CHAPTER 3. RULES FOR DERIVATIVES

3.3 The Chain Rule

Comments. The Chain Rule is the most interesting one we have learned so far, and it is one of the most complicated we will learn. It is one of the rules we will learn for taking the derivative of a combination of functions. In particular, this rule will tell us how to take the derivative when one function is inside of another.

Example 1. (Based on Hughes-Hallett, 5e, 3.3 Example 1)
Let $G$ be the amount of gas, in gallons, consumed by a car on a particular trip, let $s$ be the distance traveled, in miles, and let $t$ be the amount of time that has elapsed, in hours.

Then $G$ is a function of $s$ and $s$ is function of $t$. We can combine these statements and conclude that $G$ is a function of $t$.

Let 0.05 gallons of gas be consumed for each mile traveled, and suppose that the car is traveling at $30 \text{mi/hr}$. How fast is gas being consumed? Calculate your answer in Leibniz notation, and give units.

Solution. The information we are given can be summarized this way

\[ \frac{dG}{ds} = 0.05 \text{gal/mi}, \]
\[ \frac{ds}{dt} = 30 \text{mi/hr}, \]

and we want to find this:

\[ \frac{dG}{dt} = ? \]

If we give these numbers and units the usual interpretation, we have this:

- We use 0.05 gallons by driving one mile,
- We drive 30 miles in one hour,
- How many gallons will we use in one hour?

The right way to answer this question is to combine the given information with multiplication:

\[
\text{gallons in one hour} = 30 \text{ miles in one hour} \times 0.05 \text{ gallons in one mile} = 1.5 \text{ gal/hr}.
\]

We can summarize this calculation in Leibniz notation this way:

\[ \frac{dG}{dt} = \frac{dG}{ds} \cdot \frac{ds}{dt}. \]

Rule (Chain Rule in Leibniz Notation). If $y$ is a function of $z$ and $z$ is a function of $t$ then

\[ \frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dt}. \]

Example 2. Break each of the following complicated functions into a combination of two simple functions. In each case write the result as a function of $z$, where $z$ is the “inside” function.

(a) $y = \sqrt{t^2 + 2t}$
Solution. (a) 
\begin{align*}
y & = \sqrt{z} \\
z & = t^2 + 2t
\end{align*}

(b) 
\begin{align*}
y & = 5z^8 \\
z & = 2t + 7
\end{align*}

(c) 
\begin{align*}
y & = \frac{11}{z} \\
z & = t^2 + 1
\end{align*}

(d) 
\begin{align*}
y & = -7.2e^z \\
z & = t^2
\end{align*}

(e) 
\begin{align*}
y & = \frac{1}{2} \ln(z) \\
z & = 3t^2 + 5
\end{align*}

Example 3. Find the derivatives of the following functions. Use z for the inside function, and use the Leibniz notation for the chain rule.

(a) \( y = 5(-3t^2 + 2t + 7)^{11} \).

(b) \( y = \frac{7}{3} \ln(t^3 + t) \).

Solution. (a) 
\begin{align*}
y & = 5z^{11}, \quad z = -3t^2 + 2t + 7 \\
\frac{dy}{dt} & = \frac{dy}{dz} \cdot \frac{dz}{dt} \\
& = 5 \cdot 11z^{10}(-6t + 2) \\
& = 55(-3t^2 + 2t + 7)^{10}(-6t + 2)
\end{align*}

(b) 
\begin{align*}
y & = \frac{7}{3} \ln(z), \quad z = t^3 + t \\
\frac{dy}{dt} & = \frac{dy}{dz} \cdot \frac{dz}{dt} \\
& = \frac{7}{3} \cdot \frac{1}{z} \cdot (3t^2 + 1) \\
& = \frac{7}{3} \cdot \frac{1}{t^3 + t} \cdot (3t^2 + 1)
\end{align*}
Comments. To some degree, a lot of people learn the chain rule one function at a time, as it applies to the outside function. In other words, they learn how the chain rule works on a case-by-case basis. There’s nothing wrong with this (as long as the student can also use the chain rule in a new or general situation that doesn’t fit any of the case-by-cases). In any case, here’s how the chain rule looks for various functions that we know.

Rule. The following table shows the derivative of a variety of basic functions, and then a chain rule version for each basic function.

<table>
<thead>
<tr>
<th>Usual version</th>
<th>Chain rule version</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d}{dx} \frac{1}{x} = - \frac{1}{x^2}$</td>
<td>$\frac{d}{dx} \frac{1}{x} = - \frac{1}{x^2} \cdot \frac{d}{dx} x$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$</td>
<td>$\frac{d}{dx} \sqrt{\sqrt{x}} = \frac{1}{2\sqrt{\sqrt{x}}} \cdot \frac{d}{dx} \sqrt{x}$</td>
</tr>
<tr>
<td>$\frac{d}{dx} x^n = nx^{n-1}$</td>
<td>$\frac{d}{dx} x^n = nx^{n-1} \cdot \frac{d}{dx} x$</td>
</tr>
<tr>
<td>$\frac{d}{dx} e^x = e^x$</td>
<td>$\frac{d}{dx} e^x = e^x \cdot \frac{d}{dx} x$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \ln(x) = \frac{1}{x}$</td>
<td>$\frac{d}{dx} \ln(x) = \frac{1}{x} \cdot \frac{d}{dx} x$</td>
</tr>
</tbody>
</table>

Example 4. Find the following derivatives

(a) $\frac{d}{dt} (3t^2 - 7t)^{11}$

(b) $\frac{d}{dt} \sqrt{t^2 - 1}$

(c) $\frac{d}{dt} \frac{1}{10t - 4}$

(d) $\frac{d}{dt} e^{1.03t - 1}$

(e) $\frac{d}{dt} \ln(2t + e^t)$

Solution. In this problem I’m not going to use the z notation, but you don’t have to follow me. What I mean is: if you are at all confused by the steps that take place, go ahead and use z and write out the extra steps where you can clearly label and see what happened. But, with practice people can often keep track of some stuff in their head. Below I do each problem twice, once just showing with a box where the inside stuff is in each function, and basically using the box notation for the chain rule that I gave above.

(a) 

$$\frac{d}{dt} (3t^2 - 7t)^{11} = 11 (3t^2 - 7t)^{10} \cdot \frac{d}{dt} (3t^2 - 7t)$$

$$= 11(3t^2 - 7t)^{10} \cdot (6t - 7)$$
(b) \[
\frac{d}{dt} \sqrt{t^2 - 1} = \frac{1}{2} (t^2 - 1)^{-1/2} \cdot \frac{d}{dt} (t^2 - 1) = \frac{1}{2} (t^2 - 1)^{-1/2} \cdot (2t)
\]

(c) \[
\frac{d}{dt} \frac{1}{10t - 4} = -\frac{1}{(10t - 4)^2} \cdot \frac{d}{dt} (10t - 4) = -\frac{1}{(10t - 4)^2} \cdot (10)
\]

(d) \[
\frac{d}{dt} e^{1.03t - 1} = e^{1.03t - 1} \cdot \frac{d}{dt} (1.03t - 1) = e^{1.03t - 1} \cdot (1.03)
\]

(e) \[
\frac{d}{dt} \ln (2t + e^t) = \frac{1}{2t + e^t} \cdot \frac{d}{dt} (2t + e^t) = \frac{1}{2t + e^t} \cdot (2 + e^t)
\]

It may be worth mentioning that as people practice the chain rule, at least for simple examples, rather than using \(y\) and \(z\), or even using boxes and filling them in, they tend to just focus their attention on part of the formula at a time, and say the right words in their head to prompt them to do the right thing at each step. Roughly, they things like “take the derivative of the outside” and make sure they are looking at the outside function while they do this. Then they say “don’t change the inside” while they write the inside again, without changing it. Then they say “multiply by the derivative of the inside” and do that. The results can be color coded, as shown below.

\[
\frac{d}{dt} (3t^2 - 7t)^{11} = 11(3t^2 - 7t)^{10} \cdot (6t - 7)
\]

\[
\frac{d}{dt} \sqrt{t^2 - 1} = \frac{1}{2} (t^2 - 1)^{-1/2} \cdot (2t)
\]
CHAPTER 3. RULES FOR DERIVATIVES

\[
\frac{d}{dt} \frac{1}{10t - 4} = \frac{-1}{(10t - 4)^2} \quad (10)
\]

- deriv. of outside don’t change inside
- deriv. of inside

\[
\frac{d}{dt} e^{1.03t - 1} = e^{1.03t - 1} \cdot (1.03) \quad (1.03)
\]

- deriv. of outside don’t change inside
- deriv. of inside

\[
\frac{d}{dt} \ln (2t + e^t) = \frac{1}{(2t + e^t)} \cdot (2 + e^t)
\]

- deriv. of outside don’t change inside
- deriv. of inside