1.7 Exponential Growth and Decay

Comments. This section is designed to give the reader practice with an important family of problems. The main idea is to take a formula such as \( f(t) = Ce^{rt} \), and use it to model some given data. Often the data will described as part of an applied problem.

There are many, many applications of exponential growth and decay: population growth, doubling time, half-life, compound interest, present and future value, to mention a few.

In this section we distinguish between two ways of interpreting percentage change, as shown

<table>
<thead>
<tr>
<th>If ( r ) is the discrete annual/monthly/hourly/per-time-period percentage change (written as a decimal), then we use</th>
<th>( y = Ca^t ) with ( a = 1 + r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( r ) is the continuous percentage change (written as a decimal) then we use</td>
<td>( y = Ce^{rt} )</td>
</tr>
</tbody>
</table>

Example 1. Find \( C \) and \( r \) such that \( f(t) = Ce^{rt} \) goes through the points \((0, 7.3)\) and \((2.9, 17.8)\).

Solution. To solve for \( C \) and \( r \) we need to plug the numbers in. Recall that “\( f(t) \)” represents what we usually call “\( y \)”, and “\( t \)” represents what we usually call “\( x \)”. Thus, the point \((0, 7.3)\) can plugged in as \( y = 0 \) and \( x = 7.3 \):

\[
\begin{align*}
  f(t) &= Ce^{rt} \\
  7.3 &= Ce^{r(0)} & \text{“}y\text{”} = 7.3, \quad \text{“}x\text{”} = 0 \\
  7.3 &= C(1) \\
  C &= 7.3
\end{align*}
\]

Now we plug in \((2.9, 17.8)\), to solve for \( r \). This will take more steps:

\[
\begin{align*}
  f(t) &= 7.3e^{rt} \\
  17.8 &= 7.3e^{r(2.9)} & \text{“}y\text{”} = 17.8, \quad \text{“}x\text{”} = 2.9 \\
  17.8 &= 7.3e^{2.9r} \\
  \frac{17.8}{7.3} &= e^{2.9r} \\
  \ln(17.8/7.3) &= \ln(e^{2.9r}) \\
  \ln(17.8/7.3) &= 2.9r \\
  r &= \frac{\ln(17.8/7.3)}{2.9} \\
  &\approx 0.30735
\end{align*}
\]
Example 2. (Hughes-Hallett, 4e, 1.7#11) A cup of coffee contains 100 mg of caffeine, which leaves the body at a continuous rate of 17% per hour.

(a) Write a formula for the amount, $A$ mg, of caffeine in the body $t$ hours after drinking a cup of coffee.

(b) Find the half-life of caffeine.

Solution. (a) Since the problem says “continuous rate of . . .” we must use the formula $A = Pe^{rt}$. Since caffeine is leaving the body, the amount in the body is decreasing, and so the amount in the body is modelled by a rate of $-17\%$. Finally, we change $-17\%$ to a decimal of $r = -0.17$. Putting all this together we get

$$A = 100e^{-0.17t}.$$ 

(b) “Half-life” means how much time will go by for the amount of caffeine to be cut in half. In other words, we want to solve for $t$ such that

$$50 = 100e^{-0.17t}.$$ 

This should be familiar:

$$\frac{50}{100} = e^{-0.17t}$$

$$\frac{1}{2} = e^{-0.17t}$$

$$\ln\left(\frac{1}{2}\right) = \ln(e^{-0.17t})$$

$$\ln\left(\frac{1}{2}\right) = -0.17t$$

$$t = \frac{\ln(1/2)}{-0.17} \approx 4.077336356$$