MA 151, Spring 2014, Midterm 3 preview solutions:

1. Find the following derivatives

   (a) $\frac{d}{dx}x^7$

   (b) $\frac{d}{dx}\sqrt{x}$

   (c) $\frac{d}{dx}\frac{1}{x^3}$

   (d) $\frac{d}{dx}5e^x$

   (e) $\frac{d}{dx}\frac{1}{2}\ln(x)$

   Solution. (a) $\frac{d}{dx}x^7 = 7x^6$

   (b) $\frac{d}{dx}\sqrt{x} = \frac{1}{2}x^{-1} \quad \left(\text{or } \frac{1}{2\sqrt{x}}\right)$.

   (c) $\frac{d}{dx}\frac{1}{x^3} = -3x^{-4} \quad \left(\text{or } \frac{-3}{x^4}\right)$.

   (d) $\frac{d}{dx}5e^x = 5e^x$

   (e) $\frac{d}{dx}\frac{1}{2}\ln(x) = \frac{1}{2}\cdot\frac{1}{x} \quad \left(\text{or } \frac{1}{2x}\right)$.

2. Find the equation of the tangent line to $f(x) = -x^4 - 32\cdot\frac{1}{x^2}$ at the point $x = 4$. Simplify the numbers in your answer.

   Solution.

$$
\begin{aligned}
y &= m(x - x_0) + y_0 \\
x_0 &= 4 \\
y_0 &= f(4) \\
&= -(4)^4 - 32\cdot\frac{1}{4^2} \\
&= -256 - \frac{32}{16} \\
&= -258 \\
f'(x) &= -4x^3 - 32(-2)x^{-3} \\
&= -4x^3 + 64\cdot\frac{1}{x^3} \\
m &= f'(4) \\
&= -4(4)^3 + 64\cdot\frac{1}{4^3} \\
&= -256 + \frac{64}{64} \\
&= -255 \\
y &= -255(x - 4) - 258
\end{aligned}
$$

3. Find the following derivatives, do not simplify your answers.

(a) $\dfrac{d}{dq}(23e^{3q+1} - 5\ln(5q+11))$

(b) $\dfrac{d}{dx}(2x^2 - 5x)(3e^x + \ln(x))$

(c) $\dfrac{d}{dx}\dfrac{5 - 4x + 9x^2}{2 + 10x + e^x}$

Solution. (a)

$$\frac{d}{dx}e^{\square} = e^{\square} \cdot \square'$$

$$\frac{d}{dx}\ln(\square) = \frac{1}{\square} \cdot \square'$$

$$\frac{d}{dq}(23e^{3q+1} - 5\ln(5q+11)) = 23e^{3q+1}(3) - 5\frac{1}{5q+11}(5)$$

(b)

$$\underbrace{(2x^2 - 5x)}_{f}\underbrace{(3e^x + \ln(x))}_{g}$$

$$f' = 4x - 5$$

$$g' = 3e^x + \frac{1}{x}$$

$$f'g + fg' = (4x-5)(3e^x + \ln(x)) + (2x^2 - 5x)(3e^x + \frac{1}{x})$$

(c)

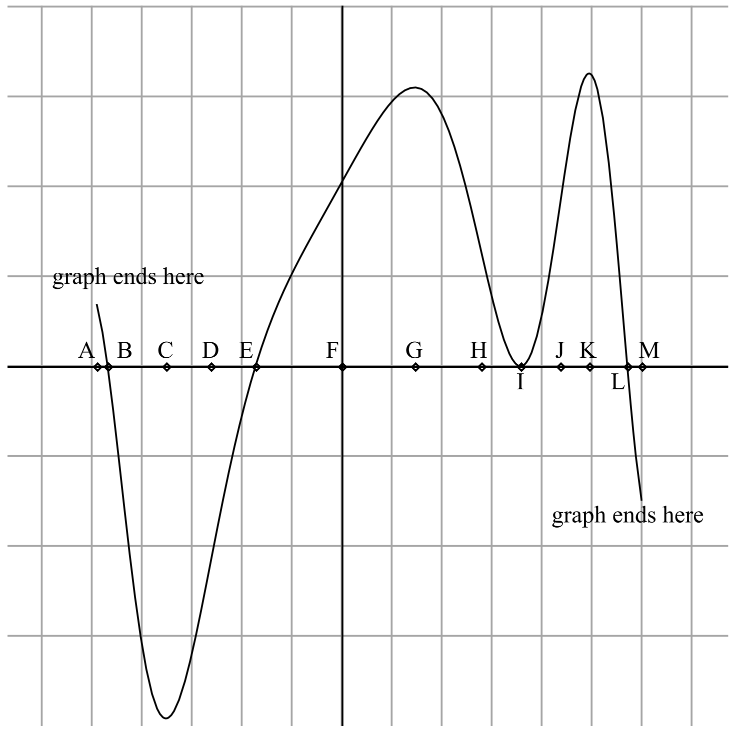
$$\frac{\overbrace{5 - 4x + 9x^2}^{f}}{\underbrace{2 + 10x + e^x}_{g}}$$

$$f' = 18x - 4$$

$$g' = e^x + 10$$

$$\frac{f'g - fg'}{g^2} = \frac{(18x-4)(2+10x+e^x) - (5-4x+9x^2)(e^x+10)}{(2+10x+e^x)^2}$$

4. Assume that the following is a graph of $f(x)$.

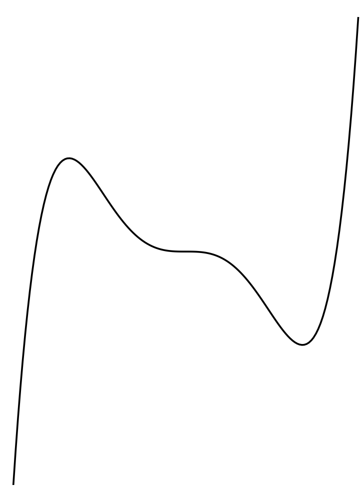


List all the critical points of $f(x)$, and identify each as a local max/min/neither.

Solution.
The critical points are where $f(x)$ has zero slope. These are C, G, I and K. The local max's are G and K. The local mins are C and I.

5. Let $f(x) = 3x^5 - 5x^3$.

   A plot of $f(x)$ is shown below (but somehow my computer seems to be broken: it doesn't show any of the axes, or numbers or tickmarks. Sorry.)



   (a) Find the critical points of $f(x)$.
   (b) Make a "1D# table" (1st Derivative Number Line Table) that shows the first derivative test and the conclusions that it gives you.

Solution. (a)

$$
\begin{aligned}
f'(x) &= 15x^4 - 15x^2 \\
15x^4 - 15x^2 &= 0 \\
15x^2(x^2 - 1) &= 0 \\
15x^2 = 0 \text{ or } x^2 - 1 &= 0 \\
x = 0 \text{ or } x^2 &= 1 \\
x = 0 \text{ or } x &= \pm 1
\end{aligned}
$$

(b)

	l. max $x = -1$		neither $x = 0$		l. min $x = 1$	
$f \nearrow$		$f \searrow$		$f \searrow$		$f \nearrow$
$f' > 0$	$f' = 0$	$f' < 0$	$f' = 0$	$f' < 0$	$f' = 0$	$f > 0$

6. Let $f(x) = \frac{8}{3}x^3 - 36x^2 + 10000$.
   (a) Find the critical points of $f(x)$.
   (b) Apply the second derivative test to each of the the critical points and identify each critical point as a local max/min/neither. Be sure to write down the steps you use to apply the second derivative test.

Solution. (a)

$$
\begin{aligned}
f'(x) &= 8x^2 - 72x \\
8x^2 - 72x &= 0 \\
8x(x - 9) &= 0 \\
x &= 0 \text{ or } x = 9
\end{aligned}
$$

(b)

$$
\begin{aligned}
f''(x) &= 16x - 72 \\
f''(0) &= -72 \\
-72 &< 0 \quad \Longrightarrow x = 0 \text{ l. max} \\
f''(9) &= 16(9) - 72 \\
&= 72 \\
72 &> 0 \quad \Longrightarrow x = 9\text{l. min}
\end{aligned}
$$

7. Find the inflection points of $f(x) = x^5 - 5x^4 + 35$.

Solution.

$$
\begin{aligned}
f'(x) &= 5x^4 - 20x^3 \\
f''(x) &= 20x^3 - 60x^2 \\
0 &= 20x^3 - 60x^2 \\
20x^3 - 60x^2 &= 0 \\
20x^2(x - 3) &= 0 \\
20x^2 = 0 \text{ or } x - 3 &= 0 \\
x = 0 \text{ or } x &= 3
\end{aligned}
$$

8. Let $f(x) = x^2 \ln(x) - 10x\ln(x) + 5x$. Suppose that the only critical points are $x = e^{-1/2}$ and $x = 5$. Use the Global Max/Min test to find the Global maximum and minimum (both x-value and y-value) on the interval $0.1 \le x \le 10$. (Note: double and triple check that you've entered your function in the calculator correctly.)

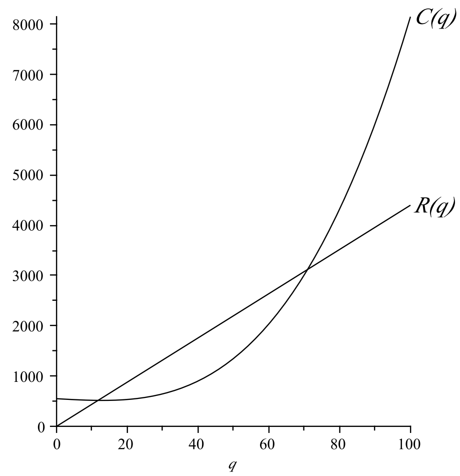
Solution.

x	y
$e^{-1/2}$	-5.9
5	-15.2
0.1	2.8
10	50

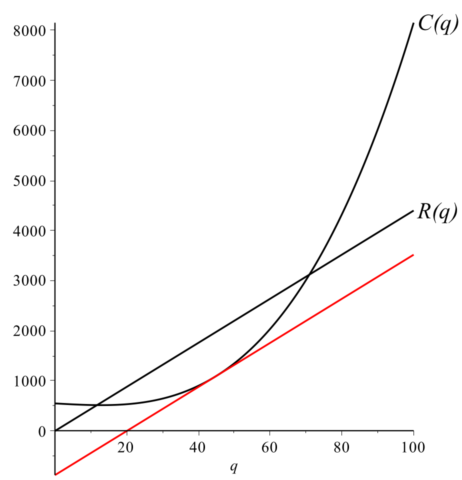
Therefore

$$\text{G. max } x = 10, y = 50$$
$$\text{G. min } x = 5, y = -15.2$$

9. Shown below are a cost and a revenue curve. Estimate the production level that maximizes profit and estimate the profit at that point.



Solution. We put our ruler along the graph of $C(q)$ and make a tangent line that is parallel to $R(q)$:



It appears that this tangent happens at $q \approx 45$.

Note that there are no other points where $C(q)$ is parallel to $R(q)$, and so we don't need to check any further to see that this is a maximum point for the profit. However, if we did need to check we would say this: Just to the left of 45 we have that $MR > MC$ and so profit is increasing; Just to the right of 45 we have that $MR < MC$ and so profit is decreasing. Thus, we have a max.

10. Suppose that your T-shirt company makes a revenue of $R(q) = 10q$ and has a cost of $C(q) = 0.003q^3 + 5.6q + 3.4$. At what quantity is profit maximized? (Note: find this value algebraically, but don't bother applying a test to make sure that the q you find is a maximum as opposed to a minimum.)

Solution.

$$\begin{aligned}
\pi(q) &= R(q) - C(q) \\
&= 10q - \left(0.003q^3 + 5.6q + 3.4\right) \\
&= -0.003q^3 + 4.4q - 3.4 \\
\pi'(q) &= -0.009q^2 + 4.4 \\
-0.009q^2 + 4.4 &= 0 \\
0.009q^2 &= 4.4 \\
q^2 &= \frac{4.4}{0.009} \\
q^2 &= 488.89 \\
q &= \pm\sqrt{488.89} \\
q &= \pm 22.1 \\
q &= 22.1
\end{aligned}$$

11. Suppose a manufacturer is making 3000 units of some item. They are selling the item for \$15 per unit, the marginal cost is \$22, and the total cost is \$41,111. If we increase production above 3000, and assuming that we sell all the units we produce, which of the following would increase? Which would decrease? Which would be impossible to tell? Why?

    (a) Revenue.
    (b) Profit.
    (c) Total cost.
    (d) Marginal cost.
    (e) Marginal revenue.

Solution. (a) Revenue will increase, because we assume that we will sell all our units.
(b) Profit will decrease, because $MR < MC$.
(c) Total cost will increase, because $MC > 0$.
(d) We can't say if MC will increase or decrease.
(e) We can't say if MR will increase or decrease.