frac{d}{dx}(uv) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}
\]
- “Word Notation”: The derivative of the first, times the second, plus the first, times the derivative of the second.
Figure 3.1: Intuitive, graphical, understanding of product rule

$f = \text{length of rectangle}, \ g = \text{height}, \ fg = \text{area.}
(fg)' = \text{rate of change} \approx \text{change of area (purple)}

$f' \approx \text{change length}, \ f'g \approx \text{change length} \times \text{height (blue)}, \ g' \approx \text{change width}, \ fg' \approx \text{change height} \times \text{length (red)}.$

Comments.

\[
\text{purple} \approx \text{red} + \text{blue}:
(fg)' = f' \cdot g + f \cdot g'.
\]

Example 2. Find the derivative of $x^2e^{5x}$.

Solution.

\[
f = x^2 \quad g = e^{5x} \\
f' = 2x \quad g' = 5e^{5x} \
\]

\[
f'g + fg' = 2xe^{5x} + x^25e^x
\]

Example 3. Find $\frac{d}{dx}(3x + e^x)(x^2 - 4e^x)$.

Solution. Identify $f$ and $g$:

\[
\underbrace{(3x + e^x)}_f \underbrace{(x^2 - 4e^x)}_g
\]

Calculate $f'$ and $g'$:

\[
f' = 3 + e^x \quad g' = 2x - 4e^x
\]

Put it all together using the product rule:

\[
f'g + fg' = (3 + e^x)(x^2 - 4e^x) + (3x + e^x)(2x - 4e^x).
\]

Example 4. Find $\frac{d}{dx}(x + e^{3x-1})\left(\frac{1}{x} + x\right)$.
Solution. Identify $f$ and $g$:

\[
\left( \frac{x + e^{3x-1}}{x} \right) = \frac{1 + x}{x}
\]

Calculate $f'$ and $g'$

\[
f' = 1 + e^{3x-1} \cdot 3
\]
\[
g' = -\frac{1}{x^2} + 1
\]

Put it all together using the product rule:

\[
f' \cdot g + f \cdot g' = (1 + 3e^{3x-1}) \cdot \left( \frac{1}{x} + x \right) + (x + e^{3x-1}) \cdot \left( -\frac{1}{x^2} + 1 \right)
\]

**Quotient Rule**

| Function Notation: $\left( \frac{f}{g} \right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$ |
| Leibniz Notation: $\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{du}{dx} \cdot v - u \cdot \frac{du}{dx}$ |

“Word Notation”:

The derivative of the top, times the bottom, minus
the top times the derivative of the bottom, everything
over the bottom squared.

Example 5. Find $\frac{d}{dx} \frac{2x + 1}{e^x + x}$.

Solution. Identify $f$ and $g$

\[
2x + 1 \leftarrow f
\]
\[
e^x + x \leftarrow g
\]

Calculate $f'$ and $g'$

\[
f' = 2
\]
\[
g' = e^x + 1
\]

Put it all together using the quotient rule

\[
\frac{f' \cdot g - f \cdot g'}{g^2} = \frac{2(e^x + x) - (2x + 1)(e^x + 1)}{(e^x + x)^2}
\]

Example 6. Find the derivative of $\frac{t + \sqrt{t}}{t^2 + \ln(t)}$.

Solution. Identify $f$ and $g$

\[
t + \sqrt{t} \leftarrow f
\]
\[
t^2 + \ln(t) \leftarrow g
\]
Calculate $f'$ and $g'$

$$f' = 1 + \frac{1}{2\sqrt{t}}$$
$$g' = 2t + \frac{1}{t}$$

Put it all together using the quotient rule

$$\frac{f' \cdot g - f \cdot g'}{g^2} = \frac{(1 + \frac{1}{2\sqrt{t}})(t^2 + \ln(t)) - (t + \sqrt{t})(2t + \frac{1}{t})}{(t^2 + \ln(t))^2}$$