1. (Hughes-Hallett 4.2 #22) An exponentially growing animal population numbers 500 at time $t = 0$; two years later, it is 1500. Find a formula for the size of the population in t years and find the size of the population at $t = 5$.

   Solution:

$$\begin{aligned} P &= Ce^{rt} \\ 1500 &= 500e^{r2} \\ 3 &= e^{2r} \\ 2r &= \ln(3) \\ r &= \tfrac{1}{2}\ln(3) \approx 0.549 \\ P &= 500e^{0.549t} \\ P(5) &= 500e^{0.549(5)} \\ &= 7782.3 \end{aligned}$$

or

$$\begin{aligned} P &= Ca^{t} \\ 1500 &= 500a^2 \\ 3 &= a^2 \\ a &= \sqrt{3} \approx 1.732 \\ P &= 500(1.732)^t \\ P(5) &= 500(1.732)^5 \\ &= 7793 \end{aligned}$$

   Note: the answers would be the same if we used more accuracy in our values for r and a.

2. Let $f(x) = \sqrt{x} + x$ and $g(x) = e^x + x$.

   (a) Find $f(x)g(x)$
   (b) Find $f(x)/g(x)$.
   (c) Find $f(g(x))$.
   (d) Find $g(f(x))$.

   Solution:

   (a) $f(x)g(x) = (\sqrt{x} + x)(e^x + x)$
   (b) $f(x)/g(x) = \dfrac{\sqrt{x} + x}{e^x + x}$.
   (c) $f(g(x)) = f(e^x + x) = \sqrt{e^x + x} + (e^x + x)$.
   (d) $g(f(x)) = g(\sqrt{x} + x) = e^{(\sqrt{x}+x)} + (\sqrt{x} + x)$.

3. Find a function $f(x)$ that is proportional to the 3rd power of x and that satisfies $f(2) = 10$.

   Solution:

$$\begin{aligned} f(x) &= kx^3 \\ 10 &= k(2)^3 \\ k &= 10/8 = 5/4 \\ f(x) &= \frac{5}{4}x^3 \end{aligned}$$

4. Let $f(x) = xe^x$. Find an approximation of $f'(2.5)$ by filling in the following table using the same rules as on the quiz[1].

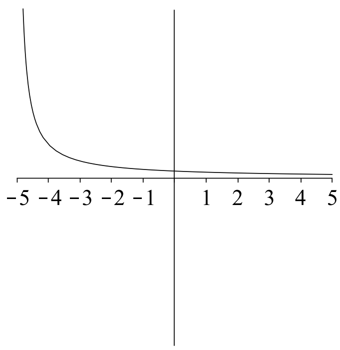
x	$f(x)$	$\dfrac{f(2.5)-f(x)}{2.5-x}$
2.4	26.45562331	40.00611590
2.49	30.03257754	42.36573600
2.499	30.41362358	42.61132000
2.4999	30.45197130	42.63600000
2.5	30.45623490	undefined
2.6	35.00571890	45.49484000
2.51	30.88537445	42.91395500
2.501	30.49890106	42.66616000
2.5001	30.46049905	42.64150000

Our best guesses come from the lines with $x = 2.4999$ and $x = 2.5001$. From these we can say $f'(2.5) \approx 42.6$.

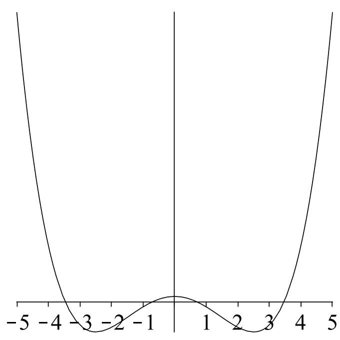
[1]You should plug numbers in for x that are close to 2.5. They should be close enough that you get two approximations of $f'(2.5)$ that are the same to the first decimal place. You should always use at least 8 digits in every step of the calculation.

5. Shown below are eight functions. Four of them are derivatives of the other four. Identify which function is the derivative of which other function; justify each answer by referring to a specific x-value, indicate whether the function is flat/increasing/decreasing and whether the derivative is 0/positive/negative at this x-value. For instance, you could say “(u) is the derivative of (v), because at $x = 1$ we have (v) is increasing and (u) is positive,” or you could be more abbreviated and say the same thing with “u=v$'$, @ $x = 1$, v$\uparrow$, u=$+$.”

(a)



(b)



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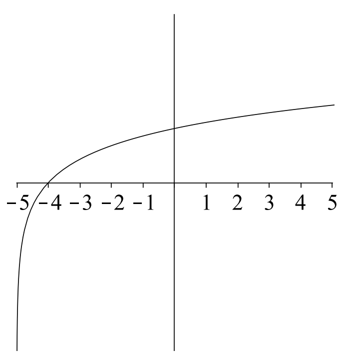
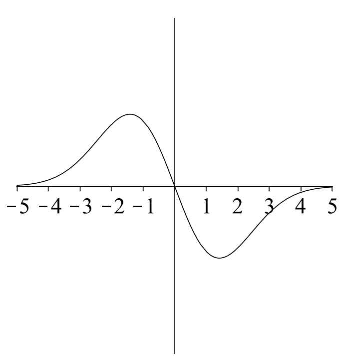
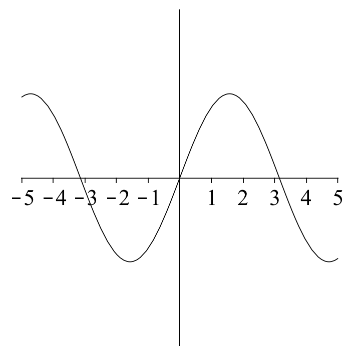


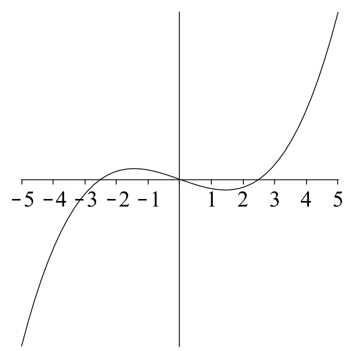
(d)



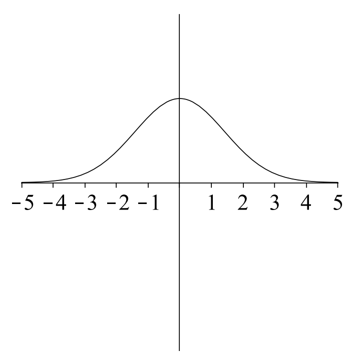
(e)



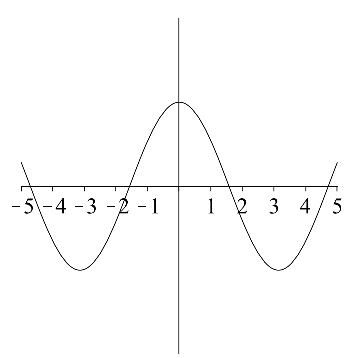
(f)



(g)



(h)



Solution:

Make sure you don't reverse which is the derivative and which is the original function.

- $a = c'$, @ $x = 0$, $c \uparrow$ and $a = +$. (Note: to make sure that this is the right answer you need to check more points. E.g. @ $x = -4$, $c \uparrow$ and $a = +$. @ $x = 5$, $c \uparrow$ and $a = +$.)
- $f = b'$, @ $x = 1$, $b \downarrow$ and $f = -$. (Note: to make sure that this is the right answer you need to check more points. E.g. @ $x = -4$, $b \downarrow$ and $f = -$. @ $x = 5$, $b \uparrow$ and $f = +$.)
- $h = e'$, @ $x = 1$, $e \uparrow$ and $h = +$. (Note: to make sure that this is the right answer you need to check more points. E.g. @ $x = -4$, $e \downarrow$ and $h = -$. @ $x = 4$, $e \downarrow$ and $h = -$.)
- $d = g'$, @ $x = 1$, $g \downarrow$ and $d = -$. (Note: to make sure that this is the right answer you need to check more points. E.g. @ $x = -3$, $g \uparrow$ and $d = +$. @ $x = 3$, $g \downarrow$ and $d = -$.)

6. Leibniz notation and interpretation. (Based on Hughes-Hallett, 4e, #10) On a certain day, CBS Evening News had a 4.3 rating. (Ratings measure the number of viewers.) News executives estimated that on that day a 0.1 drop in the ratings for the CBS Evening News corresponds to a \$$5.5$ million drop in revenue. Let R be the revenue and p be the ratings.

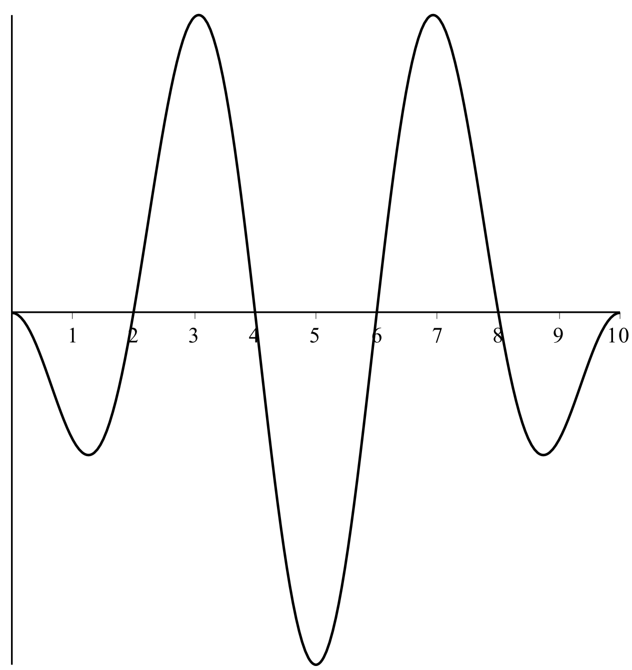
   Express the information in this problem as a derivative using Leibniz notation. Specify the units, the point at which the derivative is evaluated, and interpret your answer.

   Solution:

$$\begin{aligned} \text{rt of ch} &= \frac{-5.5}{-0.1} \\ &= 55 \\ \left.\frac{dR}{dp}\right|_{p=4.3} &= 55\,\$\ \text{mil/rating point} \end{aligned}$$

   At a viewer rating of 4.3 points, an drop of 1 rating point will decrease the revenue by \$55 million.

7. Let $f(x)$ be defined by the graph below



For each quantity below, indicate whether it is $+$, $-$ or 0

$f(0.75) = -$	$f'(0.75) = -$	$f''(0.75) = 0$
$f(1) = -$	$f'(1) = -$	$f''(1) = +$
$f(1.25) = -$	$f'(1.25) = 0$	$f''(1.25) = +$
$f(2.25) = +$	$f'(2.25) = +$	$f''(2.25) = 0$
$f(3) = +$	$f'(3) = 0$	$f''(3) = -$
$f(4) = 0$	$f'(4) = -$	$f''(4) = 0$
$f(5) = -$	$f'(5) = 0$	$f''(5) = +$
$f(6) = 0$	$f'(6) = +$	$f''(6) = 0$
$f(7.75) = +$	$f'(7.75) = -$	$f''(7.75) = 0$
$f(8.75) = -$	$f'(8.75) = 0$	$f''(8.75) = +$
$f(9.25) = -$	$f'(9.25) = +$	$f''(9.25) = 0$

For the first column, you simply look at the y-values on the graph, and see if they are above/below/or the x-axis.

For the second column, you look at the graph, and see if the slope is up/down/0.

For the third column you look at which way the graph is curving: $+$ means curving up or less down, $-$ means curving down or less up, and 0 means not curving much at all (it may be easiest to see this if it's half way between spots that are clearly curving one way and then the other).

8. Suppose $f(23) = 175$ and $f'(23) = 1.5$.
   (a) Find an estimate for $f(26)$.
   (b) Suppose now that you know, for the same function f, that $f''(x)$ is positive for all x. Is your estimate in (a) too high or too low? Justify your answer.

Solution:

(a)

$$\begin{aligned} f(26) &\approx y_0 + m\Delta x \\ &= 175 + 1.5(3) \\ &= 179.5 \end{aligned}$$

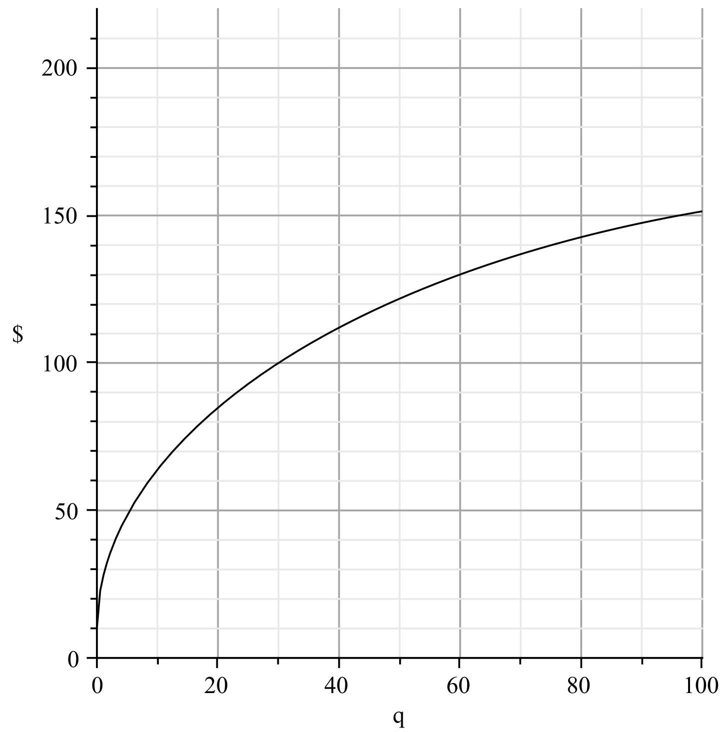
(b)

$$\begin{aligned} &f''(x) = + \\ \Rightarrow\ &f(x) = \text{ concave up} \\ \Rightarrow\ &f(x) \text{ lies above tangent line} \\ \Rightarrow\ &f(x) \text{ lies above the estimate} \\ \Rightarrow\ &\text{estimate is too low} \end{aligned}$$

9. Shown below is the graph of a total cost function of making q items. Use the graph to answer the following questions:

   (a) Find the total cost of making 50 items.

   (b) Estimate the marginal cost when $q = 50$; give units.



Solution:

(a) $C(50) \approx 120$. (Simply look at the y-value on the graph at $q = 50$.)

(b) $C'(50) \approx 0.85$. (Simply draw the tangent line, and use points on the line to calculate its slope. I used the points $(0, 80)$ and $(100, 165)$.)