Contents

Chapter 0

Review: Lines, Fractions, Exponents

0.1 Lines

Lines have more than one form of equation; here we'll recall a few of them.

Definition. In x and y coordinates, here are two forms of the equation of a line:

$$\begin{aligned} y &= mx + b && \text{slope-intercept} \quad (m = \text{slope},\ b = y\text{-intercept}) \\ y &= m(x - x_0) + y_0 && \text{point-slope} \quad (m = \text{slope},\ (x_0, y_0) = \text{given point}) \end{aligned}$$

There are a two important variations:

- A vertical line has an equation of the form $x = C$, where C is a constant.
- A horizontal line has an equation of the form $y = C$, where C is a constant.

0.2 Fractions

We used fractions in the previous subsection when we calculated slopes of lines, but we didn't really practice manipulating fractions much. All we really needed to know there was that a fraction $\frac{a}{b}$ represented a number, and/or a ratio. Now we will recall how to manipulate fractions a little bit. Here are the rules for adding, subtracting, multiplying and dividing fractions[1]:

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= \frac{ad + bc}{bd} \\ \frac{a}{b} - \frac{c}{d} &= \frac{ad - bc}{bd} \\ \frac{a}{b} \cdot \frac{c}{d} &= \frac{ac}{bd} \\ \frac{a}{b} \div \frac{c}{d} &= \frac{a}{b} \cdot \frac{d}{c} \end{aligned}$$

[1] If you like, you can take these rules as the definitions, or even as some made up formulas that tell us how to play a game. However, I don't really recommend this: I think most people want to know *why* these are the right rules, *where* they came from, and *what* they mean. However, we don't have time or space to deal with that here.

Note that in general, the following are not equal, do not make the mistake that they are:

$$\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y}$$
$$\frac{a}{x+a} \neq \frac{1}{x+1}$$

0.3 Rules of exponents

Recall the following rules:

$$a^{-b} \text{ means } \frac{1}{a^b}$$
$$a^{1/b} \text{ means } \sqrt[b]{a}$$
$$(a^n)^m = a^{nm}$$
$$a^n a^m = a^{n+m}$$
$$\frac{a^n}{a^m} = a^{n-m}$$
$$(ab)^n = a^n b^n$$

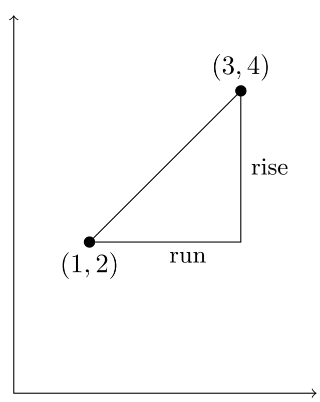
Note that in general the following are not equal (make sure you don't make the mistake of thinking they are)

$$(x+y)^2 \neq x^2 + y^2 \qquad (\text{unless } x = 0 \text{ or } y = 0)$$
$$\sqrt{x+y} \neq \sqrt{x} + \sqrt{y} \qquad (\text{unless } x = 0 \text{ or } y = 0)$$

Example 1. Find the equation of the line through the points $(1, 2)$ and $(3, 4)$.

Solution.

$$m = \frac{\text{rise}}{\text{run}} = \frac{4-2}{3-1} = \frac{2}{2} = 1$$
$$y = m(x - x_0) + y_0$$
$$y = 1(x-1) + 2 \text{ or } y = x + 1$$



This is where we ended on Monday, January 13

Chapter 1

Functions and Change

1.1 Functions

Definition. Here's the most common **function notation**: "$f(x)$". In this, f is the name of a function, x is the input, and $f(x)$ is the output. Usually we're told what $f(x)$ equals with a formula, but sometimes we're told what it equals with a table of numbers or with a graph. Sometimes we replace "f" with some familiar function such as natural log: $\ln(x)$, or sine: $\sin(x)$.

Example 1. Let $f(x)$ be defined by the table of numbers below.

x	1	2	2.5	2.9	3.1	3.5	4
$f(x)$	2	3	3.1	4	1	2.9	3

(a) Find $f(1)$. Find $f(4)$.
(b) Is it possible to find $f(1.5)$?
(c) Solve $f(x) = 1$. Solve $f(x) = 3$.

Solution. (a) $f(1)$ means that we are plugging in $x = 1$ to $f(x)$. We find 1 in the first row, and then read the output, 2 in the second row. Doing the same thing for $x = 4$ we get

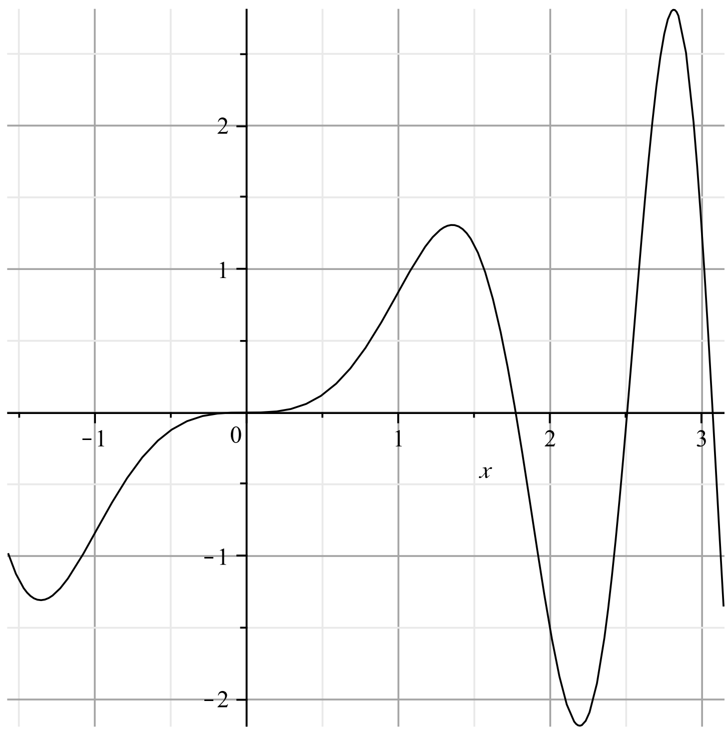
$$f(1) = 2 \qquad f(4) = 3.$$

(b) We could make a guess that $f(1.5)$ is between 2 and 3, maybe exactly 2.5. But this would just be a guess, there's no reason to think this would be exactly correct. Technically, $f(1.5)$ just isn't defined.

(c) Solving $f(x) = 1$ means that we want to ask the question: "what can we plug into f to make the output equal to 1?". In other words, we find 1 in the second row, and ask what is the number above it in the first row. We can see that $f(3.1) = 1$ so $x = 3.1$ is the solution. Doing the same thing for $f(x) = 3$ we get

$$f(x) = 1 \Longrightarrow x = 3.1$$
$$f(x) = 3 \Longrightarrow x = 2 \text{ or } x = 4$$

Example 2. Let $f(x)$ be the function defined by the graph below



(a) Find $f(0)$, $f(1)$, $f(1.5)$, $f(2)$.
(b) Solve $f(x) = 0$; solve $f(x) = 1$ (find all solutions)

Solution. (a) For each of these, we take the number inside the parentheses, find it on the x-axis, and then go straight up or down to the graph and find our answer as the y-value. For instance, we find $f(1) = 0.8$, because when $x = 1$, the y-value on the graph is 0.83.

$$f(1) = 0.8$$
$$f(0) = 0$$
$$f(1.5) \approx 1.1$$
$$f(2) \approx -1.5$$

(b) For each of these, we take the number on the right, look this up on the y-axis, then go straight left or right to the graph and find our answer as the x-value. Watch out, there is more than one solution for each. For instance, $f(x) = 0$ has one solution of $x = 0$, because the point $(0, 0)$ is on the graph. But there is another solution $x \approx 1.7$ because the point $(1.7, 0)$ is also on the graph.

$$f(x) = 0 \Rightarrow x \approx 0, 1.7, 2.5, 3.1$$
$$f(x) = 1 \Rightarrow x \approx 1.2, 1.6, 2.6, 3$$

Example 3. Let $f(x) = 3x - 7$.
(a) Find $f(1)$. Find $f(2)$.
(b) Solve $f(x) = 1$; solve $f(x) = 2$ (find all solutions).

Solution. (a)

$$f(1) = 3(1) - 7$$

$$-4$$
$$f(2) = 3(2) - 7$$
$$= -1$$

(b)

$$\begin{aligned} f(x) &= 1 \\ 3x - 7 &= 1 \\ 3x &= 8 \\ x &= 8/3 \\ f(x) &= 2 \\ 3x - 7 &= 2 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Example 4. Let $f(x) = -2x + 5$.
(a) Find $f(3)$.
(b) Solve $f(x) = 3$.

Solution. (a) $f(3) = -2(3) + 5 = -1$.
(b)

$$\begin{aligned} f(x) &= 3 \\ -2x + 5 &= 3 \\ -2x &= -2 \\ x &= 1 \end{aligned}$$

This is where we ended on Wednesday, January 15

1.2 Linear Functions

Definition. A **linear function** is a function whose graph is a straight line.

The purpose of this section is to practice using these functions in applied problems. As a really simple example, consider the function that converts from Celsius to Fahrenheit: $F = \frac{9}{5}C + 32$. In general, the problems in this section will give you information, and buried in this information will be: either two points, or a point and a slope. From this, you will find the equation of a line, and hence a linear function. Often, the problem has other questions as well, asking you to solve for something else, or interpret your answers.

Example 1. (Hughes-Hallett, 4e, 1.2#26) The number of species of coastal dune plants in Australia decreases as the latitude, in °S, increases. There are 34 species at 11 °S and 26 species at 44 °S.
(a) Find a formula for the number, N, of species of coastal dune plants in Australia as a linear function of the latitude, ℓ in °S.
(b) Give units for and interpret the slope and the vertical intercept of this function.

(c) Graph this function between $\ell = 11\,^\circ$S and $\ell = 44\,^\circ$S. (Australia lies entirely within these latitudes.)

Solution. (a) Note: the mathematical information in this problem can be summarized as: we have two points: $(11, 34)$ and $(44, 26)$. We start by calculating the slope:

$$m = \frac{\Delta N}{\Delta \ell} = \frac{34 - 26}{11 - 44} = \frac{8}{-33} = -\frac{8}{33}$$

Now we use the point-slope formula with $m = -8/33$ and the point $(11, 34)$

$$N = -\frac{8}{33}(\ell - 11) + 34$$

(b) The vertical intercept is another name for the y-intercept, so we put the equation of our line in slope-intercept form:

$$N = -\frac{8}{33}\ell + \frac{8}{33} \cdot 11 + 34 = -\frac{8}{33}\ell + \frac{8}{3} + \frac{102}{3} = -\frac{8}{33}\ell + \frac{110}{3}.$$

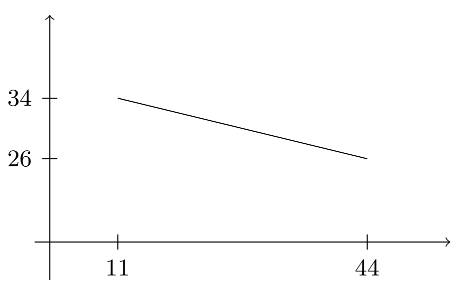
The units for slope are the units of N, divided by the units for ℓ, thus

$$m = -\frac{8}{33}, \qquad \text{units of slope} = \frac{\text{number of species}}{\text{degree latitude}}$$

Here's one way to interpret the slope: every $1\,^\circ$ latitude south that you go, the number of species will decrease by 0.24 species (where $8/33 \approx 0.24$).

The units for the vertical intercept are the units for N: number of species. This means that if Australia were to extend all the way to the equator (which it does not), you would expect to see about 37 species (where $110/3 \approx 37$)

(c)



Example 2. (Hughes-Hallett, 3e, 1.2#25) A controversial 1992 Danish study reported that men's average sperm count has decreased from 113 million per milliliter in 1940 to 66 million per milliliter in 1990.

(a) Express the average sperm count, S, as a linear function of the number of years, t, since 1940.

(b) A man's fertility is affected if his sperm count drops below about 20 million per milliliter. If the linear model found in part (a) is accurate, in what year will the average male sperm count fall below this level?

Solution. (a) We have two data points: 113 million sperm in 1940, 66 million sperm in 1990. Notice which is the input and which the output: the phrase "S, as a linear function of the number of years" means that we want the year to be the input. Finally, the phrase "the number of years, t, since 1940" means that $t = 0$

corresponds to 1940. Putting all this together, we can rephrase the problems like this:

Find line through $(0, 113)$ and $(50, 66)$

We start with the slope

$$m = \frac{113 - 66}{0 - 50} = \frac{-47}{50}.$$

Now we use the point-slope or slope-intercept equation

$$S = -\frac{47}{50}(t - 0) + 113 = -\frac{47}{50}t + 113$$

(b) We set the formula from (a) to equal 20 and solve for t:

$$\begin{aligned} 20 &= -\frac{47}{50}t + 113 \\ 20 - 113 &= -\frac{47}{50}t \\ -93 &= -\frac{47}{50}t \\ (-93)\left(-\frac{50}{47}\right) &= t \\ t &= 98.9 \end{aligned}$$

Now we turn this value for t into a year: $1940 + 98.9 = 2038.9$.

Example 3. (Hughes-Hallett, 4e, 1.2#12) A cell phone company charges a monthly fee of \$25 plus \$0.05 per minute. Find a formula for the monthly charge, C, in dollars, as a function of th enuber of minutes, m, the phone is used during the month.

Solution. We are asked to find an equation like $y = mx + b$, but the letters are different here: C instead of y and m instead of x. In other words, we are filling in this:

$$C = \_\_m + b$$

I'll give the answer first and hen explain:

$$C = 0.05m + 25$$

In the equation $C = \_\_m + b$ the variable b represents how much is paid even if you use 0 minutes. In this case, you would pay \$25 even if you use 0 minutes, so $b = 25$.

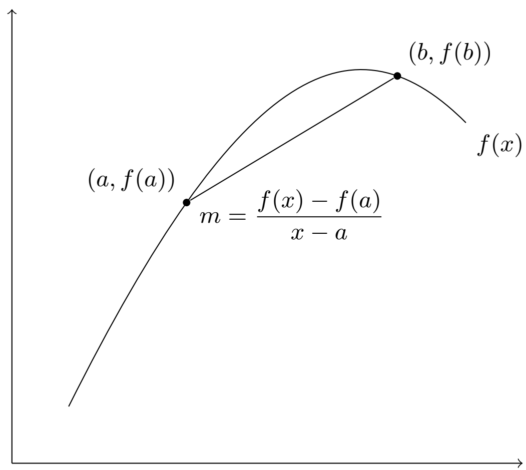
The number in front of m is the slope, and this tells you how much the cost goes up each minute. This is \$0.05.

1.3 Rates of change

Definition. Given any function $f(x)$, we define

$$\boxed{\begin{array}{l}\text{average rate of change} \\ \text{from } x = a \text{ to } x = b\end{array}} = \frac{f(b) - f(a)}{b - a}$$

We sometimes abbreviate this notation as $\frac{\Delta y}{\Delta x}$. This number equals the slope of the secant line as pictured



This is where we ended on Friday, January 17

Example 1. (Hughes-Hallett, 4e, 1.3#24) The table below shows the production of tobacco in the US, in millions of pounds.

(a) What is the average rate of change in tobacco production between 1996 and 2003? Give units and interpret your answer in terms of tobacco production.

(b) During this seven-year period, is there any interval during which the average rate of change was positive? If so, when?

Year	1996	1997	1998	1999	2000	2001	2002	2003
Production	1517	1787	1480	1293	1053	991	879	831

Solution. (a)

$$\begin{aligned}\text{avg. rt. ch.} &= \frac{P(2003) - P(1996)}{2003 - 1996} \\ &= \frac{831 - 1517}{7} \\ &= \frac{-686}{7} \\ &= -98 \quad \text{million pounds/year}\end{aligned}$$

Interpretation: each year the US tobacco production decreases by 98 million pounds.

(b) The average rate of change was positive from 1996 to 1997.

Example 2. (Hughes-Hallett, 3e, 1.3#26) The table below shows the sales, S, in millions of dollars, of Intel Corporation, a leading manufacturer of of integrated circuits:

Year	1998	1999	2000	2001	2002	2003
S	26,273	29,389	33,726	26,539	26,764	30,141

(a) What is the average rate of change from 1998 to 2003? Interpret its units and meaning.

(b) Assuming that the change continues at the same rate as in part (a), when will sales reach 40,000 million dollars?

(c) Over which intervals does it appear that the function S is increasing?

Solution. (a) $\frac{30141.0 - 26273}{2003 - 1998} \approx 773.6\,{}^{\$\text{Million}}/_{\text{Year}}$. This means that the sales are increasing by an average of \$773.6 million each year.

(b) There's more than one approach to predict when sales will reach 40,000 million. I'll do it in detail in one way, and briefly refer to two other approaches.

In the first solution, we model sales with a linear equation (it should be linear because of the phrase "Assuming that the change continues at the same rate"). We use the point-slope equation: $S = m(t - t_0) + S_0$ for $m = 777.3$, $t_0 = 2003$, $S_0 = 30,141$,

$$S = 777.3(t - 2003) + 30,141.$$

Now we solve for when $S = 40,000$

$$\begin{aligned} 40,000 &= 777.3(t - 2003) + 30,141 \\ 9859 &= 777.3(t - 2003) \\ 12.684 &= t - 2003 \\ t &= 2015.6 \end{aligned}$$

Here's another way to solve this: let $t = 0$ correspond to 1998, then we have our linear model

$$S = 777.3t + 26,273$$

(note: this is simpler because letting $t = 0$ correspond to 1998 makes $26,273$ the y-intercept). Now we solve this for $t = 17.66$. This gives a year of $1998 + 17.66 = 2015.6$.

Here's the last way to solve this. Look at the data in 2003, $S = 30,141$. The problem is asking for a change in sales of $40,000 - 30,141 = 9859$. How long will this take? We divide this change by the rate of change

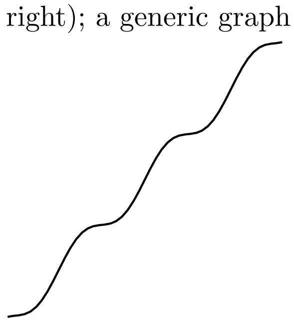
$$\frac{9859}{777.3} = 12.68$$

So, from 2003 it will take 12.68 years to reach $40,000$, this gives a year of $2003 + 12.68 = 2015.6$.

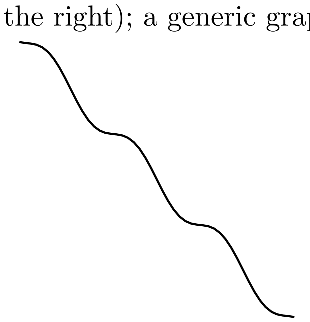
(c) It appears that S is increasing from 1998 to 2000, and then 2001 to 2003.

Definition.

We say that a function is **increasing** if its graph goes upwards (as we move to the right); a generic graph is shown.

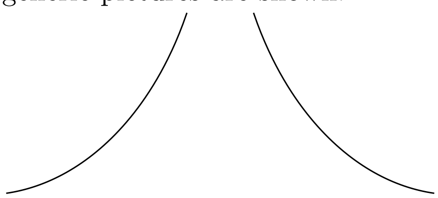


We say that a function is **decreasing** if its graph goes downwards (as we move to the right); a generic graph is shown.

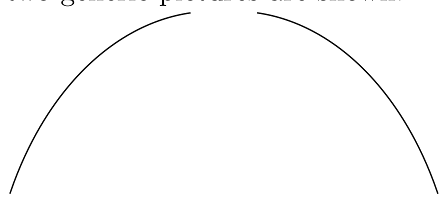


Definition.

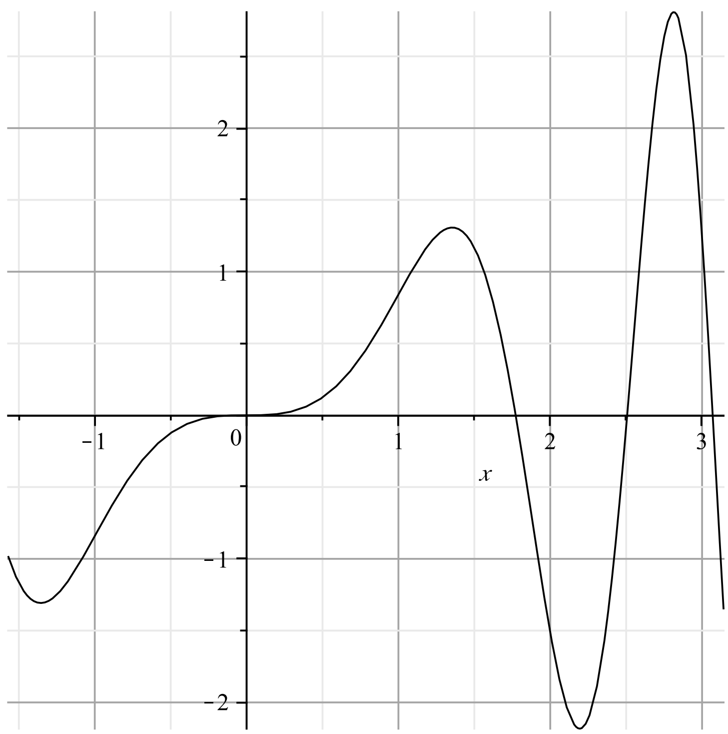
We say that $f(x)$ is **concave up** if its graph curves upwards, or less downwards, (as we move to the right); two generic pictures are shown:



We say that $f(x)$ is **concave down** if its graph curves downwards, or less upwards, (as we move to the right); two generic pictures are shown:



Example 3. Let $f(x)$ be defined by the graph below



(a) Over which intervals does it appear that $f(x)$ is increasing? Decreasing?
(b) Over which intervals does it appear that $f(x)$ is concave up? Concave down?

Solution. (a) It appears to be increasing from -1.3 to 1.3, and from 2.2 to 2.8. It appears to be decreasing from -1.6 to -1.3, from 1.3 to 2.2 and from 2.8 to 3.1.

Here is alternative, more compact, notation for the same solution:

$$f\uparrow: \quad -1.3 \le x \le 1.3 \text{ and } 2.2 \le x \le 2.8$$
$$f\downarrow: \quad -1.6 \le x \le -1.3 \text{ and } 1.3 \le x \le 2.2 \text{ and } 2.8 \le x \le 3.1$$

Here's a final, even more compact way of writing the same thing:

$$f\uparrow: \quad (-1.3, 1.3) \cup (2.2, 2.8)$$
$$f\downarrow: \quad (-1.6, -1.3) \cup (1.3, 2.2) \cup (2.8, 3.1)$$

(b) It appears to be concave up from -1.6 to -0.9, from 0 to 0.9, from 1.8 to 2.5. It appears to be concave down from -0.9 to 0, from 0.9 to 1.8 and from 2.5 to 3.1.

Here is alternative, more compact, notation for the same solution:

$$f \text{ C.U.}: \quad -1.6 \le x \le -0.9 \text{ and } 0 \le x \le 0.9 \text{ and } 1.8 \le x \le 2.5$$
$$f \text{ C.D.}: \quad -0.9 \le x \le 0 \text{ and } 0.9 \le x \le 1.8 \text{ and } 2.5 \le x \le 3.1$$

Here's a final, even more compact way of writing the same thing:

$$f \text{ C.U.}: \quad (-1.6, -0.9) \cup (0, 0.9) \cup (1.8, 2.5)$$
$$f \text{ C.D.}: \quad (-0.9, 0) \cup (0.9, 1.8) \cup (2.5, 3.1)$$

This is where we ended on Wednesday, January 22

1.4 Applications of Functions to Economics

Definition. The **cost function** gives the total cost of producing a quantity of some good. The standard notation is: $q =$ quantity, $C(q) =$ cost. In this section, unless we explicitly say otherwise, we assume that $C(q)$ is a linear function. Then the **fixed cost** is defined as $C(0)$ and the **variable cost** is define as the slope of $C(q)$.

The **revenue function** gives the total revenue received for a quantity of some good. Typical notation: $q =$ quantity, $R(q) =$ revenue. Note that $R(q) = pq$, where p is the price per item.

The **profit function** gives the total profit for a quantity of some good, i.e. revenue minus cost. Typical notation: $q =$ quantity, $\pi(q) =$ profit.

A **break-even point** is a quantity that produces 0 profit.

This is where we ended on Friday, January 24

Example 1. (Hughes-Hallett, 4e, 1.4#15(a)) Production costs for manufacturing running shoes consist of a fixed overhead of \$650,000 plus variable costs of \$20 per pair of shoes. Each pair of shoes sells for \$70.

Find the total cost, $C(q)$, the total revenue, $R(q)$, and the total profit, $\pi(q)$, as a function of the number of pairs of shoes produced, q, and the break even point.

Solution. $C(q) = \$650,000 + \$20q$

$R(q) = \$70q$

$\pi(q) = R(q) - C(q) = 70q - (650000 + 20q) = -650000 + 50q$

$0 = 6500000 + 50q \Rightarrow q = 13000.$

Definition. Marginal cost, **marginal revenue**, and **marginal profit** are names for the slope of cost, revenue, and profit, respectively. For non-linear functions, we will interpret the word "slope" here to mean derivative (once we have learned derivative). Another way to understand marginal cost/revenue/profit, is that it tells you how much the cost/revenue/profit changes when you increase the quantity q by 1 (you can even use this to calculate the marginal cost/revenue/profit, but this is more work than taking the slope or derivative).

Example 2. (Hughes-Hallett, 4e, 1.4#15(b)) Find the marginal cost, marginal revenue and marginal profit for the shoe company (see Example 1).

Solution. Marginal cost = slope of $C(q) = 20$.

Marginal revenue = slope of $R(q) = 70$.

Marginal profit = slope of $\pi(q) = 50$.

Definition. The **supply curve** is the graph relating the quantity q that manufacturers are willing to supply, to the price p for which the item can be sold. Note that price is viewed as the input variable (however economists usually put p on the vertical axis based on historical tradition). As price increases, the quantity q also increases.

The **demand curve** is the graph relating the quantity q that consumers are willing to buy, to the price p. Note that price is viewed as the input variable (however economists usually put p on the vertical axis based on historical tradition). As price increases, the quantity q decreases.

The **equilibrium point** is the intersection of the supply and demand curves. The price value of this intersection is called the **equilibrium price** and the quantity value is called the **equilibrium quantity**. Typical notations for these values are p^* and q^* respectively. It is assumed that markets tend to move towards, and settle in, equilibrium points.

Example 3. (Hughes-Hallett, 4e,1.4#24) One of the tables below represents a supply curve; the other represents a demand curve.

(a) Which table represents which curve? Why?
(b) At a price of \$155, approximately how many items would consumers purchase?
(c) At a price of \$155, approximately how many items would manufacturers supply?
(d) Will the market push prices higher or lower than \$155?
(e) What would the price have to be if you wanted consumers to buy at least 20 items?
(f) What would the price have to be if you wanted manufacturers to supply at least 20 items?

I :

p (\$/unit)	182	167	153	143	133	125	118
q (quantity)	5	10	15	20	25	30	35

II :

p (\$/unit)	6	35	66	110	166	235	316
q (quantity)	5	10	15	20	25	30	35

Solution. (a) Table I is demand because $p \uparrow$ makes $q \downarrow$.

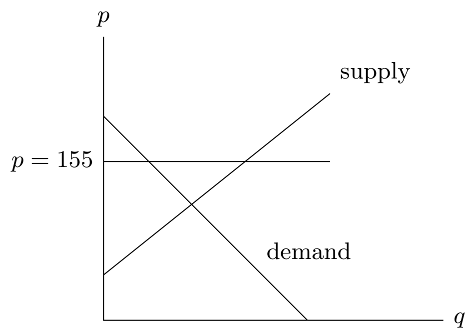
Table II is supply because $p \uparrow$ makes $q \uparrow$.

(b) ≈ 14

(c) ≈ 24

(d) The market will push prices lower. There are (at least) two ways to see this.

Graphically, we have something like the graph below



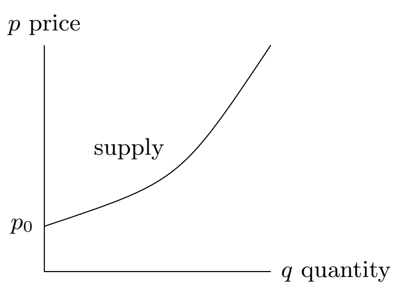
where the line $p = 155$ intersects the two curves *above* the equilibrium point (the intersection) because that's where we have a smaller value for q on the demand curve. Since the market will tend to push things to equilibrium, this means the price needs to come down towards the intersection point.

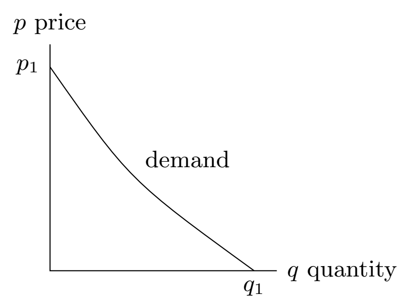
Here's another way to find the same answer for this problem: since the suppliers have put 24 items on the market, but people only want to buy 14, there are too many items. What to do? Lower prices or have a sale.

(e) \$143

(d) \$110

Example 4. Below are some generic supply and demand graphs. Interpret the economic meaning of the vertical and horizontal intercepts.





Solution. For p_0 we have $q = 0$. Thus, at this price (or below), the manufacturers are not willing to make any of the item.

For p_1 we have $q = 0$. Thus, at this price (or above) the consumer will not buy any of the item. For q_0 we have $p = 0$. Thus, this is the quantity that could be given away for free.

This is where we ended on Monday, January 27

Example 5. (Hughes-Hallett, 4e, 1.4#25) A company produces and sells shirts. The fixed costs are \$7000 and the variable costs are \$5 per shirt.

(a) Shirts are sold for \$12 each. Find cost and revenue as functions of the quantity of shirts, q.

(b) The company is considering changing the selling price of the shirts. Demand is $q = 2000 - 40p$, where p is price in dollars and q is the number of shirts. What quantity is sold at the current price of \$12? What profit is realized at this price?

(c) Use the demand equation to write cost and revenue as a function of the price, p. Then write profit as a function of price.
(d) Graph profit against price. Find the price that maximizes profits. What is this profit?

Solution. (a)

$$\begin{aligned} C(q) &= 7000 + 5q, \\ R(q) &= 12q. \end{aligned}$$

(b)

$$\begin{aligned} q(12) &= 2000 - 40 * 12 = 1520, \\ \pi(1520) &= R(1520) - C(1520) \\ &= 3640. \end{aligned}$$

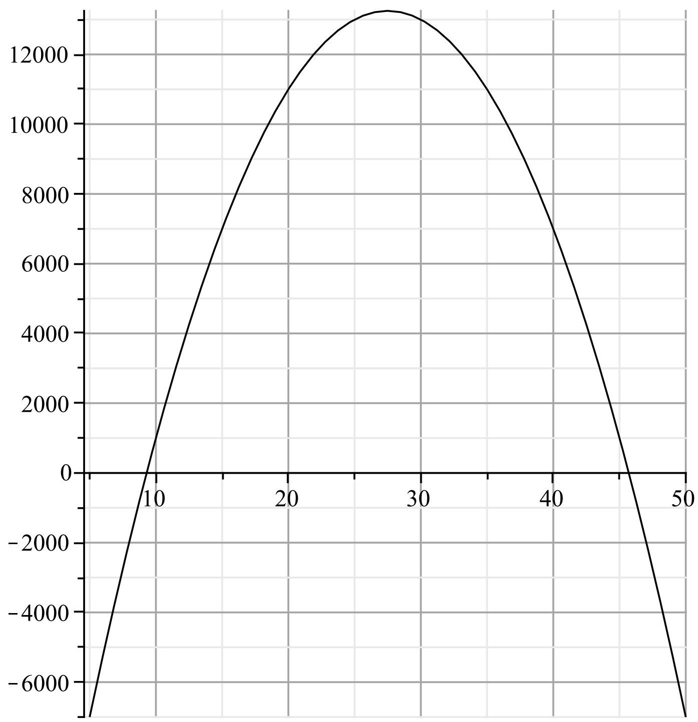
(c)

$$\begin{aligned} C(p) &= C(2000 - 40p) \\ &= 7000 + 5(2000 - 40p) \\ &= 17000 - 200p \end{aligned}$$

Note that $R(p)$ is not $12(2000 - 40p)$ because the 12 has changed and is now represented by the variable p. Thus

$$\begin{aligned} R(p) &= R(2000 - 40p) \\ &= p(2000 - 40p) \\ &= 2000p - 40p^2 \\ \pi(p) &= R(p) - C(p) \\ &= -40p^2 + 2200p - 17000. \end{aligned}$$

(d) A graph is shown below:



It appears that the maximum is at about \$27.50, and the profit is about \$13, 250.

Definition. Some special functions:

- A **depreciation function** is a function $V(t)$, that calculates a decreasing value of some item as a function of of time.
- A **tax adjusted** supply or demand curve takes the original curve, and substitutes a different value of p into the formula for q. One substitutes the new value of p that the manufacturer or the consumer sees after the tax.

  For instance, if a tax of \$5 is placed on each item, then when a consumer pays \$$p$, the manufacture only receives $p - 5$. Thus, we substitute $p - 5$ into the supply equation.

  If a sales tax of 0.08% is placed on each sale, then when a consumer sees a price of \$$p$, they actually end up paying $1.08p$ to purchase the item. Thus, we substitute $1.08p$ into the demand equation.

Example 6. (Hughes-Hallet, 4e, 1.4#38) In Example 8 (from the text), the demand and supply curves are given by $q = 100 - 2p$ and $q = 3p - 50$, respectively; the equilibrium price is \$30 and the equilibrium quantity is 40 units. A sales tax of 5% is imposed on the consumer.

(a) Find the equation of the new demand and supply curves.
(b) Find the new equilibrium price and quantity.
(c) How much is paid on taxes on each unit? How much of this is paid by the consumer and how much by the producer?
(d) How much tax does the government collect?

Solution. (a) If the price the consumer sees is \$$p$ then they actually pay \$$1.05p$ after the sales tax. Thus, the number of units the consumers will buy is

$$q = 100 - 2\big(1.05p\big)$$

$$q = 100 - 2.1p$$

The supply curve is unchanged because the manufacturer still receives \$$p$ from each unit sold.

(b) The equilibrium point is now found from intersecting the new demand curve with the unchanged supply curve:

$$\begin{aligned} \text{demand} &= \text{supply} \\ 100 - 2.1p &= 3p - 50 \\ 150 &= 5.1p \\ p &= \frac{150}{5.1} \approx \$29.41 \end{aligned}$$

The equilibrium quantity is $q = 100 - 2.1(29.41) = 38.23$.

(c) Each unit sells for \$29.41. Of this amount, 5% is taxes, i.e. \$1.47 is the total amount of taxes per unit. In some sense, the consumer pays all of this and the produce pays none of it. But the right way to interpret this question is by comparing the prices to the pre-tax equilibrium.

Pre-tax, the supplier got \$30 per unit, and now they get \$29.41 per unit, costing the supplier \$0.59 per unit.

Pre-tax the consumer paid \$30 per unit, and now they pay $1.05(29.41) =$ \$30.88 per unit, costing the consumer \$0.88 per unit.

Thus, the total tax per unit of \$1.47 is split into $\$0.59 + \$0.88$.

(d) The government collects \$1.47 in taxes per unit. There are a total of 38.23 units being sold for a total tax collection of

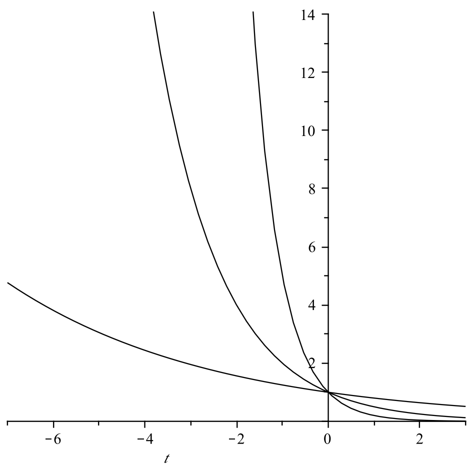
$$38.23(\$1.47) = \$56.2.$$

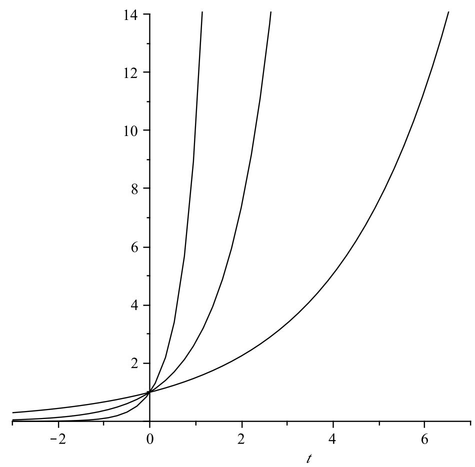
This is where we ended on Wednesday, January 29

1.5 Exponential Functions

Definition. An **exponential function** is one of the form $f(x) = Ca^t$ where C and a are constants, and $a > 0$. Note that C is the y-intercept of $f(t)$. We can always write a as $a = 1 + r$. In this case, we call r the percentage rate of change. Note that r is the decimal representation of the percentage. Note that r can be negative, in which case a is less than 1, and the function $f(x)$ is decreasing.

All exponential functions have about the same shape: growing if $a > 1$ and shrinking if $a < 1$. We picture some below:





0.8^t, 0.5^t, 0.2^t and 10^t, e^t and 1.5^t.

Example 1. In fall of 2009 the annual tuition at Loyola University Maryland was[1] \$36,510. In fall of 2013 the annual tuition was[2] \$41,850. Over this time the tuition grew exponentially with a percentage rate of growth of 3.47%.

(a) Assuming that the tuition continues to grow at the same rate, what will it be in fall of 2016?

(b) Using a graph, predict when the annual tuition will be $\$50,000$.

Solution. (a) We model the tuition with the following equation

$$T = 36510(1.035)^t$$

where t is the number of years after 2009.

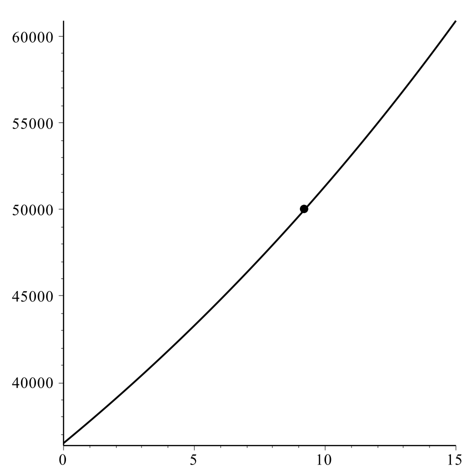
In 2016 we will have $t = 7$ and so tuition will probably be

$$T = 36510(1.0347)^7 \approx \$46,357$$

(b) We want to solve

$$36510(1.0347)^t = 50000.$$

We start by graphing $36510(1.0347)^t$ and then find when the y-value equals 50000. Shown below is a plot, with a point marking the y-value 50000.



[1]From the Loyola 2009–2010 undergraduate catalogue.
[2]From the Loyola 2013–20114 undergraduate catalogue.

The corresponding t-value is approximately 9.5. So the tuition will be 50000 in the year corresponding to $t = 10$, i.e. 2019.

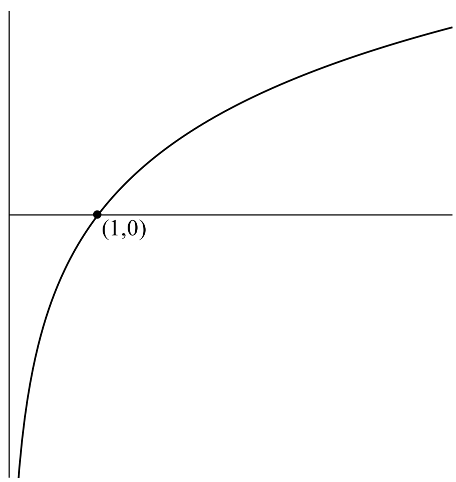
1.6 Natural Logarithm

Definition. Let e be a number that's approximately 2.71828.... The **natural logarithm** function $\ln(x)$ is the inverse function of e^x. In other words

$$\ln(e^x) = x \text{ and } e^{\ln(x)} = x.$$

Note: "x" represents anything in the above formulas. So in fact we also have $\ln(e^y) = y$, $\ln(e^z) = z$, $\ln(e^\square) = \square$ and $\ln(e^{2x^2-5x}) = 2x^2 - 5x$. Similar comments apply to $e^{\ln(x)}$.

To remember what the graph of $\ln(x)$ looks like, take the graph of e^x and reflect it over the line $y = x$:



Note that the graph is not defined for an x that is negative or 0. Also, the x intercept is $x = 1$.

Fact. Properties of logarithms.[3]

1. $\ln(ab) = \ln(a) + \ln(b)$
2. $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$
3. $\ln(a^b) = b\ln(a)$

Example 1. Solve

$$5e^{2x} = 7.$$

Solution. We have

$$e^{2x} = 7/5$$

[3]Property (1) comes from $e^a \cdot e^b = e^{a+b}$. In other words, for e^x, multiplying the outputs is the same as adding the inputs. For $\ln(x)$, it's the reverse. Property (2) comes from $\frac{e^a}{e^b} = e^{a-b}$. In other words, for e^x, dividing the outputs is the same as subtracting the inputs. For $\ln(x)$, it's the reverse. Property (3) comes from $(e^a)^b = e^{ab}$. In other words, for e^x, raising the output to the power b is the same as multiplying the input by b. For $\ln(x)$, it's the reverse.

$$\begin{aligned} \ln(e^{2x}) &= \ln(7/5) \\ 2x &= \ln(7/5) \\ x &= \frac{1}{2}\ln(7/5) \end{aligned}$$

Example 2. Solve for t using natural logarithms,

$$10 = 6e^{0.5t}.$$

Solution.

$$\begin{aligned} \frac{10}{6} &= e^{0.5t} \\ \ln(10/6) &= \ln(e^{0.5t}) \\ \ln(10/6) &= 0.5t \\ t &= 2\ln(10/6) \\ t &\approx 1.022 \end{aligned}$$

Example 3. We return to the model of Loyola University Maryland's tuition presented in Example 1 in Section 1.5. The tuition was $\$36,510$ in fall 2009 and has grown at an annual rate 3.47%.

Using natural logs, find when the tuition is predicted to be $\$53,000$.

Solution. The tuition is modelled by the equation

$$T(t) = 36510(1.0347)^t$$

where t is years since 2009. Now we solve $T(t) = 53000$.

$$\begin{aligned} T(t) &= 53000 \\ 36510(1.0347)^t &= 53000 \\ (1.0347)^t &= 53000/36510 \\ \ln\left((1.0347)^t\right) &= \ln(53000/36510) \\ t\ln(1.0347) &= \ln(53000/36510) \\ t &= \frac{\ln(53000/36510)}{\ln(1.0347)} \\ t &\approx 10.9 \end{aligned}$$

We round (up) to $t = 11$. Thus, tuition is expected to be $\$53,000$ in 2020.

This is where we ended on Friday, January 31

1.7 Exponential Growth and Decay

This section is designed to give the reader practice with an important family of problems. The main idea is to take a formula such as $f(t) = Ce^{rt}$, and use it to model some given data. Often the data will described as part of an applied problem.

There are many, many applications of exponential growth and decay: population growth, doubling time, half-life, compound interest, present and future value, to mention a few.

In this section we distinguish between two ways of interpreting percentage change, as shown

If r is the annual/monthly/hourly/ per-time-period percentage change (written as a decimal), then we use $$y = Ca^t \text{ with } a = 1 + r$$
If r is the continuous percentage change (written as a decimal) then we use $$y = Ce^{rt}$$

Example 1. Find C and r such that $f(t) = Ce^{rt}$ goes through the points $(0, 7.3)$ and $(2.9, 17.8)$.

Solution. In this problem, since one of the points has $t = 0$, this will make it easier to solve first for C. The point $(0, 7.3)$ means that we should have $f(0) = 7.3$. Plugging these numbers in we get

$$7.3 = Ce^0 \Rightarrow C = 7.3$$

Now we plug in $(2.9, 17.8)$, to solve for r. This will take more steps

$$\begin{aligned} f(2.9) = 17.8 \Rightarrow \qquad 17.8 &= 7.3e^{r2.9} \qquad (1.1) \\ 2.438 &= e^{2.9r} \\ \ln(2.43856) &= \ln\left(e^{2.9r}\right) \\ \ln(2.43856) &= 2.9r \\ r &= \frac{\ln(2.43856)}{2.9} \approx 0.30735 \end{aligned}$$

Example 2. (Hughes-Hallett, 4e, 1.7#11) A cup of coffee contains 100 mg of caffeine, which leaves the body at a continuous rate of 17% per hour.

(a) Write a formula for the amount, A mg, of caffeine in the body t hours after drinking a cup of coffee.

(b) Find the half-life of caffeine.

Solution. (a) $A = 100e^{-0.17t}$

(b)

$$\begin{aligned} 50 &= 100e^{-0.17t} \\ \frac{1}{2} &= e^{-0.17t} \\ \ln(1/2) &= \ln(e^{-0.17t}) \\ \ln(1/2) &= -0.17t \\ t &= \frac{\ln(1/2)}{-0.17} \approx 4.077336356 \end{aligned}$$

This is where we ended on Monday, February 3

1.8 New Functions From Old

We can combine old functions to make new functions. We can add, subtract, multiply, divide, and plug one inside of another. The notation $f(g(x))$ means that $g(x)$ is the input for $f(x)$. We illustrate these combinations by example.

Example 1. Let $f(x) = 3x - 2$ and let $g(x) = x^2 + x$. Find the following functions:
(a) $f(x) + g(x)$
(b) $f(x)g(x)$
(c) $f(x)/g(x)$
(d) $f(g(x))$
(e) $g(f(x))$.

Solution. (a)

$$f(x) + g(x) = 3x - 2 + x^2 + x$$

(b)

$$f(x)g(x) = (3x - 2)(x^2 + x)$$

(c)

$$f(x)/g(x) = \frac{3x - 2}{x^2 + x}$$

(d)

$$\begin{aligned} f(g(x)) &= f(x^2 + x) && \text{next, replace } x \text{ in } 3x - 2 \text{ with } x^2 + x \\ &= 3(x^2 + x) - 2 \\ &= 3x^2 + 3x - 2 \end{aligned}$$

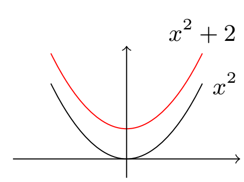
(e)

$$\begin{aligned} g(f(x)) &= g(3x - 2) && \text{next, replace each } x \text{ in } x^2 + x \text{ with } 3x - 2 \\ &= (3x - 2)^2 + (3x - 2) \\ &= 9x^2 - 12x + 4 + 6x - 2 \\ &= 9x^2 - 6x + 2 \end{aligned}$$

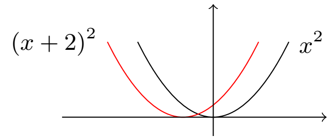
Example 2. Let $f(x) = x^2$. In each case start by finding the formula for the indicated function, and then figure out what the graph looks like (using your calculator or what you know about graphs of parabolas). Afterwards describe the graph geometrically as it compares to the original graph of x^2.
(a) $f(x) + 2$.
(b) $f(x + 2)$.
(c) $f(x - 2)$.
(d) $2f(x)$.
(e) $f(2x)$.

Solution.

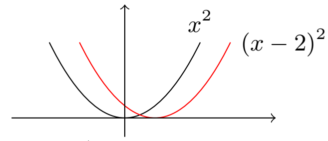
(a) $f(x) + 2 = x^2 + 2$.
The standard parabola has been shifted up by 2.



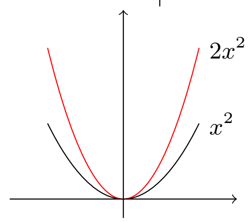
(b) $f(x+2) = (x+2)^2$
The standard parabola has been shifted left by 2.



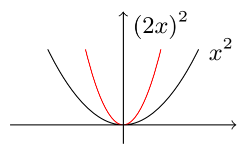
(c) $f(x-2) = (x-2)^2$.
The standard parabola has been shifted right by 2.



(d) $2f(x) = 2x^2$.
The standard parabola has been stretched by 2.



(e) $f(2x) = (2x)^2$.
The standard parabola has been horizontally squished by 2.



Example 3. Let $f(x) = e^x$ and $g(x) = x^2$. Find: (a) $f(g(1))$, (b) $g(f(1))$, (c) $f(g(x))$.

Solution. Start from the inside out, replace x in each formula with whatever is inside the parentheses of f or g:

$$\begin{aligned}
(a) \qquad f(g(1)) &= f(1^2) \\
&= e^1 \\
&= e \\
(b) \qquad g(f(1)) &= g(e^1) \\
&= (e)^2 \\
&= e^2 \\
(c) \qquad f(g(x)) &= f(x^2) \\
&= e^{(x^2)} \\
&= e^{x^2}
\end{aligned}$$

This is where we ended on Monday, February 10

Example 4. In Hughes-Hallet, 4e, 1.4#36, we started with supply $q = 0.5p - 25$, demand $q = 165 - 0.5p$, and an equilibrium point of \$190 and 70 items.

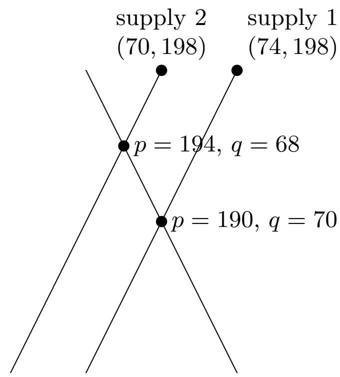
Then a \$8.00 per item tax was imposed on the supplier. This changed the supply curve to $q = 0.5p - 29$ and gave a new equilibrium point of \$194 and 68 items.

Interpret the above information in terms of shifted curves.

Solution. Let the original supply curve be given by $f(p) = 0.5p - 25$. Then the new supply curve is given by substituting in $p - 8$:

$$f(p-8) = 0.5(p-8) - 25 = 0.5p - 29.$$

Comparing the graph of $f(p)$ and $f(p-8)$ we see that the second is a shifted version of the first; the shift can be viewed as a shift left by 4 or down by 8.



Because the second supply curve has been shifted, it intersects the demand curve at a different point, a little to the left and higher than the first intersection.

This is where we ended on Wednesday, February 12

1.9 Power Functions

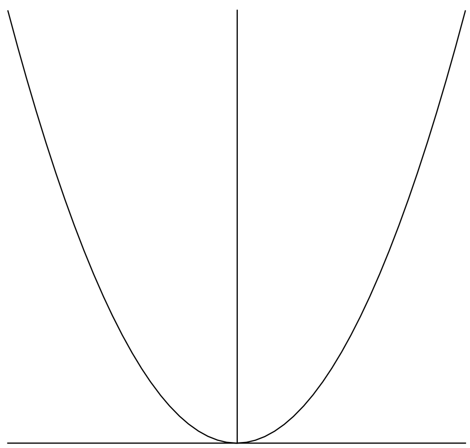
Recall the following rules:

$$\begin{aligned} a^{-b} &\text{ means } \frac{1}{a^b} \\ a^{1/b} &\text{ means } \sqrt[b]{a} \\ (a^n)^m &= a^{nm} \\ a^n a^m &= a^{n+m} \\ \frac{a^n}{a^m} &= a^{n-m} \\ (ab)^n &= a^n b^n \end{aligned}$$

Note that in general the following are not equal (make sure you don't make the mistake of thinking they are)

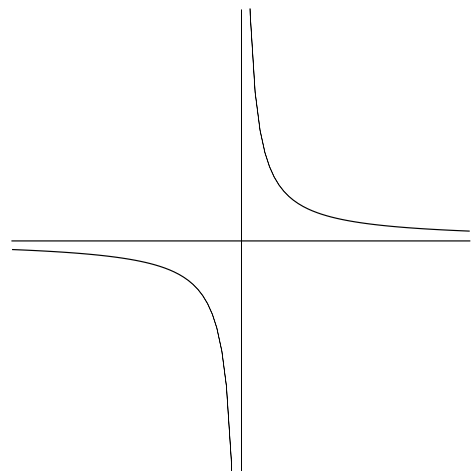
$$\begin{array}{ccccc} (x+y)^2 & \neq & x^2+y^2 & & (\text{unless } x=0 \text{ or } y=0) \\ \sqrt{x+y} & \neq & \sqrt{x}+\sqrt{y} & & (\text{unless } x=0 \text{ or } y=0) \end{array}$$

You should recall what the graphs of some basic power functions look like: x^2, $\frac{1}{x}$, x^3 and $\sqrt{x}$. You should be able to figure out the shape without a calculator, and also just be able to recognize them if the graphs are shown.



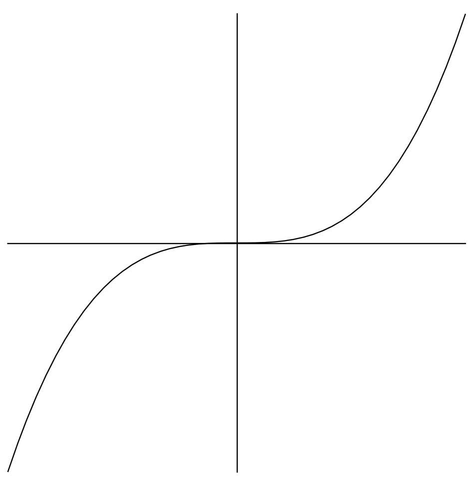
x^2

Notes: positve and negative x-values make the same y-values (because squaring stuff makes it positive). A little bit bigger x-values make a lot bigger y-values (like $2^2 = 4$, but $5^2 = 25$).



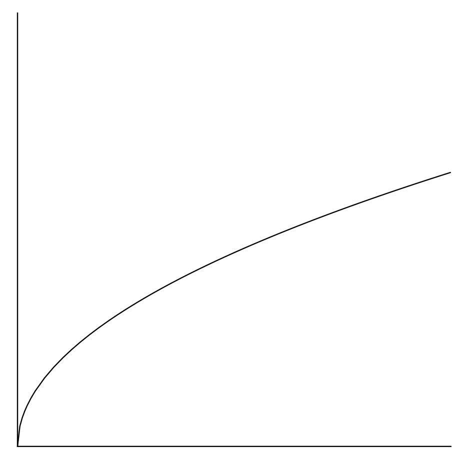
$\frac{1}{x}$

Notes: Bigger x-values make y-values close to 0 (like $1/100 = 0.01$). There's a vertical asymptote at $x = 0$ (this is what happens when you try to divide 1 by 0).



x^3

Notes: It looks kind of like x^2 on the right, but it's negative on the left.

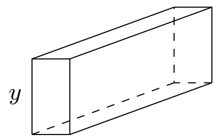


$\sqrt{x}$

Notes: It has exactly the same shape as left half of x^2, but turned on it's side.

Definition. A **power function** is one of the form $f(x) = Cx^n$, where C is a constant, called the **proportionality** constant, and n is real number. We call n the **power** and we say that x^n is a **power** of x.

Example 1. It is known that the strength S of a wooden board is proportional to the square of the width y of the board. Write a formula for S.



Solution. We have $S = Cy^2$ for some constant C.

Example 2. (Hughes-Hallett, 4e, 1.9#18) The surface area of a mammal, S, satisfies the equation $S = kM^{2/3}$, where M is the body mass, and the constant of proportionality k depends on the body shape of the mammal. A human of body mass 70 kg has a surface area of $18,600\,\mathrm{cm}^2$. Find the constant of proportionality for humans. Find the surface area of a human with body mass 60 kg.

Solution. We solve for k by plugging in 70 and 18,600:

$$\begin{aligned} S &= kM^{2/3} \\ 18600 &= k70^{2/3} \\ k &= \frac{18600}{70^{2/3}} \\ &\approx 1095.08 \end{aligned}$$

Now that we know k, we use it in the formula:

$$\begin{aligned} S &= 1095.08\, M^{2/3} \\ S &= 1095.08(60)^{2/3} \\ &\approx 16,783\,\mathrm{cm}^2 \end{aligned}$$

Chapter 2

The Derivative

2.1 Tangent and Velocity Problems

The problems in this section are at the heart of Calculus and lead directly to the main idea in Calculus, limits. But even more important than the problems themselves is the technique we use to solve them. Generalizing this technique leads directly to limits and derivatives.

Example 1. The height of a thrown ball is given by the following function:

$$p(t) = -4.9t^2 + 3.5t + 2$$

where t is in seconds and p is in meters. Find an approximation of the velocity at $t = 2.3$.

Solution. We start with the definition of velocity:

$$\text{Velocity} = \frac{\text{change in distance}}{\text{change in time}}.$$

To calculate change in distance we need to use *two* values for time.

Let's start with $t = 2.2$ and $t = 2.3$:

$$p(2.3) = -15.871,\ p(2.2) = -14.016 \quad \Rightarrow \quad \text{vel } = \frac{-15.871 - (-14.016)}{2.3 - 2.29} = -18.55\,\text{m/s}.$$

To make our answer more accurate, we can repeat the calculation, but this time using $t = 2.29$, instead of 2.2, because 2.29 is closer to 2.3. Here's what we get

$$p(2.3) = -15.871,\ p(2.29) =, 15.681 \quad \Rightarrow \quad \text{vel } = \frac{-15.871 - (15.681)}{2.3 - 2.29} = 18.99\,\text{m/s}.$$

Of course this might be accurate enough. But we don't know for sure until we go one step further. Let's use $t = 2.2999$

$$p(2.3) = -15.871,\ p(2.2999) = -15.869, \quad \Rightarrow \quad \text{vel } = \frac{-15.871 - (-15.869)}{2.3 - 2.2999} = -19.04\,\text{m/s}.$$

Based on the above calculations, our best guess is $-19.04\,\text{m/s}$.

Now I'll show you a way to repeat these calculations but to do them faster, with more accuracy, and letting the calculator take care of more of the details.

First, we enter the original formula in `Y1`:

$$\texttt{Y1} = -4.9\texttt{x}^2 + 3.5\texttt{x} + 2$$

Now, we create a function to do the calculations like $\frac{p(2.3) - p(2.29)}{2.3 - 2.29}$. The second number can change, so we use `x` for this number:

$$\texttt{Y2} = (\texttt{Y1}(2.3) - \texttt{Y1}(\texttt{x}))/(2.3 - \texttt{x})$$

To enter "`Y1`" in this formula go to `VARS`, `Y-VARS`, `1:Function`, `1:Y1`.

Now you could use `Y1` directly to repeat some of the above calculations, but I'll show you how to make a table that does it more quickly, especially if you want to do it for three or four or five numbers.

First, go to `TBLSET` (that's `2ND`, `WINDOW`). Make sure you have `Indpnt: Auto Ask` selected (this means that the calculator will ask you for the x-value, the "independent" variable).

Then go to `TABLE` (that's `2ND`, `GRAPH`) and enter some of the above x-values, and maybe some other ones too. You should get something like this:

x	Y1	Y2
2.3	-15.87	ERR :
2.29	-15.68	-18.99
2.299	-15.85	-19.04
2.2999	-15.87	-19.04

Make sure you remember that `Y2` is calculating things like $\frac{p(2.3) - p(2.29)}{2.3 - 2.29}$. The calculator gives an error in the first line because this formula would give division by 0 if $x = 2.3$. Also, if you want more accuracy than what is shown in the table, you can put your cursor on the number and the calculator will show you all the digits.

This is where we ended on Monday, February 17

There are two really hugely important conclusions we will draw from this example: (1) Sometimes we can make a sequence of approximations that appear to be getting closer and closer to the correct answer; (2) A difference quotient (i.e. a fraction with a subtraction on top and on the bottom) may be an important example of a thing that we can approximate like this.

In fact, both of these observations are extremely profound and important. The first one becomes, in its final from, the concept of a mathematical limit. The second one becomes the derivative.

Finally, note that even the partial solution we have of this problem is a monumental step forward in the history of science. The very idea that you could quantify physical processes, and do mathematics with these quantities, was incomprehensible to almost everyone, until Galileo. After Galileo it took another 100 years before people understood that you could start with one quantified process, position in this case, and do something to it to calculate its rate of change.

Definition. The **instantaneous velocity** is the limit of average velocities over shorter and shorter time intervals.

This definition merely records what we did in the previous example. However, we now want to *extend* what we have done, by applying the same idea, but to other functions besides position.

Definition. Given a function f, and an input number a, we define the following:

$$f'(a) = \text{the limit of average rates of change of } f \text{ over shorter and shorter intervals that contain } a$$

We call $f'(a)$ the **derivative** of f at a (we also call it the **instantaneous rate of change** of f at a, and sometimes just the **rate of change** of f at number a).

Example 2. The quantity of some drug in a person's blood is given by $Q = 500(0.9)^t$, with Q in mg and t in hours. Estimate the rate of change of the quantity of drug at time $t = 1.5$.

Solution. Recall that "rate of change" means $Q'(1.5)$ and is found by calculating average rates of change using shorter and shorter intervals that contain 1.5. For convenience, we will fix one our t-values at 1.5, and change the other t-value[1]

Here's one calculation using 1.5 and 1.4:

$$Q(1.5) \approx 426.9,\ Q(1.4) \approx 431.4,\ Q'(1.5) \approx \frac{426.9 - 431.4}{1.5 - 1.4} = -45\,\text{mg/hr}.$$

Now we repeat, to get more accuracy, with 1.5 and 1.49. Actually, there's something wrong with the following calculation, see if you can see what it is:

$$Q(1.5) \approx 426.9,\ Q(1.49) \approx 427.4,\ Q'(1.5) \approx \frac{426.9 - 427.4}{1.5 - 1.49} = \frac{-0.5}{0.01} = -50\,\text{mg/hr}.$$

The problem is that we want a little more accuracy in our final answer, but the 0.5 is all we have on top. All the other numbers got cancelled when we subtracted $426.9 - 427.4$. We have four digits here, but at least two of them cancel, leaving at most two digits left. If we want more digits of accuracy to be left, we need to include a lot more digits to the right of what is cancelled. Let's try that calculation again with more digits:

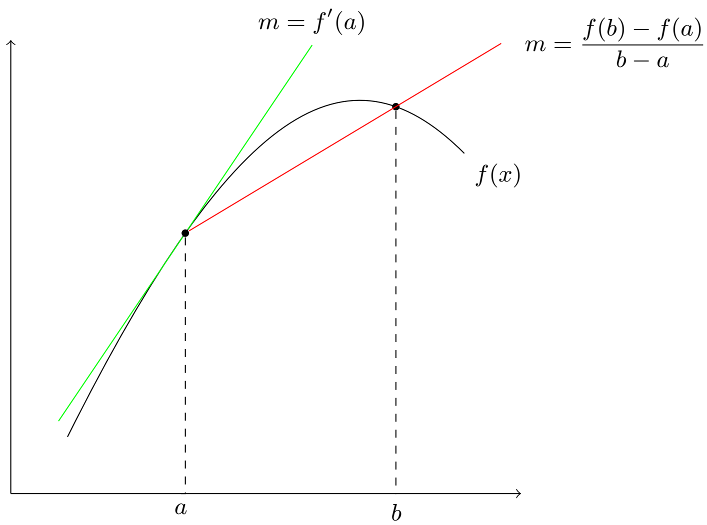
$$Q(1.5) \approx 426.9074841,\ Q(1.49) \approx 427.3575131,$$
$$Q'(1.5) \approx \frac{426.9074841 - 427.3575131}{1.5 - 1.49} = \frac{-.4500290}{0.01} = -45.00\,\text{mg/hr}$$

Based on this calculation our best guess for the answer is -45 mg/hr.

Fact. The derivative of f at a, in other words $f'(a)$, equals the slope of the tangent line of f at a (in other words the slope of the line that is tangent to the graph of f at the point $(a, f(a))$).

[1] As opposed to using two t-values that squeeze in on 1.5, such as 1.4 and 1.6

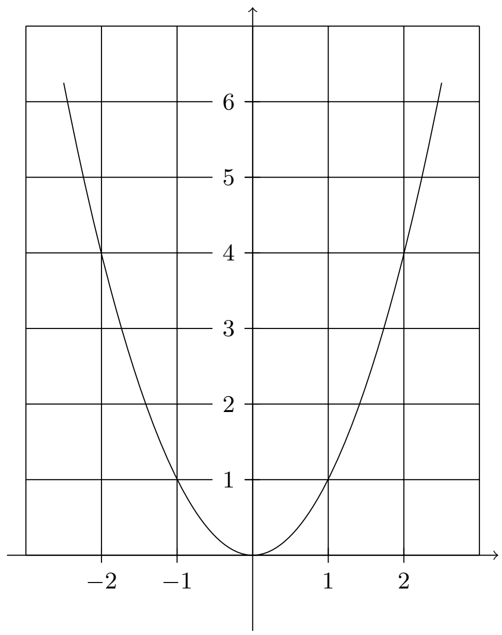
Explanation. Recall that the rate of change $\dfrac{f(b)-f(a)}{a-b}$ represents the slope through two points on the graph of $f(x)$ (i.e. the slope of a secant line). Thus, the derivative represents the slope of a line that we get by moving the two points closer and closer together. If you move b closer and closer to a, in the picture below, the secant line (in red) moves closer and closer to the tangent line (in green):



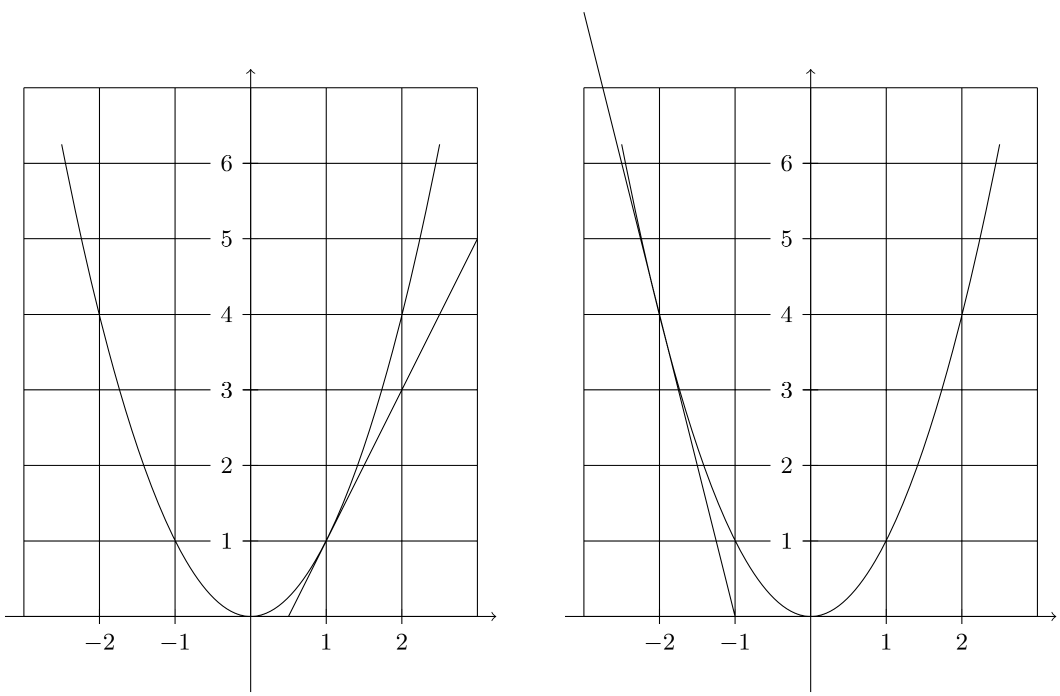
□

People frequently say "the slope of f at a" instead of "the slope of the tangent line of f at $x = a$" and there is nothing wrong with this provided one remembers the role of the tangent line.

Example 3. Using the graph of $f(x)$ shown below, estimate $f'(-2)$ and $f'(1)$. (Hint: the best way to do this is to draw a tangent line at the point, extend it either as long as possible, or until it hits a point on the grid that has a clear value. Calculate the slope of the tangent line using either points at/near the end of the line, or points with a clear value on the grid.)



Solution. We start by drawing tangent lines at $x = 1$. This is a line that should just touch the curve at one point, namely $(1, 1)$. The line should have the same slope as the curve at that point. We do the same thing at $(-2, 4)$. This is what it looks like:



Now, $f'(1)$ is the slope of the first line. We use the points at the end of the line to estimate the slope (we use points at the end, as opposed to the middle, to minimize the effect of any errors we make when we read off the values for the coordinates). It looks to me like we get $\Delta y \approx 5$ and $\Delta x \approx 2.5$. Thus, $m \approx 2$ and so $f'(1) \approx 2$.

We do a similar drawing for $f'(-2)$, but this time at the point $(-2, 4)$. This time it looks like $\Delta y \approx -8$ and $\Delta x \approx 2$, so $m \approx -4$ and $f'(-2) \approx -4$.

This is where we ended on Wednesday, February 19

2.2 The derivative as a function

In the previous section we defined $f'(a)$, the derivative of f at the *number* a. Therefore, in that section, we were just looking at the derivative at one point at a time. in this section, we look at the derivative at more than one point at a time.

Definition. Given any function f, we define the **derivative function** f' as follows:

$$f'(x) = \text{the derivative of } f \text{ at } x$$

Example 1. Let $p(t) = -4.9t^2 + 3.5t + 2$ (as in Section 2.1, Example 1). Later in the course we will show that $p'(t) = -9.8t + 3.5$. Assume for now that this is true.
(a) Find the velocity at $t = 2.3$.
(b) Find when the velocity will be 0.

Solution. Recall that "derivative" means rate of change and that if you find the rate of change position, that gives you velocity. Thus, we have the velocity

$$v(t) = -9.8t + 3.5\ .$$

(a) This is $v(2.3) = -9.8(2.3) + 3.5 = -19.04\,{}^{\mathrm{m}}\!/\!_{\mathrm{s}}$.
(b) Since $v(t) = -9.8t + 3.5$, we want to solve $0 = -9.8t + 3.5$. This is easy

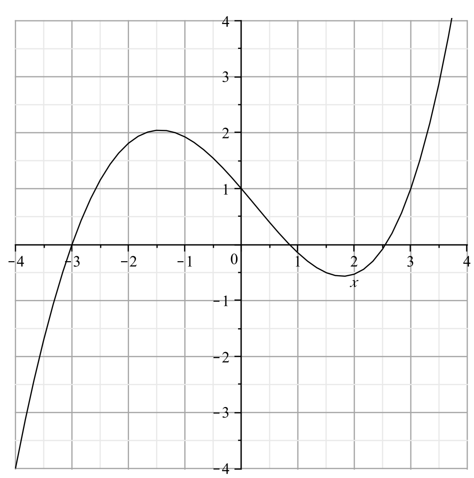
$$-9.8t + 3.5 = 0 \;\Rightarrow\; -9.8t = -3.5 \;\Rightarrow\; t = \frac{-3.5}{-9.8} = 0.357$$

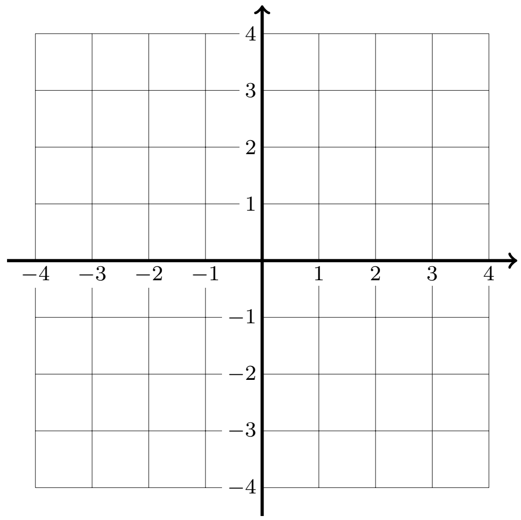
Note: to solve the second question, we need to have a function for the derivative, not just be able to find the derivative at specific points in time.

Now we describe what the derivative tells us graphically:

If $f'(x) > 0$ then f is *increasing* around x
If $f'(x) < 0$ then f is *decreasing* around x
If $f'(x) = 0$ then the graph of f is horizontal at x

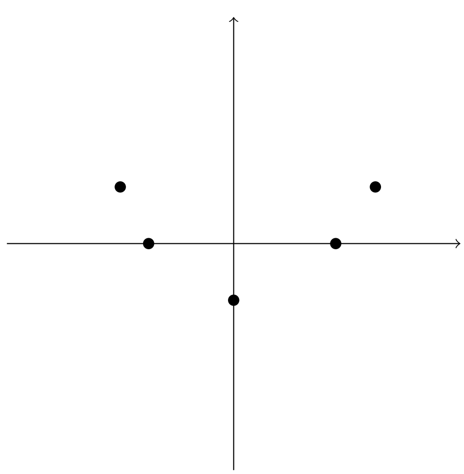
Example 2. (Based on Hughes-Hallett, 4e, 2.2#4) Based on the following graph of the function $f(x)$, make a rough sketch of the graph of $f'(x)$.



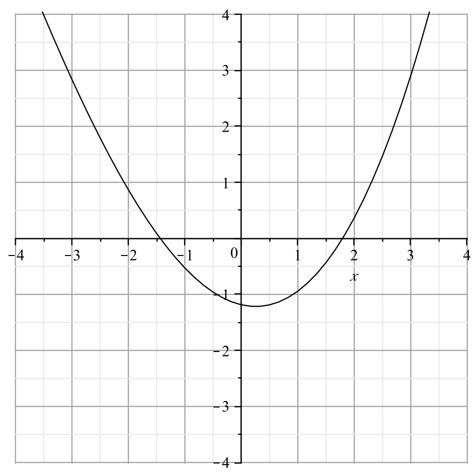


Solution. We could repeat the steps we did in Example 3 for, say, 5 points, and find $f'(x)$ as accurately as possible at these 5 points. But, since we are making a rough sketch, we don't need to be quite as accurate. We'll repeat the same some of the same steps, but you don't need to calculate the slope of each tangent line quite as accurately.

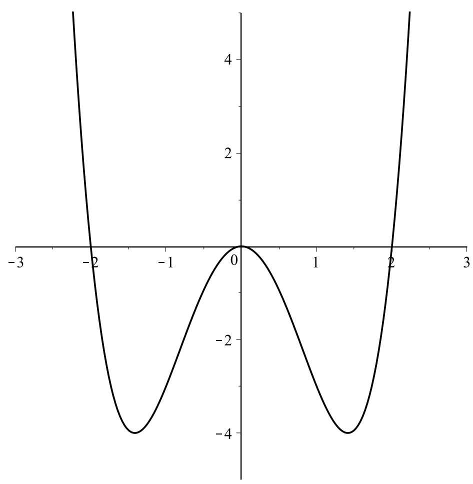
For instance, it looks like the slope is zero at $x = -1.5$ and $x = 1.8$, i.e. $f'(-1.5) \approx 0$ and $f'(1.8) \approx 0$. Between these two points, it looks like the slope should be negative. For instance $f'(0)$ should be negative, perhaps about -1, i.e. $f'(0) \approx -1$. To the left of $x = -1.5$ we should have a positive slope, for instance $f'(-2) \approx 1$. To the right of -1.8 we have positive slope, for instance $f'(2.5) \approx 1$. So far, we have the following points



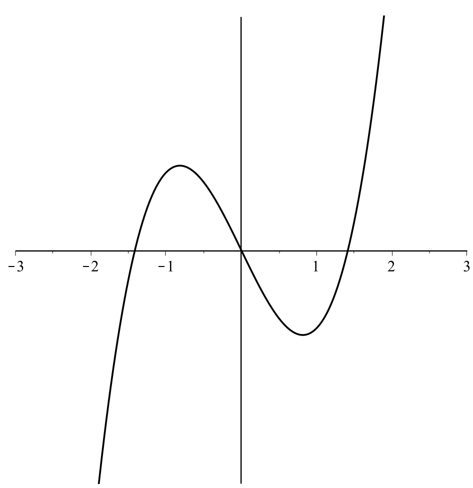
Now, add to these points a nice smooth curve, that extends upwards on both sides. What you get by hand might resemble what is shown below (which is based on the actual formula I used to make the graphs).



Example 3. Make a rough sketch of the derivative of the following graph



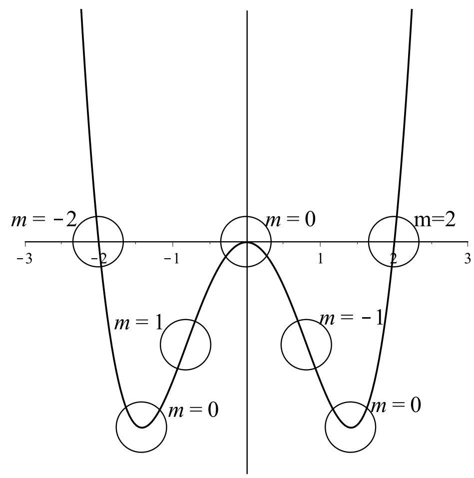
Solution. We will justify below that the following is a reasonable sketch of the derivative



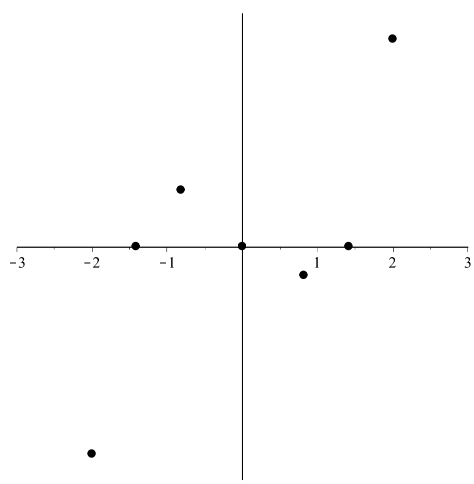
(I left off the tickmarks on the y-axis because in this problem we weren't meant to have answers that were that numerically accurate: the graph should mereley have the right shape).

To see that this is a reasonable sketch, start by identifying various slopes on the original graph. Here's the thought process I went through for this:

- Start with where the slope is 0 (marked below with "$m = 0$"),
- Identify one spot where the slope is positive (marked below with "$m = 1$", where "1" is kind of made up),
- Identify one spot where the slope is negative (marked below with "$m = -1$"),
- Identify other spots where the slope is positive or negative, and figure out if it's steeper or shallower than the first spots (marked below with $m = 2$ and $m = -2$ where I used "2" because it's steeper than 1).



Now we have 7 slopes that are labelled. Turn these into y-values on the graph of the deriavtive:



Now just put a smooth curve through the points and you get what we claimed was the solution above.

2.3 Variations on the derivative

In this section we learn variations on how we write derivatives, interpret them and use them. I'll put all three concepts right up front, and then we'll do our examples.

- **Leibniz notation for derivative:** $f'(x) = \dfrac{df}{dx}$, $V'(t) = \dfrac{dV}{dt}$, $C'(q) = \dfrac{dC}{dq}$, etc.

  This notation has some advantages over the prime notation
  - It suggests a ratio,
  - It makes explicit the role of both the input and output variables,
  - It makes the chain rule (coming later) look nice,
  - We can use notation $\dfrac{d}{dx}$ (or $\dfrac{d}{dt}$, or $\dfrac{d}{dq}$, etc) to mean 'take the derivative of ...".
  - It reads as "the derivative of, ..., with respect to x"
  - this notation is easier to see than $'$.

  To plug in a number, such as $x = 5$, into the notation $f'(x)$, we write $f'(5)$. To plug in this number in Leibniz notation we write $\left.\dfrac{df}{dx}\right|_{x=5}$.
- **Units Rates of Change:**
  - The derivative of any function equals the rate of change of that function, with the units given by $\dfrac{\text{units of output}}{\text{units of input}}$.
- **Linear Approximation** The linear approximation of $f(x)$ near $x = a$ is the tangent line function, where $x = a$ is the point it's tangent at. In other words, it is

$$\begin{array}{ccccc} f(x) \approx & f(a) & + & f'(a) & \Delta x \\ & \| & & \| & \| \\ & y_0 & + & m & (x - x_0) \end{array}$$

  or to state it one more way,

$$f(x) \approx y, \quad y = m(x - x_0) + y_0.$$

This is where we ended on Friday, February 21

Example 1. (Hughes-Hallett, 4e, 2.3#5) The cost, $C = f(w)$, in dollars, of buying a chemical is a function of the weight bought, w, in pounds.

(a) In the statement $f(12) = 5$, what are the units of the 12? What are the units of the 5? What is C? Explain what this is saying about the cost of buying the chemical.

(b) Do you expect the derivative f' to be positive or negative? Why?

(c) Rewrite the statement $f'(12) = 0.4$ in Leibniz notation. What are the units of the 12? What are the units of the 0.4? Explain what this is saying about the cost of buying the chemical?

Solution. (a) Since 12 is the input, we have $w = 12$ pounds. Since 5 is the output, we have $C = \$5$. Thus, it costs \$5 to buy 12 pounds of chemical.

(b) I expect the derivative to be positive. It should cost more to buy more pounds of chemicals. Thus, the function should be increasing, and so f' should be positive.

(c) $f'(12) = 0.4$ can be written as $\left.\dfrac{dC}{dw}\right]_{w=12} = 0.4$. The units of 12 are still pounds. The units of 0.4 are $\dfrac{\text{units of } C}{\text{units of } w} = \dfrac{\$}{\text{pound}}$. This is saying that buying each additional pound of chemical costs an additional \$0.4.

Example 2. (Hughes-Hallett, 4e, 2.3#29) For some painkillers, the size of the dose, D, given depends on the weight of the patient, W. Thus, $D = f(W)$, where D is in milligrams and W is in pounds.

(a) Interpret the statements $f(140) = 120$ and $f'(140) = 3$ in terms of this painkiller.

(b) Use the information in the statements of part (a) to estimate $f(145)$.

Solution. (a) The statement $f(140) = 120$ means that a 140 pound patient has a dose of 120 mg. To interpret $f'(140) = 3$ we start with Leibniz notation and units:

$$\frac{dD}{dW} = 3\,{}^{\text{mg}}/_{\text{pound}}$$

This means that the dose should increase by 3 mg for each additional pound of weight of the patient.

(b) To estimate $f(145)$ means "estimate the dose for a 145 pound patient". We know that a 140 pound patient gets 120 mg, and that each additional pound of weight increases the dose by 3 mg. Thus, the dose should be

$$120 + 3 \cdot 5 = 135$$

Note this is an example of the formula $y = y_0 + f'(x)\Delta x$:

$$\begin{array}{cccc} y = & y_0 + & f'(x) & \Delta x \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 135 = & 120 + & 3 & \cdot\ 5 \end{array}$$

Example 3. (Hughes-Hallett, 4e, 2.3#9) The temperature, T, in degrees Fahrenheit, of a cold yam placed in a hot oven is given by $T = f(t)$, where t is the time in minutes since the yam was put in the oven.

(a) What is the sign of $f'(t)$? Why?

(b) What are the units of $f'(20)$? What is the practical meaning of the statement $f'(20) = 2$?

Solution.

(a) $f'(t)$ is positive, because the temperature of the yam increases.

(b) The units of $f'(20)$ are °F/min. Practically, $f'(20) = 2$ means that after 20 minutes, the temperature of the yam will increase by 2 °F for each eadditional minute.

2.4 The second derivative

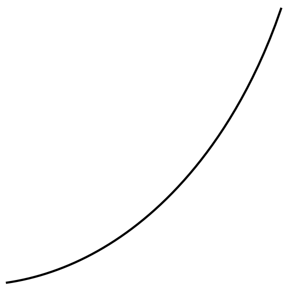
Definition. Recall that we can start with one function, $f(x)$, and define a second function $f'(x)$, the derivative function of $f(x)$. We can repeat this process and define the **second derivative**

$$\begin{aligned} f''(x) &= \text{the derivative of } f' \text{ at } x \\ &= \frac{d^2 f}{dx^2} \end{aligned}$$

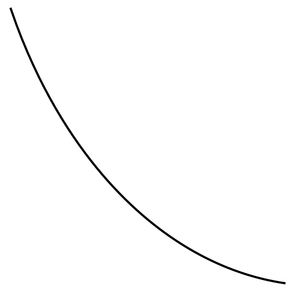
Now we describe what the second derivative tells us graphically:

If $f''(x) > 0$ then f' is *increasing* around x, and so f is concave up
If $f''(x) < 0$ then f' is *decreasing* around x, and so f is concave down

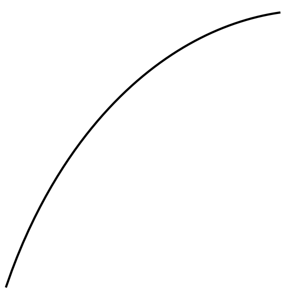
Recall that there are *four* pictures that relate concave up/down to increasing/decreasing.



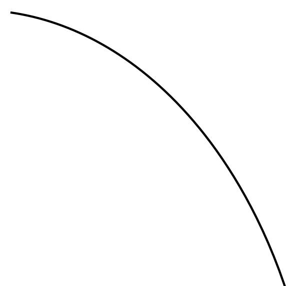
$f'(x) > 0, \quad f''(x) > 0$



$f'(x) < 0, \quad f''(x) > 0$



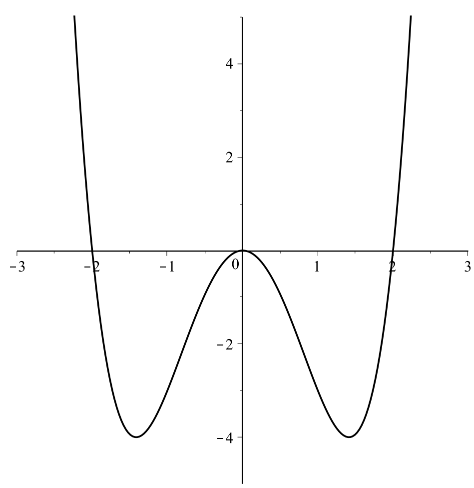
$f'(x) > 0, \quad f''(x) < 0$



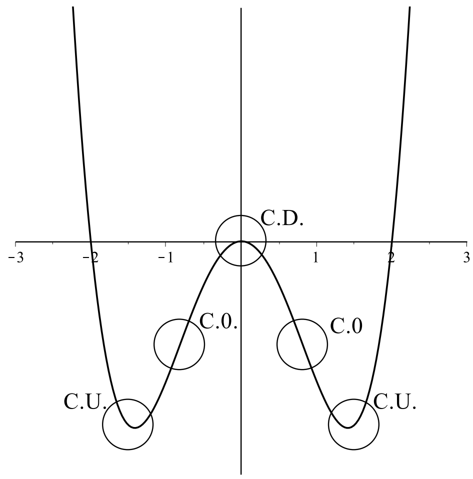
$f'(x) < 0, \quad f''(x) < 0$

This is where we ended on Monday, February 24

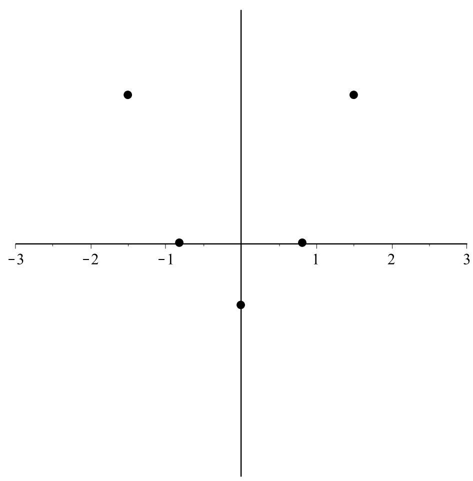
Example 1. Make a rough sketch of the *second* derivative of the following graph



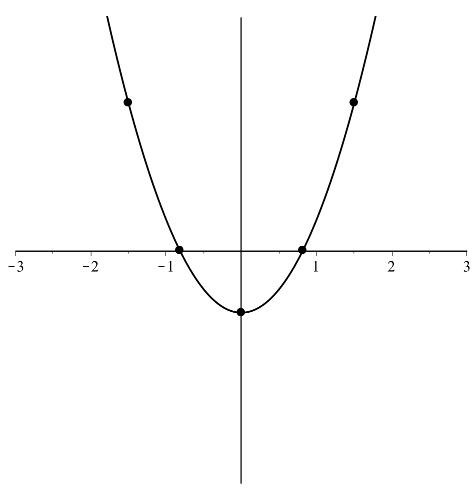
Solution. We start by identifying what the concavity is at 5 crucial parts of the original graph.



Now we turn those spots where we figured out the concavity, into 5 points on the graph we're making. Remember: turn the concavity on the original graph into y-values on the new graph (we'll make up y-values, say $y = 1$ for concave up and $y = -1$ for concave down):



Now, we simply make a smooth curve through the points we've found



Fact. The second derivative effects linear approximations as follows. Recall that the linear approximation of $f(x)$ near $x = a$ is given by the function $y = f'(a)(x - a) + f(a)$.

- If $f''(a) > 0$ then the graph of $f(x)$ lies *above* its tangent line and therefore $y = f'(a)(x - a) + f(a)$ is too *low* of an approximation.
- If $f''(a) < 0$ then the graph of $f(x)$ lies *below* its tangent line and therefore $y = f'(a)(x - a) + f(a)$ is too *high* of an approximation.

Example 2. In Example 2 from Section 2.3 we had the following:

$$\begin{aligned} f(140) &= 120 && \text{140 lb patient , 120 mg dose} \\ f'(140) &= 3 && \text{3 mg/lb change in dosage per pound} \end{aligned}$$

and we used it to estimate $f(145)$ as follows:

$$f(145) \approx \underset{y_0}{120} + \underset{+m\Delta x}{3\,(5)}$$

Suppose we add some additional information to this. Suppose we know that $f''(140) = 0.2$. Is the estimate we found earlier for $f(145)$ too high or too low?

Solution. Since 0.2 is positive, this means that $f(x)$ is concave up at $x = 140$. Thus, the graph of $f(x)$ lies *above* the tangent line. The estimate we found earlier is the value on the tangent line, $y_0 + m\Delta x$. Thus, the estimate is below the true value, i.e. the estimate we found earlier is too *low*.

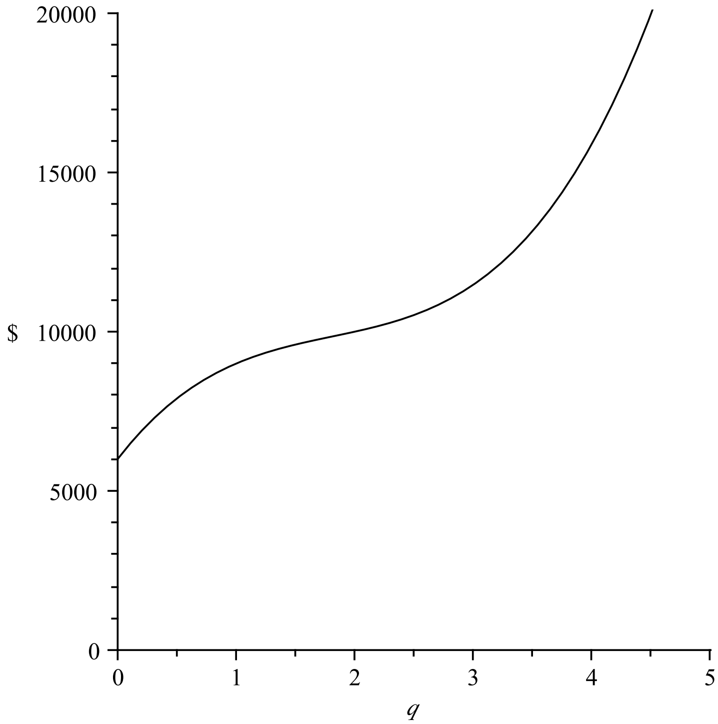
This is where we ended on Wednesday, February 26

2.5 Marginal Cost and Revenue

Recall that marginal cost is the rate of change of cost and marginal revenue is the rate of change of revenue. Now that we know what the derivative is, we can define these differently.

$$\text{MC (Marginal Cost)} = C'(q) \quad \approx C(q+1) - C(q)$$
$$\text{MR (Marginal Revenue)} = R'(q) \quad \approx R(q+1) - R(q)$$

Example 1. (Hughes-Hallett, 4e, 2.5 Ex. 3) The graph of a cost function is shown below. Does it cost more to make the 500th item or the 2000th? (This means just that item, not all the items 1, 2, 3, ..., 500, together.) At approximately what production level is marginal cost the smallest? What is the total cost at this level?



Solution. The cost to make the 500th item is the marginal cost. It is (closely approximated by) the slope of $C(q)$ at $q = 0.5$ (i.e $q = 0.5$thousand). We are not asked to estimate this slope, but just compare it to the slope at $q = 2$. In other words, which is bigger $C'(0.5)$ or $C'(2)$?

It is fairly easy to see that $C'(0.5) > C'(2)$. Thus, the 500th item costs more.

In fact, it appears that $C'(q)$ is smallest at $q = 2$. i.e. the marginal cost is smallest when 2000 items are made. The total cost at this level is $10000.

Example 2. (Hughes-Hallet, 4e, 2.5#11) Let $C(q)$ represent the cost and $R(q)$ represent the revenue, in dollars, of producing q items.

(a) If $C(50) = 4300$ and $C'(50) = 24$, estimate $C(52)$.

(b) If $C'(50) = 24$ and $R'(50) = 35$, approximately how much profit is earned by the 51st item?

(c) If $C'(100) = 38$ and $R'(100) = 35$, should the company produce the 101st item? Why or why not?

Solution. (a)

$$\begin{aligned} C(52) &\approx y_0 + m\Delta x \\ &= 4300 + 24(2) \\ &= 4348 \end{aligned}$$

(b) The 51st item will cost $24 and bring in a revenue of $35. Thus, the profit on that item will be $11.

(c) The 101st item will cost $38 and bring in a revenue of $35. Thus, it will produce a net loss. The company should not make it.

Chapter 3

Rules for Derivatives

3.1 Shortcuts for powers of x, e^x, constants, sums, and differences

Recall that Leibniz notation, such as $\frac{df}{dx}$ reads as "the derivative of f with respect to x", or "take the derivative of f with respect to x". There is a useful variation of this: we let $\frac{d}{dx}$ stand for the phrase "the derivative of". Using this notation, we state our first bunch of derivative rules.

Constant rule: $\frac{d}{dx}C = 0$ where C is a constant

Linear rule: $\frac{d}{dx}(mx + b) = m$

Constant multiple rule: $\frac{d}{dx}(C \cdot f(x)) = C \cdot f'(x)$, where C is a constant

Sum and Difference rule: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Power rule: $\frac{d}{dx}x^n = nx^{n-1}$ for any real number n

Combining the above rules we can take the derivative of any polynomial.

Example 1. Find the derivative of $y = 3.7x^5 - 253x^4 + 10x^2 + 7$; use at most one of the above rules at a time, and indicate which rule this is.

Solution.

$$\begin{aligned}
\frac{d}{dx}(3.7x^5 - 253x^4 + 10x^2 + 7) &= (3.7x^5)' - (253x^4)' + (10x^2)' + \frac{d}{dx}7 && \text{(sum and difference rule)}\\
&= 3.7(x^5)' - 253(x^4)' + 10(x^2)' + 0 && \text{(Constant multiple rule and constant rule)}\\
&= 3.7 \cdot 5x^4 - 253 \cdot 4x^3 + 10 \cdot 2x^1 && \text{(power rule)}\\
&= 18.5x^4 - 1012x^3 + 20x && \text{(cleaning up)}
\end{aligned}$$

This is where we ended on Friday, February 28

Example 2. We return to the problem posed in Section 2.1, Example 1 and Section 2.2, Example 1.

Recall that the ball had a position given by $p(t) = -4.9t^2 + 3.5t + 2$.

Find a formula for the velocity of the ball at time t.

Solution.

$$\begin{aligned} \text{vel} &= \text{deriv of position} \\ v(t) &= p'(t) \\ &= \frac{d}{dt}(-4.9t^2 + 3.5t + 2) \\ &= -4.9(2)t^1 + 3.5(1)t^0 + 0 \\ &= -9.8t + 3.5 \end{aligned}$$

Example 3. Find the derivative of each of the following functions.

(a) $y = 3.5x^7$
(b) $y = -2.5x^{-11.5}$
(c) $y = 5x^4 + 7x^3 - 12x^2 + 8x + 9$
(d) $f(x) = 2\sqrt{x}$
(e) $g(t) = 7\sqrt[5]{t}$
(f) $h(z) = \dfrac{11}{z^3}$
(g) $f(x) = 3.5x^2 + \dfrac{7}{x^2} - 11\sqrt{x}$
(h) $g(t) = at^2 + bt + c$ (assume that a, b and c are unknown constants).

Solution.

(a)

$$\begin{aligned} y' &= \frac{d}{dx}(3.5x^7) \\ &= 3.5(7)x^6 \\ &= 24.5x^6 \end{aligned}$$

(b)

$$\begin{aligned} y' &= \frac{d}{dx}(-2.5x^{-11.5}) \\ &= -2.5(-11.5)x^{-12.5} && \text{note: } -11.5 - 1 = -12.5 \end{aligned}$$

(c)

$$\begin{aligned} y' &= \frac{d}{dx}(5x^4 + 7x^3 - 12x^2 + 8x + 9) \\ &= 20x^3 + 21x^2 - 24x + 8 \end{aligned}$$

(d)

$$\begin{aligned} f(x) &= 2\sqrt{x} = 2x^{1/2} && \text{at least rewrite it like this in your head} \\ f'(x) &= 2\left(\tfrac{1}{2}\right)x^{-1/2} && \text{note: } \tfrac{1}{2} - 1 = -\tfrac{1}{2} \\ &= x^{-1/2} \end{aligned}$$

(e)

$$\begin{aligned} g(t) &= 7\sqrt[5]{x} = 7t^{1/5} && \text{at least rewrite it like this in your head} \\ g'(t) &= 7\left(\tfrac{1}{5}\right)t^{-4/5} && \text{note: } \tfrac{1}{5} - 1 = -\tfrac{4}{5} \\ &= \frac{7}{5}t^{-4/5} \end{aligned}$$

(f)

$$\begin{aligned} h(z) &= \frac{11}{z^3} = 11z^{-3} && \text{at least rewrite it like this in your head} \\ h'(z) &= 11(-3)z^{-4} \\ &= -33z^{-4} \end{aligned}$$

(g)

$$\begin{aligned} f(x) &= 3.5x^2 + \frac{7}{x^2} - 11\sqrt{x} \\ &= 3.5x^2 + 7x^{-2} - 11x^{1/2} && \text{at least rewrite it like this in your head} \\ f'(x) &= 3.5(2)x^1 + 7(-2)x^{-3} - 11\left(\tfrac{1}{2}\right)x^{-1/2} \\ & 7x - 14x^{-3} - \frac{11}{2}x^{-1/2} \end{aligned}$$

(h)

$$\begin{aligned} g'(t) &= \frac{d}{dt}(at^2 + bt + c) \\ &= 2at + b && \text{just treat } a, b \text{ and } c \text{ the same way you would treat any constant} \end{aligned}$$

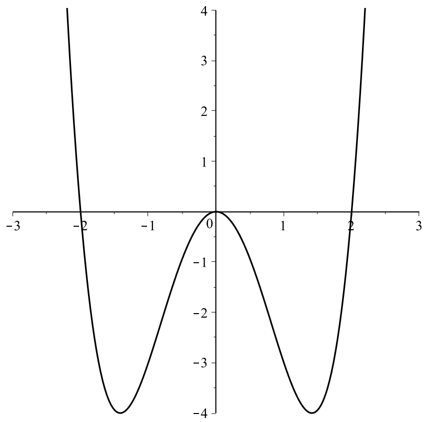
This is where we ended on Monday, March 10

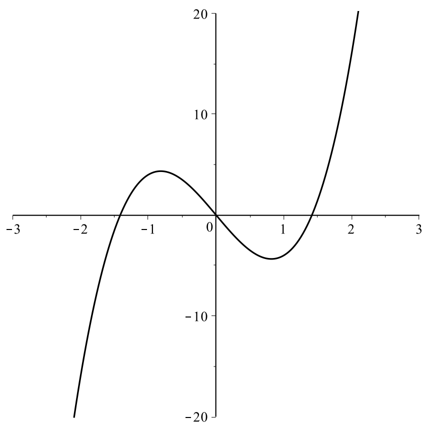
Example 4. Let $f(x) = x^4 - 4x^2$. Calculate $f'(x)$, $f''(x)$, and graph $f(x)$, $f'(x)$ and $f''(x)$. Compare your results to Examples 2.2, Ex. 3 and 2.4, Ex 1.

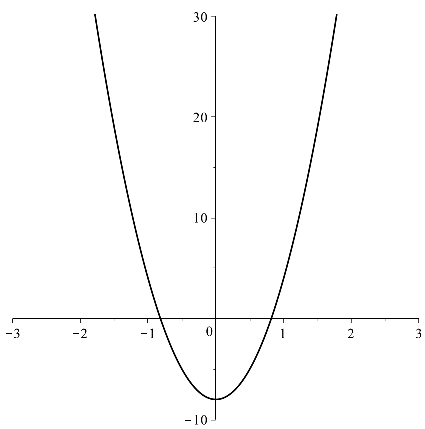
Solution.

$$\begin{aligned} f(x) &= x^4 - 4x^2 \\ f'(x) &= \frac{d}{dx}(x^4 - 4x^2) \\ &= 4x^3 - 8x \\ f''(x) &= \frac{d}{dx}(4x^3 - 8x) \\ &= 12x^2 - 8 \end{aligned}$$

Here are the graphs:







Definition. The **tangent line** to a function $f(x)$ at a point $x = a$ is given by

$$y = m(x - x_0) + y_0$$

where

$$x_0 = a,$$
$$y_0 = f(a),$$
$$m = f'(a).$$

You can combine the previous steps into one really short formula and say

$$y = f'(a)(x - a) + f(a).$$

Some students have a hard time seeing how to use the formula for the tangent line, at least at first. As an alternative, here's a recipe:

Step 0. You are given a function $f(x)$ and a number $x = a$.
Step 1. Plug $x = a$ into $f(x)$ and calculate this number.
Step 2. Find $f'(x)$; this is a formula.
Step 3. Plug $x = a$ into $f'(x)$ and calculate this number.
Step 4. Your formula is $y = m(x - x_0) + y_0$ where $x_0 = a$, y_0 is the number from step 1 and m is the number from step 3.

Example 5. Find the equation of the tangent line at $x = 5$ of $f(x) = 2x^2 - x + 3$.

Solution.

$$\begin{aligned} y &= m(x - x_0) + y_0 \\ x_0 &= a \\ y_0 &= f(5) = 2(5)^2 - 5 + 3 = 48 \\ f'(x) &= 4x - 1 \\ m &= 4(5) - 1 = 19 \\ & \boxed{y = 19(x - 5) + 48} \end{aligned}$$

3.2 Derivatives of exponentials and logarthms

$$\begin{aligned} &\text{Exponential rule:} \quad \frac{d}{dx}e^x = e^x \\ &\text{General Exponential rule:} \quad \frac{d}{dx}a^x = \ln(a)a^x \\ &\text{Logarithm rule:} \quad \frac{d}{dx}\ln(x) = \frac{1}{x} \end{aligned}$$

Example 1. The human population of the entire world can be modeled by

$$P = 6.8(1.011)^t$$

where P is in billions, and t is the year with $t = 0$ corresponding to 2010 (source Wikipedia).

Find the estimated rate of growth in 2020, and interpret your answer, with units.

Solution. Recall that

$$\text{rate of growth} \ = \ \text{derivative.}$$

Also, the year 2020 means that $t = 10$. Thus, we need to calculate $P'(10)$. *Always* find the derivative as a formula first, and then plug in the number. We use the general exponential rule

$$P'(t) = 6.8 \ln(1.011)(1.011)^t$$

and plug in $t = 10$

$$P'(10) = 6.8 \ln(1.011)(1.011)^{10} \approx 0.08299 \,\text{B/year}$$

In 2020 the population will be increasing by 82.99 million people each year.

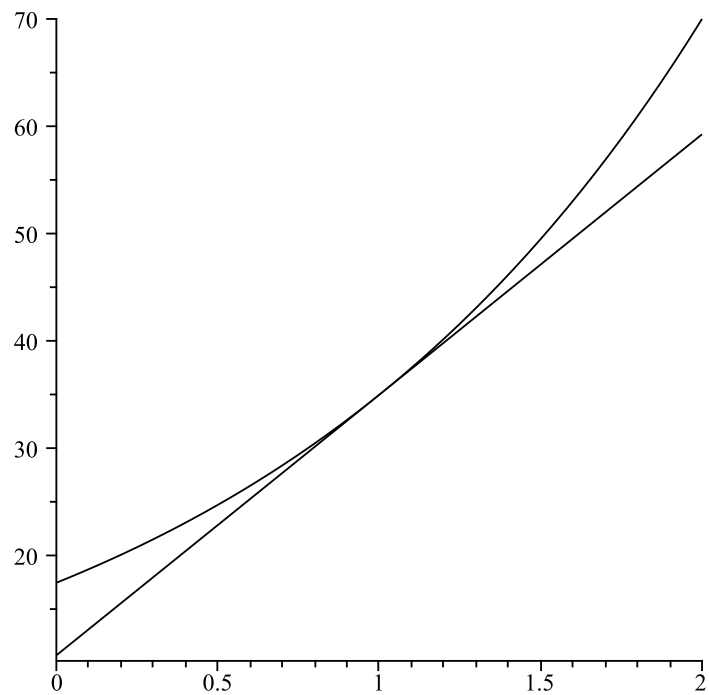
This is where we ended on Wednesday, March 12

Example 2. Find the equation of the tangent line of $f(x) = 17.5 \cdot 2^x$ at $x = 1$, and show a graph of the function $f(x)$ and the tangent line.

Solution. We have $y_0 = f(1) = 17.5(2) = 35$.
We have $f'(x) = 17.5\ln(2)2^x$
We have $m = f'(1) = 17.5\ln(2) \cdot 2 = 35\ln(2)$.
Putting it all together we have

$$y = 35\ln(2)(x - 1) + 35$$

Here's the graph



This is where we ended on Wednesday, March 19

3.3 The Chain Rule

The Chain Rule is the most interesting one we have learned so far, and it is one of the most complicated we will learn. It is one of the rules we will learn for taking the derivative of a combination of a functions. In particular, this rule will tell us how to take the derivative when one function is *inside* of another.

Example 1. (Based on Hughes-Hallett, 4e, 3.3 Example 1) The amount of gas, G, in gallons, consumed by a car depends on the distance traveled, s, in miles. But, suppose we want to know how much gas is consumed each *hour*, not each mile? Well, the distance traveled, s depends on the time traveled, t, in hours. Let 0.05 gallons of gas be consumed for each mile traveled, and suppose that the car is traveling at $30\,\mathrm{mi/hr}$. How fast is gas being consumed? Give units.

Solution. The information we are given can be summarized this way

$$\frac{dG}{ds} = 0.05 \,{}^{\text{gal}}\!/\!{}_{\text{mi}},$$
$$\frac{ds}{dt} = 30 \,{}^{\text{mi}}\!/\!{}_{\text{hr}},$$

and we want to find this:

$$\frac{dG}{dt} = ?$$

If we give these numbers and units the usual interpretation, we have this:

- We use 0.05 gallons by driving one mile,
- We drive 30 miles in one hour,
- How many gallons will we use in one hour?

The right way to answer this question is to combine the given information with multiplication:

$$\begin{aligned} \text{gallons in one hour } &= 30 \text{ miles in one hour } \times 0.05 \text{gallons in one mile} \\ & 30 \,{}^{\text{mi}}\!/\!{}_{\text{hr}} \times 0.5 \,{}^{\text{gal}}\!/\!{}_{\text{mi}} \\ &= 1.5 \,{}^{\text{gal}}\!/\!{}_{\text{hr}}. \end{aligned}$$

We can summarize this calculation in Leibniz notation this way:

$$\frac{dG}{dt} = \frac{dG}{ds} \cdot \frac{ds}{dt}.$$

Rule (Chain Rule in Leibniz Notation)**.** If y is a function of z and z is a function of t then

$$\frac{dy}{dz} = \frac{dy}{dz} \cdot \frac{dz}{dt}.$$

Example 2. Write each of the following functions as a function of z, where z is the "inside" function.

(a) $y = \sqrt{x^2 + 2x}$
(b) $y = 5(2x + 7)^8$
(c) $y = \dfrac{11}{x^2 + 1}$
(d) $y = -7.2e^{x^2}$
(e) $y = \frac{1}{2} \ln(3x^2 + 5)$

Solution. (a)

$$z = x^2 + 2x$$
$$y = \sqrt{z}$$

(b)

$$z = 2x + 7$$
$$y = 5z^8$$

(c)

$$\begin{aligned} z &= x^2 + 1 \\ y &= \frac{11}{z} \end{aligned}$$

(d)

$$\begin{aligned} z &= x^2 \\ y &= -7.2e^z \end{aligned}$$

(e)

$$\begin{aligned} z &= 3x^2 + 5 \\ y &= \tfrac{1}{2}\ln(z) \end{aligned}$$

This is where we ended on Friday, March 21

Example 3. Find the derivatives of the following functions. Use z for the inside function, and use the Leibniz notation for the chain rule.
(a) $y = 5(-3x^2 + 2x + 7)^{11}$.
(b) $y = \frac{7}{3}\ln(x^3 + x)$.

Solution. (a)

$$\begin{aligned} z &= -3x^2 + 2x + 7 \\ y &= 5z^{11} \\ \frac{dy}{dx} &= \frac{dy}{dz} \cdot \frac{dz}{dx} \\ &= 5 \cdot 11z^{10} \cdot (-6x + 2) \\ &= 55(-3x^2 + 2x + 7)^{11}(-6x + 2) \end{aligned}$$

(b)

$$\begin{aligned} z &= x^3 + x \\ y &= \tfrac{7}{3}\ln(z) \\ \frac{dy}{dx} &= \frac{dy}{dz} \cdot \frac{dz}{dx} \\ &= \tfrac{7}{3} \cdot \frac{1}{z} \cdot (3x^2 + 1) \\ &= \tfrac{7}{3} \cdot \frac{1}{x^3 + x} \cdot (3x^2 + 1) \\ &= \frac{7(3x^2 + 1)}{3(x^3 + x)} \end{aligned}$$

To some degree, a lot of people learn the chain rule one function at a time, as it applies to the outside function. In other words, they learn how the chain rule works on a case-by-case basis. There's nothing wrong with this (as long as the student can also use the chain rule in a new or general situation that doesn't fit any of the case-by-cases). In any case, here's how the chain rule looks for various functions that we know.

Rule. The following table shows the derivative of a variety of basic functions, and then a chain rule version for each basic fuction.

Usual version	Chain rule version
$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$	$\frac{d}{dx}\frac{1}{\square} = -\frac{1}{\square^2}\cdot\frac{d}{dx}\square$
$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$	$\frac{d}{dx}\sqrt{\square} = \frac{1}{2\sqrt{\square}}\cdot\frac{d}{dx}\square$
$\frac{d}{dx}x^n = nx^{n-1}$	$\frac{d}{dx}\square^n = n\square^{n-1}\cdot\frac{d}{dx}\square$
$\frac{d}{dx}e^x = e^x$	$\frac{d}{dx}e^{\square} = e^{\square}\cdot\frac{d}{dx}\square$
$\frac{d}{dx}\ln(x) = \frac{1}{x}$	$\frac{d}{dx}\ln(\square) = \frac{1}{\square}\cdot\frac{d}{dx}\square$

Example 4. Find the following derivatives

(a) $\frac{d}{dx}(3x-7)^{11}$

(b) $\frac{d}{dx}\sqrt{1.9x-4.3}$

(c) $\frac{d}{dx}\frac{1}{10x-4}$

(d) $\frac{d}{dx}e^{1.03x-1}$

(e) $\frac{d}{dx}\ln(2x+e^2)$

Solution. (a)

$$\begin{aligned}\frac{d}{dx}\left(\boxed{3x-7}\right)^{11} &= 11\left(\boxed{3x-7}\right)^{10}\cdot\frac{d}{dx}\boxed{3x-7}\\ &= 11(3x-7)^{10}\cdot(3)\end{aligned}$$

(b)

$$\begin{aligned}\frac{d}{dx}\sqrt{\boxed{1.9x-4.3}} &= \frac{1}{2}\left(\boxed{1.9x-4.3}\right)^{-1/2}\cdot\frac{d}{dx}\boxed{1.9x-4.3}\\ &= \frac{1}{2}(1.9x-4.3)^{-1/2}\cdot(1.9)\end{aligned}$$

(c)

$$\begin{aligned}\frac{d}{dx}\frac{1}{\boxed{10x-4}} &= \frac{-1}{(\boxed{10x-4})^2}\cdot\frac{d}{dx}\boxed{10x-4}\\ &= \frac{-1}{(10x-4)^2}\cdot(10)\end{aligned}$$

(d)

$$\begin{aligned}\frac{d}{dx}e^{\boxed{1.03x-1}} &= e^{1.03x-1}\cdot\frac{d}{dx}\boxed{1.03x-1}\\ &= e^{1.03x-1}\cdot(1.03)\end{aligned}$$

(e)

$$\begin{aligned}\frac{d}{dx}\ln\left(\boxed{2x+e^2}\right) &= \frac{1}{\boxed{2x+e^2}}\cdot\frac{d}{dx}\boxed{2x+e^2}\\ &= \frac{1}{2x+e^2}\cdot(2)\end{aligned}$$

This is where we ended on Monday, March 24

3.4 Product and Quotient Rules

Product rule

Function Notation: $(f \cdot g)' = f' \cdot g + f \cdot g'$

Leibniz Notation: $\frac{d}{dx}uv = \frac{du}{dx}\cdot v + u\cdot\frac{dv}{dx}$

"Word Notation": The derivative of the first, times the second, plus the first, times the derivative of the second.

Example 1. In Fall 2013, the undergraduate enrollment at Loyola University Maryland was 3875 and the tution was \$41850 per year (information taken from the 2013–2014 Loyola Catalogue). Hypothetically, suppose that the school is thinking of increasing it's tuition by \$125. Suppose that this would cause the enrollment to decrease by 3 students.

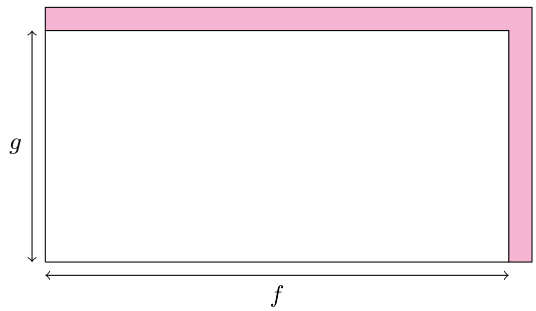
What would be the change in revenue?

Solution. Let ΔR be the change in revenue, $\Delta T = 125$ the change in tution, and $\Delta S = -3$ the change in number of students. The first thing to note is that

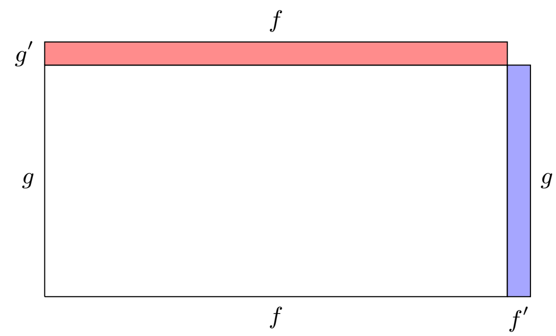
$$\Delta R \neq \Delta S * \Delta T!!!$$

In fact, if you did this calculation you would find that ΔR is negative, which is wrong.

Figure 3.1: Intuitive, graphical, understanding of product rule



f = length of rectangle, g = height,
fg = area.
$(fg)'$ = rate of change
$\approx$ change of area (purple)



$f' \approx$ change length,
$f'g \approx$ change length $\times$ height (blue),
$g' \approx$ change width,
$fg' \approx$ change height $\times$ length (red).

purple $\approx$ red + blue:

$$(fg)' = f' \cdot g + f \cdot g'$$

The problem is that ΔT needs to be combined with all the students, not just the 3 students who don't come. After all, most of the students will be paying more tuition, and so this will increase the revenue. Similarly ΔS needs to be combined with the entire tution, not the change in tuition.

Here is the right way to find ΔR:

$$\begin{aligned}
\Delta R &= \text{new revenue } - \text{ old revenue} \\
&= (T + \Delta T) \cdot (S + \Delta S) - T \cdot S \\
&= T \cdot S + T \cdot \Delta S + \Delta T \cdot S + \Delta T \cdot \Delta S - T \cdot S \\
&= T \cdot \Delta S + \Delta T \cdot S + \Delta T \cdot \Delta S \\
&= 41850(-3) + 125(3875) + 125(-3) \\
&= 358,825 - 375 \\
&= 358,450
\end{aligned}$$

There is one more thing to note here:

$$\Delta R \approx 358,825 = T \cdot \Delta S + \Delta T \cdot S.$$

This approximation is pretty accurate because ΔT and ΔS are both relatively small, and it would be even more accurate if ΔT and ΔS were smaller. If we imagine what happens as ΔT and ΔS both become so small that they approach 0, then we get the product rule:

$$R' = T \cdot S' + T \cdot S'.$$

Example 2. Find $\dfrac{d}{dx}(3x + e^x)(x^2 - 4e^x)$.

Solution. Identify f and g:

$$\underbrace{(3x+e^x)}_{f}\underbrace{(x^2-4e^x)}_{g}$$

Calculate f' and g':

$$\begin{aligned} f' &= 3+e^x \\ g' &= 2x-4e^x \end{aligned}$$

Put it all together using the product rule:

$$f'\cdot g + f\cdot g' = (3+e^x)(x^2-4e^x)+(3x+e^x)(2x-4e^x).$$

Example 3. Find $\dfrac{d}{dx}(x+e^{3x-1})\left(\dfrac{1}{x}+x\right)$.

Solution. Identify f and g:

$$\underbrace{(x+e^{3x-1})}_{f}\underbrace{\left(\frac{1}{x}+x\right)}_{g}$$

Calculate f' and g'

$$\begin{aligned} f' &= 1+e^{3x-1}\cdot 3 \\ g' &= -\frac{1}{x^2}+1 \end{aligned}$$

Put it all together using the product rule:

$$f'\cdot g + f\cdot g' = (1+3e^{3x-1})\cdot\left(\frac{1}{x}+x\right)+(x+e^{3x-1})\cdot\left(-\frac{1}{x^2}+1\right)$$

Quotient Rule

Function Notation: $\left(\dfrac{f}{g}\right)' = \dfrac{f'\cdot g - f\cdot g'}{g^2}$

Leibniz Notation: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{\frac{du}{dx}\cdot v - u\cdot\frac{du}{dx}}{v^2}$

"Word Notation":

The derivative of the top, times the bottom, minus the top times the derivative of the bottom, everything over the bottom squared.

Example 4. Find $\dfrac{d}{dx}\dfrac{2x+1}{e^x+x}$.

Solution. Identify f and g

$$\frac{2x+1 \leftarrow f}{e^x + x \leftarrow g}$$

Calculate f' and g'

$$f' = 2$$
$$g' = e^x + 1$$

Put it all together using the quotient rule

$$\frac{f' \cdot g - f \cdot g'}{g^2} = \frac{2(e^x + x) - (2x+1)(e^x + 1)}{(e^x + x)^2}$$

Example 5. Find the derivative of $\dfrac{t + \sqrt{t}}{t^2 + \ln(t)}$.

Solution. Identify f and g

$$\frac{t + \sqrt{t} \leftarrow f}{t^2 + \ln(t) \leftarrow g}$$

Calculate f' and g'

$$f' = 1 + \frac{1}{2\sqrt{t}}$$
$$g' = 2t + \frac{1}{t}$$

Put it all together using the quotient rule

$$\frac{f' \cdot g - f \cdot g'}{g^2} = \frac{(1 + \frac{1}{2\sqrt{t}})(t^2 + \ln(t)) - (t + \sqrt{t})(2t + \frac{1}{t})}{(t^2 + \ln(t))^2}$$

This is where we ended on Wednesday, March 26

Chapter 4

Using the Derivative

4.1 Local Max and Mins

Definition. Let $x = c$ be in the domain of $f(x)$.

$x = c$ is a	local maximum	if $f(x) \leq f(c)$ for all x near c. (we allow endpoints)
$x = c$ is a	local minimum	if $f(x) \geq f(c)$ for all x near c. (we allow endpoints)

Definition. If $x = c$ is in the domain of f and $f'(c) = 0$, then we call $x = c$ a **critical point**. We also call the (x, y)-point $(c, f(c))$ a critical point. We call $f(c)$ the **critical value**.

Theorem 1 (First derivative test)**.** To find the local max/mins of a function $f(x)$ do the following.

1. First find the critical points.
2. Figure out whether $f'(x)$ is $+$ or $-$ on each side of each critical point (four cases, lots of pictures):

$f'(x) = +, -$ left of c	right of c	outcome
$+$	$-$	$\Rightarrow x = c$ local max
$-$	$+$	$\Rightarrow x = c$ local min
$+$	$+$	$\Rightarrow x = c$ neither
$-$	$-$	$\Rightarrow x = c$ neither

Example 1. Let $f(x) = 4x^3 + 3x^2 - 6x$.

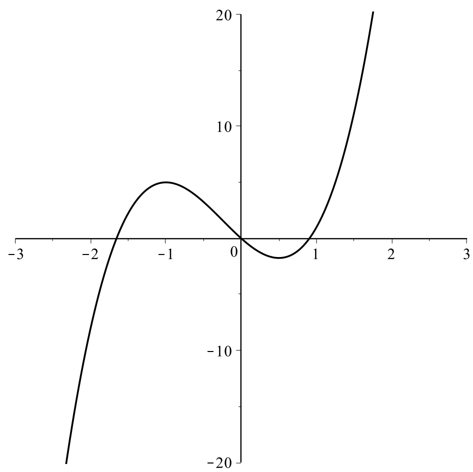
(a) Using your calculator, graph $f(x)$. Use a window that shows all the "interesting" features (in particular it should show the local max/mins).

(b) Describe the critical points, the local max and mins, and where the function is increasing/decreasing.

(c) Take the first derivative of $f(x)$, find the critical points algebraically.

(d) Summarize your work in a "1D# table" (1st Derivative Number Line Table) table that shows the first derivative test and the conclusions that it gives you.

Solution. (a) The graph of $f(x)$ is shown below



(b) (i) The critical points are where $m = 0$. This appears to be at $x = -1$ and $x = 1/2$.
(ii) There is a local max at $x = -1$ and a local min at $x = 1/2$.
(iii) f is increasing to the left of $x = -1$ and to the right of $x = 1/2$. Conversely, f is decreasing between $x = -1$ and $x = 1/2$.

(c)

$$
\begin{aligned}
f(x) &= 4x^3 + 3x^2 - 6x \\
f'(x) &= 12x^2 + 6x - 6 \\
f'(x) &= 0 \\
12x^2 + 6x - 6 &= 0 \\
6(2x^2 + x - 1) &= 0 \\
6(2x - 1)(x + 1) &= 0 \\
(2x - 1) &= 0 \Rightarrow 2x = 1 \Rightarrow x = 1/2 \\
\text{or} \quad (x + 1) &= 0 \Rightarrow x = -1
\end{aligned}
$$

(d)

	l. max $x = -1$		l. min $x = 1/2$	
$f \nearrow$			$f \searrow$			$f \nearrow$
$f' > 0$	$f' = 0$	$f' < 0$	$f' = 0$	$f' < 0$

This is where we ended on Friday, March 28

Example 2. Let $f(x) = e^{-2x} + 5x$.

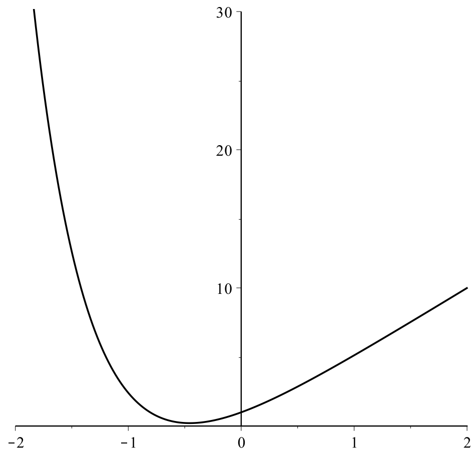
(a) Using your calculator, graph $f(x)$. Use a window that shows all the "interesting" features (in particular it should show the local max/mins).

(b) Describe the critical points, the local max and mins, and where the function is increasing/decreasing.

(c) Take the first derivative of $f(x)$, find the critical points algebraically.

(d) Summarize your work in a "1D# table" (1st Derivative Number Line Table) table that shows the first derivative test and the conclusions that it gives you.

Solution. (a) The graph of $f(x)$ is shown below



(b) (i) The critical points are where $m = 0$. This appears to be at $x = -1/2$.

(ii) There is a local min at $x = -1/2$.

(iii) f is decreasing to the left of $x = -1/2$ and increasing to the right of $x = -1/2$.

(c)

$$\begin{aligned}
f(x) &= e^{-2x} + 5x \\
f'(x) &= -2e^{-2x} + 5 \\
f'(x) &= 0 \\
-2e^{-2x} + 5 &= 0 \\
2e^{-2x} &= 5 \\
e^{-2x} &= 5/2 \\
\ln(e^{-2x}) &= \ln(5/2) \\
-2x &= \ln(5/2) \\
x &= -\frac{1}{2}\ln(5/2) \\
x &\approx -0.46
\end{aligned}$$

(d)

l. min

$x = -0.46$

$f \searrow$ | $f \nearrow$

$f' < 0$ $\quad f' = 0 \quad$ $f' > 0$

Theorem 2 (Second derivative test)**.** To find the local max/mins of a function $f(x)$ try the following.

1. First find the critical points.

2. Figure out whether $f''(c)$ is + or − (three cases):

$f''(c)$	outcome
+	local min
−	local max
0 or DNE	test says nothing

Example 3. The function $f(x) = \dfrac{\ln(x)}{x}$ has a critical point at $x = e$. Use the second derivative test to identify it as a local max/min.

Solution. We start by finding the first derivative, using the quotient rule. Since we state our quotient rule using "f" and "g", maybe it would be nice to use a different letter for the original function. Let's use "y".

$$y = \frac{\ln(x) \leftarrow f}{x \quad \leftarrow g}$$

Now let's calculate f' and g':

$$\begin{aligned} f' &= \frac{1}{x} \\ g' &= 1 \end{aligned}$$

Now let's write down y':

$$y' = \frac{\frac{1}{x} \cdot x - \ln(x) \cdot 1}{(x)^2}.$$

Now is the time to start simplifying things after taking the derivative. Why? Because we can; that is, this formula can simplify a fair amount. Also, because we need to do something else with it, namely take the derivative again.

$$\begin{aligned} y' &= \frac{\frac{1}{x} \cdot x - \ln(x) \cdot 1}{(x)^2} \\ &= \frac{1 - \ln(x)}{x^2} \end{aligned}$$

To take the derivative again, let's label f and g:

$$y' = \frac{1 - \ln(x) \leftarrow f}{x^2 \quad \leftarrow g}$$

Now we calculate f' and g':

$$\begin{aligned} f' &= -\frac{1}{x} \\ g' &= 2x \end{aligned}$$

Now we wrte down y'':

$$y'' = \frac{\left(-\frac{1}{x}\right)(x^2) - (1 - \ln(x))(2x)}{(x^2)^2}.$$

We don't have a lot of reason to simplify this, unless it would make it easier to plug a number into. Maybe a little simplification would help:

$$y'' = \frac{-x - 2x(1 - \ln(x))}{x^4}.$$

Now, we plug in $x = e$:

$$y''(e) = \frac{-e - 2e(1 - \ln(e))}{e^4}.$$

This is complicated enough that I'm more worried I'll make a mistake plugging it into my calculator than by simplifying it. So, I'll simplify it:

$$\begin{aligned} y''(e) &= \frac{-e - 2e(1 - \ln(e))}{e^4} \\ &= \frac{-e - 2e(1 - 1)}{e^4} \\ &= \frac{-e - 0}{e^4} \\ &= -\frac{1}{e^3} \\ &= -\# \end{aligned}$$

From this, we see that $x = e$ is a local max.

4.2 Inflection points

Definition. For a function $f(x)$, an **inflection point** is a number $x = c$ such that $f(x)$ changes concavity at $x = c$.

We find inflection points the same we way find local max/mins: (1) take the second derivative, (2) set it equal to 0, (3) solve this equation, (4) confirm your answers by looking at the graph.

Example 1. For the function $f(x) = 0.00001x^4 - 1000x^3 - 10000000x^2$ do the following:

(a) Take the first derivative and find the critical points algebraically.
(b) Take the second derivative and find the inflection points algebraically.
(c) Graph $f(x)$ (use one or more windows that shows all the "interesting" features (in particular it should show the local max/mins and the graph should be close enough that you can read the values) and identify each critical point as a local maximum, minimum or neither

Solution. (a)

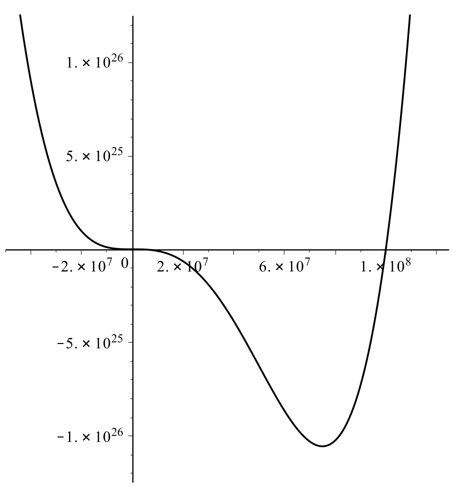
$$\begin{aligned} f(x) &= 0.00001x^4 - 1000x^3 - 10000000x^2 \\ f'(x) &= 0.00004x^3 - 3000x^2 - 20000000x \\ f'(x) &= 0 \\ 0.00004x^3 - 3000x^2 - 20000000x &= 0 \end{aligned}$$

$$\begin{aligned} x(0.00004x^2 - 3000x - 20000000) &= 0 \\ x &= 0 \\ \text{or } x &= \frac{3000 \pm \sqrt{(-3000)^2 - 4(0.00004)(20000000)}}{2(0.00004)} \\ x &\approx -6666.074179, 0., 7.500666607 \times 10^7 \end{aligned}$$

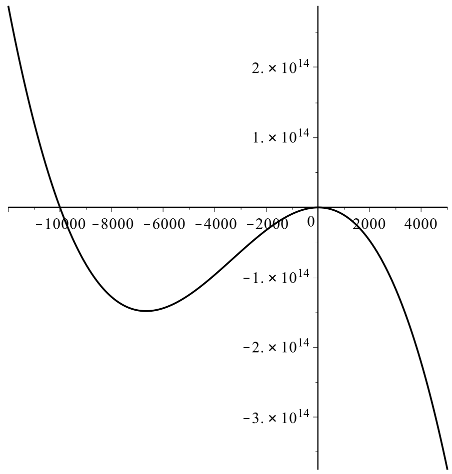
(b)

$$\begin{aligned} f'(x) &= 0.00004x^3 - 3000x^2 - 20000000x \\ f''(x) &= 0.00012x^2 - 6000x - 20000000 \\ f''(x) &= 0 \\ 0.00012x^2 - 6000x - 20000000 &= 0 \\ x &= \frac{6000 \pm \sqrt{(-6000)^2 - 4(0.00012)(20000000)}}{2(0.00012)} \\ x &\approx -3333.111141, 5.000333311 \times 10^7 \end{aligned}$$

(c) Now, if we try to graph $f(x)$, we quickly find out that the standard window shows us nothing. We should at least have our x-range include our two inflection points. Thus, we could try $\pm 1 \times 10^8$. This comes pretty close to being a good window, and the following shows a slight adjustment:



This graph shows us most of the interisting information, but we can't really see what's going on around the origin. We know from part (a) that there's a critical point at $x \approx -6666$ and $x = 0$. To see these critical points we need to have a window that goes to the left of -6666 and to the right of 0. How far to the left and right? Well, how about we double -6666 to get `xmin` $= -12000$. That's about 5000 to the left, and then we can go about 5000 to the right too:



This is where we ended on Monday, March 31

This is where we ended on Wednesday, April 2

4.3 Global max and min

Definition. Let $x = c$ be in the domain of $f(x)$.

$x = c$ is a global maximum	if $f(x) \leq f(c)$ for all x in the domain of $f(x)$. (we allow endpoints)
$x = c$ is a global minimum	if $f(x) \geq f(c)$ for all x in the domain of $f(x)$. (we allow endpoints)

"Global" max/mins are also sometimes called "absolute" max/mins.

Theorem 1 (Global max/min test (aka "closed interval method"))**.** To find the absolute max/min of a function $f(x)$ on an interval $[a, b]$, do the following.

1. Find the critical points of $f(x)$ in the interval $[a, b]$.
2. Calculate the y-value of $f(x)$ at each critical point.
3. Calculate the y-value of $f(x)$ at each endpoint.
4. The absolute max value is the biggest y-value from steps 2 and 3. The absolute min value is the smallest y-value from steps 2 and 3.

Example 1. Find the absolute max/mins of $f(x) = x + \dfrac{1}{x}$, on the interval $[0.2, 4]$.

Solution. Step 1, Critical points:

$$\begin{aligned} f(x) &= x + \frac{1}{x} \\ f'(x) &= 1 - \frac{1}{x^2} \\ f'(x) &= 0 \\ 1 - \frac{1}{x^2} &= 0 \\ 1 &= \frac{1}{x^2} \end{aligned}$$

$$x^2 = 1$$
$$x = \pm\sqrt{1}$$
$$x = 1 \text{ (because } -1 \text{ isn't in the domain)}$$

Steps 2 and 3, y-values:

x	y-value (of $f(x)$)
0.2	≈ 5.2
1	≈ 2
4	≈ 4.25

Step 4, biggest and smallest:

G. min	$x = 1, y = 2$
G. max	$x = 0.2, y = 5.2$

Example 2. Find the absolute max/mins of $f(x) = x^3 - 10x^2 + 5x + 10$ on the interval $[-2, 10]$.

Solution. Step 1, Critical points:

$$f(x) = x^3 - 10x^2 + 5x + 10$$
$$f'(x) = 3x^2 - 20x + 5$$
$$f'(x) = 0$$
$$3x^2 - 20x + 5 = 0$$
$$x = \frac{20 \pm \sqrt{400 - 4 \cdot 3 \cdot 5}}{6}$$
$$x = \frac{20 \pm \sqrt{340}}{6} = \frac{10 \pm \sqrt{85}}{3} \approx 0.2, 6.4$$

Steps 2 and 3, y-values:

x	y-value (of $f(x)$)
-2	≈ -47
0.2	≈ 10
6.4	≈ -105
10	≈ 60

Step 4, biggest and smallest:

G. max	$x = 10, y = 60$
G. min	$x = 6.4, y = -105$

4.4 Optimizing Cost and Revenue

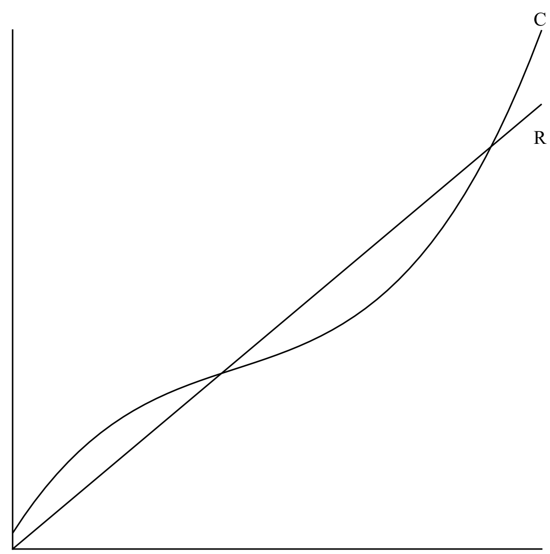
In this section we apply some of what we've learned for max/mins to topics from economics and business: cost, revenue and profit.

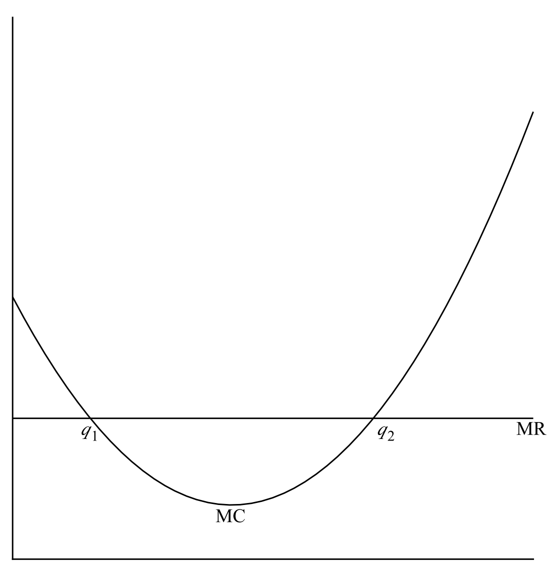
Recall that $\pi(q)$ is the profit function, and that $\pi(q) = R(q) - C(q)$.

- If $MC = MR$ then $\pi'(q) = 0$ and so π is at a critical point (*maybe* maximum)

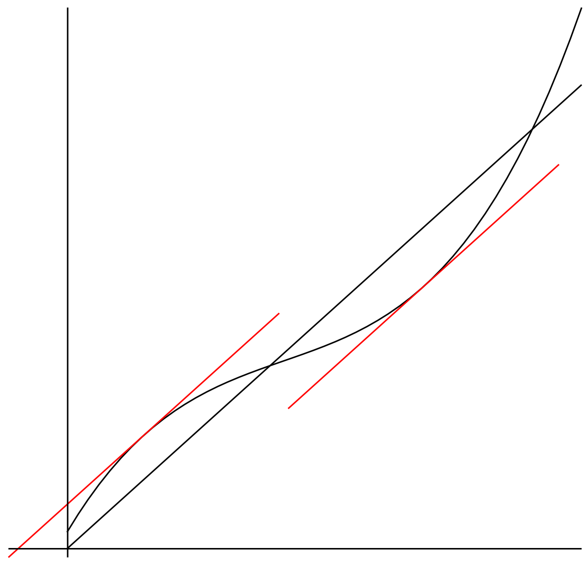
- If $MR > MC$ then $\pi'(q) > 0$ and so increasing q will increase π.
- If $MR < MC$ then $\pi'(q) < 0$ and so increasing q will decrease π.

Example 1. Consider the following two graphs, showing cost and revenue, and marginal cost and marginal revenue. (a) Interpret the significance of q_1 and q_2 on the graph of R and C. (b) Identify which point is a maximum for profit, and explain.





Solution. (a) In the top graph, q_1 and q_2 correspond to points with the biggest gaps between C and R. But there's another, better way to see them. They are the points where the tangent line of C is parallel to R, makred below in red:



(b) We claim that q_2 is the local maximum. The best way to see this is as follows: a little to the left of this point, we have that $MR > MC$, and thus π is increasing; a little to the right of this point, we have that $MR < MC$, and thus π is decreasing. In other words, π increases up to $\pi(q_2)$, and then decreases. That means q_2 is a maximum.

This is where we ended on Friday, April 4

Example 2. (*Based* on Hughes-Hallett, 4e, 4.4#7) The table below shows marginal cost MC and marginal revenue, MR.

(a) Use the marginal cost and marginal revenue at a production of $q = 5000$ to determine whether production should be increased or decreased from 5000. (Explain, in writing.)

(b) Estimate the production level that maximizes profit. (Explain, in writing.)

q	5000	6000	7000	8000	9000	10000	11000	12000	13000	14000	15000
MR	60	58	56	55	54	53	53	53	53	53	53
MC	48	52	54	55	58	63	60	57	54	53	52

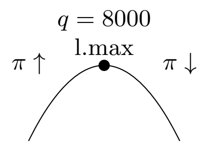
Solution. (a) At $q = 5000$ we should increase production. At $q = 5000$ we have $MR = 60$ and $MC = 48$. Since $MR > MC$ we know that profit will increase.

(b) Before giving the real answer, let me mention some decoy answers.

False answer: The maximum is at $q = 5000$ because that's where the biggest gap is between MR and MC, with MR on top. This is false because profit is maximum at the biggest gap between $\underline{R}$ and $\underline{C}$, not MR and MC.

False answer: the maximum is at $q = 14000$ because that's where $MR = MC$ and the production level q is largest. This is false because $MR = MC$ means we have a critical point for profit, it could be a local maximum or a local minumum, but we don't know. It may often be true that making the production level q larger increases profit, but not always.

Real answer: the maximum is at $q = 8000$. At this point we have $MR = MC$. Just to the left of 8000 we have $MR > MC$ and so profit is increasing. Just to the right of 8000 we have $MR < MC$ and so profit is decreasing. Thus we have



(Note, the same reasoning used here will show that $q = 14000$ is a local *min* for profit.)

4.5 Average Cost

Definition. The **average cost** is the total cost divided by the number of items, in other words

$$a(q) = \frac{C(q)}{q}$$

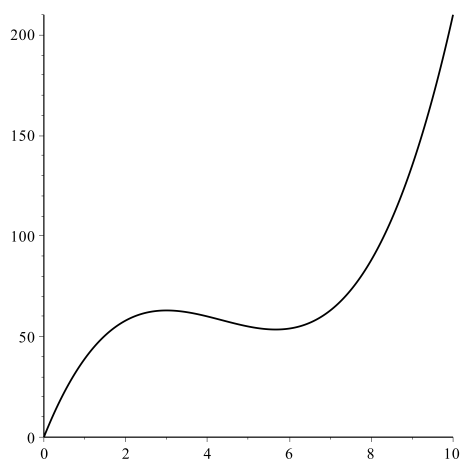
where q is the number of items, $C(q)$ is the total cost, and $a(q)$ is the average cost.

Example 1. The total cost of T-shirts is $C(q) = q^3 - 13q^2 + 51q$ for $0 \le q \le 10$ (for the sake of realism we'll suppose that q is measured in thousands, but this doesn't affect the problem one way or the other).

(a) Estimate using the graph of $C(q)$ where $a(q)$ has a minimum.

(b) Solve for q using Calculus and algebra to minimize $a(q)$.

Solution. (a) We start with a graph of $C(q)$

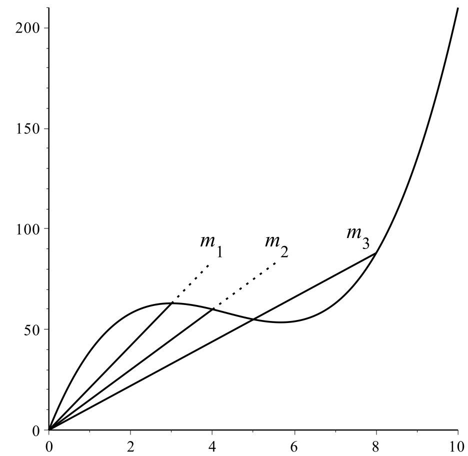


but this graph does not show $a(q)$ directly. Here is the trick for finding $a(q)$ in this graph. Note that $a(q) = \dfrac{C(q)}{q}$ is a ratio. How can we picture a ratio? Usually as a slope. To see how to do this, we can rewrite $a(q)$ this way

$$a(q) = \frac{C(q)}{q}$$

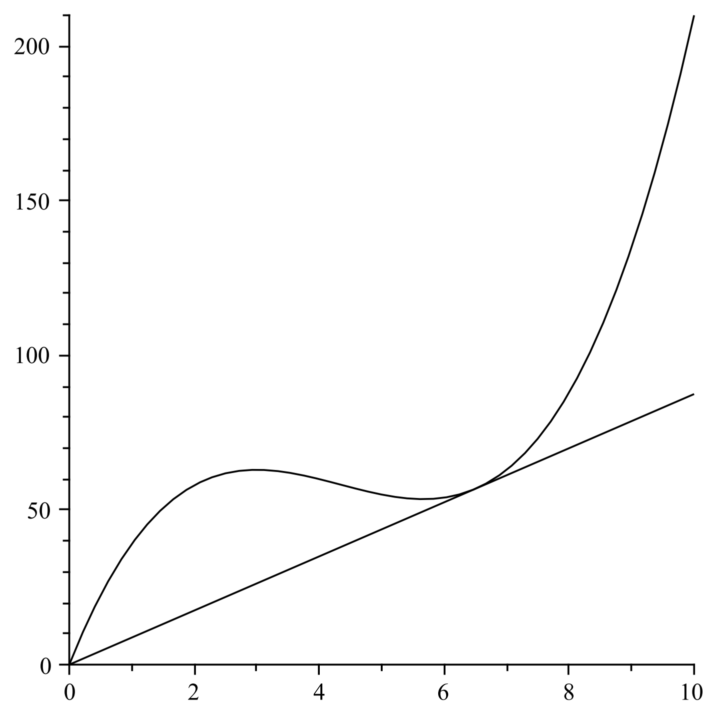
$$= \frac{C(q) - 0}{q - 0}$$
$$= \text{slope through } (0,0) \text{ and } (q, C(q))$$

In other words we draw the line through $(0,0)$ and a point on the graph. Here are three lines like this:



Which one of these lines has the smallest slope? The one that is shallowest, m_3, the one that shows $a(8) = \dfrac{C(8)}{8}$.

So, how do we find the minimum value of $a(q)$? We want to find the smallest slope that can be made by lines like those pictured above. Get out your ruler, or ID, or some other straight edge. Fix one end at the origin, and start the other end on the x-axis. Keeping the end at the origin fixed, rotate the ruler up until you find the first graphh point that the ruller touches. This is pictured below:



It appears that the value where a line like this touches the graph is at $q \approx 6.5$.

(b) Now we do this problem algebraically with formulas and derivatives:

$$\begin{aligned} a(q) &= \frac{C(q)}{q} \\ &= \frac{q^3 - 13q^2 + 51q}{q} \\ &= q^2 - 13q + 51 \\ a'(q) &= 2q - 13 \\ 2q - 13 &= 0 \\ q &= 13/2 \\ q &= 6.5 \end{aligned}$$

This is where we ended on Monday, April 7

We can combine the conclusions of the previous example (including what can be seen in the graph) as follows:

If $a(q) = MC$ then $a(q)$ is at a critical point (*maybe* minimum)
If $a(q) < MC$ then increasing q will increase $a(q)$
If $a(q) > MC$ then increasing q will decrease $a(q)$

Example 2. (Based on Hughes-Hallett, 3e, 4.5#10) The marginal cost at a production level of 2000 units of an item is \$10 per unit and the total cost is \$30000. If the production level were increased slightly above 2000, would the following quantities increase or decrease, or is it impossible to tell? Why?

(a) Average cost (b) Profit

Solution. The average cost at this production level is \$15 per unit. This is more than the marginal cost. Therefore, increasing q will cause $a(q)$ to increase.

We do not know if profit will increase or decrease, because we do not know what the marginal revenue is.

This is where we ended on Wednesday, April 9

4.6 Elasticity of Demand

Definition. Let q be the quantity of some product demanded (bought) when the price is p (so q is a function of p).

- **elasticity** E defined as
$$E = \left| \frac{p}{q} \cdot \frac{dq}{dp} \right|$$
- E approximated by
$$E \approx \left| \frac{\Delta q/q}{\Delta p/p} \right|$$
- E interpreted as: percentage change in demand, compared to percentage change in price.

- predicting percentage change in demand:

$$\frac{\Delta q}{q} \approx -E\frac{\Delta p}{p}$$

- $E > 1$ means **elastic demand**
  $E < 1$ means **inelastic demand**

Note: Elasticity can change with time, or with price points. Also, some people use the negative of our definition of elasticity. When in doubt about signs, make sure that any calculation that predicts change in demand gives a change in the correct direction; i.e. increasing price should decrease demand and decreasing price should increase demand.

Example 1. The elasticity of the demand for butter is 0.62. What do you expect to happen to demand if there is a 25% decrease in price? What do you expect to happen to demand if there is a 90% increase in price?

Solution. With a 25% decrease in price we expect

$$\frac{\Delta q}{q} \approx -0.62(-0.25) = 0.155 = 15.5\% \text{ increase in demand}$$

With a 90% increase in price we expect

$$\frac{\Delta q}{q} \approx -0.62(0.90) = -0.558 = -55.8\% \text{ decrease in demand}$$

Example 2. In Fall 2013, the undergraduate enrollment at Loyola University Maryland was 3875 and the tution was \$41850 per year (information taken from the 2013–2014 Loyola Catalogue). According to `http://centerforcollegeaffordability.org/archives/1336` the elasticity of demand for a 4 year college is 0.10.

(a) Will a 5% increase in tution cause total revenue to go up or go down?

(b) Can you find a way to predict this answer without repeating all the calculations?

Solution. (a) Since the elasticity is 0.10, a 5% increase in tution should cause a 0.1(0.05) decrease in attendence. Thus, the attendance is predicted to be

$$3875(1 - 0.005) \approx 3856.$$

Now we compare the old and new revenues:

$$\begin{aligned} R &= pq \\ \text{old revenue: } 3875 \times 41850 &= \$162,168,750 \\ \text{new revenue: } 3856 \times 41850(1.05) &= \$169,442,280 \end{aligned}$$

So the revenue went up by \$7,273,530.

(b) It's more efficient to calculate everything as a percentage change:

$$\% \text{ change } R = \frac{\Delta R}{R}$$

$$
\begin{aligned}
&= \frac{\Delta p \cdot q + p \cdot \Delta q}{p \cdot q} && \text{see Section 3.4, Example 1}\\
&= \frac{\Delta p \cdot q}{p \cdot q} + \frac{p \cdot \Delta q}{p \cdot q}\\
&= \frac{\Delta p}{p} + \frac{\Delta q}{q}\\
&= \frac{\Delta p}{p} - E\frac{\Delta p}{p}\\
&= (1-E)\frac{\Delta p}{p}\\
&= (1-E) \cdot \%\text{ change } p
\end{aligned}
$$

In this problem we have

$$\%\text{ change } R = (1 - 0.1)(0.05) = 0.045$$

Thus, we expect revenue to go up approximately 4.5%. (Indeed, in part (a) we saw that it went up by \$7,273,530, and this is 4.3% of \$162,168,750.)

Rule. In general, the elasticity determines whether R is an increasing function of p or not:

If $E < 1$ then increasing p will increase R
If $E > 1$ then increasing p will decrease R
If $E = 1$ then R is at a critical point.

Example 3. The demand function of T-shirts is $q = 1500 - 125p$.

(a) Find R when $p = \$5$.

(b) Find E when $p = \$5$.

(c) When $p = \$5$, find out if R is increasing or decreasing (i.e. will increaing p make R increase or decrease). Do the problem in two different ways: by using the Elasticity, and by finding R as a function of p and using the derivative.

Solution. (a)

$$
\begin{aligned}
R &= pq\\
&= 5(1500 - 125 \cdot 5)\\
&= 5 \cdot 875\\
&= \$4375
\end{aligned}
$$

(b)

$$
\begin{aligned}
E &= \left| \frac{p}{q} \cdot \frac{dp}{dq} \right|\\
&= \left| \frac{5}{875}(-125) \right|\\
&= 0.714
\end{aligned}
$$

(c) The first way to do this problem uses E:

$$E = 0.714 < 1 \Rightarrow R\uparrow.$$

The second way uses the derivative

$$\begin{aligned}
R(p) &= p(1500 - 125p) \\
\frac{dR}{dp} &= \frac{d}{dp} p(1500 - 125p) \\
&= \frac{d}{dp} 1500p - 125p^2 \\
&= 1500 - 250p \\
\left.\frac{dR}{dp}\right]_{p=5} &= 1500 - 250(5) \\
&= 250 \\
\left.\frac{dR}{dp}\right]_{p=5} &= + \qquad \Longrightarrow \quad R\uparrow
\end{aligned}$$

This is where we ended on Monday, April 14

Chapter 5

Accumulated Change: the Definite Integral

5.1 Distance and Accumulated Change

Comments. On an intuitive level, we all know that travelling at a certain velocity for a certain time means that we will travel a certain distance. Moreover, if we know the value of the velocity, and the value of the time travelled, we can come up with an estimate for the distance travelled:

$$\text{Distance} \;=\; \text{velocity} \;\times \Delta\text{time}.$$

In this section, we explore the idea of making this calculation more and more accurate, by using shorter and shorter intervals of time.

Example 1. Suppose the table below shows the velocity of a car, at 10 minute intervals. Find an upper estimate and a lower estimate for the distance the car has travelled.

t(min)	0	10	20	30	40	50	60
$v(t)$(mph)	0	22	38	35	37	32	29

Solution. We are only finding an estimate, so for each 10 minute interval, we will pick one of the velocities, and use that velocity for the whole 10 minutes. We also need to make a unit conversion: either we change time from minutes into hours, or the velocity into miles per minute. We choose to change time:

t(hours)	0	1/6	2/6	3/6	4/6	5/6	6/6
$v(t)$(mph)	0	22	38	35	37	32	29

To find the upper estimate we pick the higher velocity in each time interval. For example, let's look at just last 10 minutes, i.e. from $t = 5/6$ to $t = 6/6$:

$$\text{Upper estimate for } t = 5/6 \text{ to } t = 6/6 = 32 \times \tfrac{1}{6} = 4.83\,\text{mi}.$$

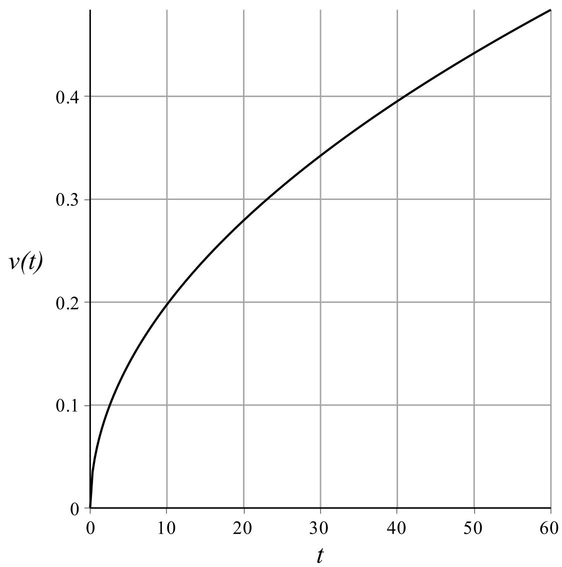
In a similar way, we pick the highest velocity for each 10 minute interval. Thus, our upper estimate is this:

$$\begin{aligned} & 0 \text{ to } \tfrac{1}{6} + \tfrac{1}{6} \text{ to } \tfrac{2}{6} + \tfrac{2}{6} \text{ to } \tfrac{3}{6} + \tfrac{3}{6} \text{ to } \tfrac{4}{6} + \tfrac{4}{6} \text{ to } \tfrac{5}{6} + \tfrac{5}{6} \text{ to } \tfrac{6}{6} \\ = {} & 22 \times \tfrac{1}{6} + 38 \times \tfrac{1}{6} + 38 \times \tfrac{1}{6} + 37 \times \tfrac{1}{6} + 37 \times \tfrac{1}{6} + 32 \times \tfrac{1}{6} \\ = {} & 34\,\text{mi} \end{aligned}$$

Now we do the same thing but pick the lower velocity in each time interval:

$$\begin{aligned} & 0 \text{ to } \tfrac{1}{6} + \tfrac{1}{6} \text{ to } \tfrac{2}{6} + \tfrac{2}{6} \text{ to } \tfrac{3}{6} + \tfrac{3}{6} \text{ to } \tfrac{4}{6} + \tfrac{4}{6} \text{ to } \tfrac{5}{6} + \tfrac{5}{6} \text{ to } \tfrac{6}{6} \\ = {} & 0 \times \tfrac{1}{6} + 22 \times \tfrac{1}{6} + 35 \times \tfrac{1}{6} + 35 \times \tfrac{1}{6} + 32 \times \tfrac{1}{6} + 29 \times \tfrac{1}{6} \\ = {} & 25.5\,\text{mi} \end{aligned}$$

Example 2. Shown below is the graph of the velocity $v(t)$ of a car, where t is minutes and v is in miles per minute.



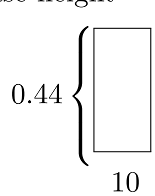
(a) Find a lower estimate for the distance the car travels in the first hour.
(b) Interpret the distance the car travels in terms of the graph of $v(t)$. Do this both for the estimate calculated in part (a) and for the actual distance the car travelled.

Solution. (a) As in the previous example, we will simply pick a velocity for each time interval, and then multiply this number times Δt. The only difference is that in this case the velocity comes from the graph instead of the table.

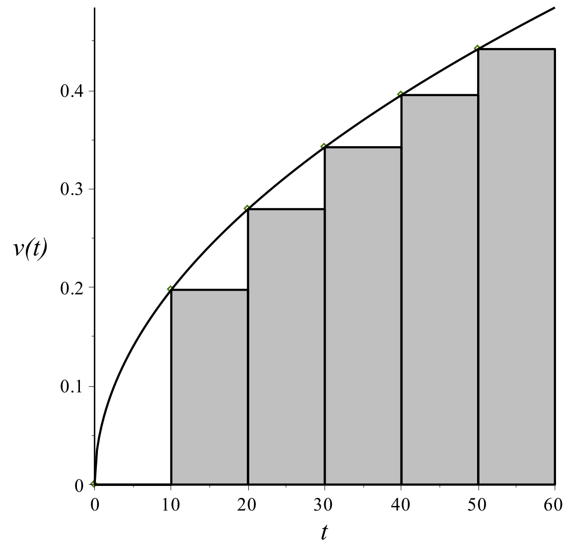
For instance, in the last 10 minutes, we use the velocity $v(50) = 0.44$ since this is the smallest velocity in the interval from $t = 50$ to $t = 60$. Here are all the numbers we use:

$$\begin{aligned} & 0 \text{ to } 10 + 10 \text{ to } 20 + 20 \text{ to } 30 + 30 \text{ to } 40 + 40 \text{ to } 50 + 50 \text{ to } 60 \\ = {} & v(0) \times 10 + v(10) \times 10 + v(20) \times 10 + v(30) \times 10 + v(40) \times 10 + v(50) \times 10 \\ = {} & 0 \times 10 + 0.20 \times 10 + 0.28 \times 10 + 0.34 \times 10 + 0.40 \times 10 + 0.44 \times 10 \\ = {} & 16.6\,\text{mi} \end{aligned}$$

(b) Each number that we used for velocity in part (a) came from the graph, and so we don't really have to do anything to interpret these numbers by themselves. But what about the products like 0.44×10? It's always possible to visual products as the area of rectangles, in this case height $= 0.44$ and width $= 10$:

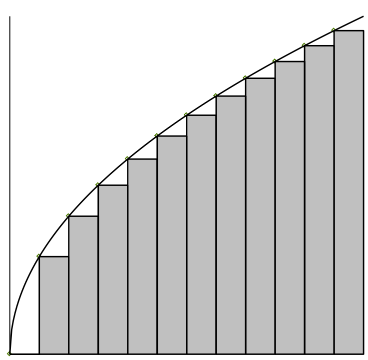


To add all the products we can visualize more than one rectangle, one for each term:

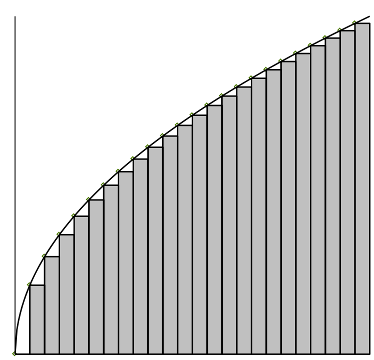


Area of 6 rectangles = sum of distances from above

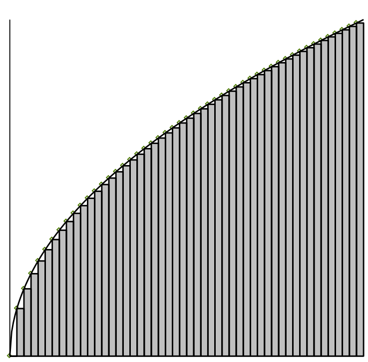
To interpret the exact area on the graph we start by imagining what happens as we use more and more rectangles, which means the same thing as dividing the times into smaller intervals. Here's what it looks like for 12 rectangles (i.e. 12 time intervals,), 24 rectangles, 50 rectangles and 100 rectangles:



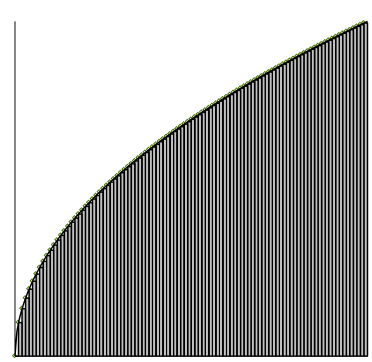
12 rectangles



24 rectangles

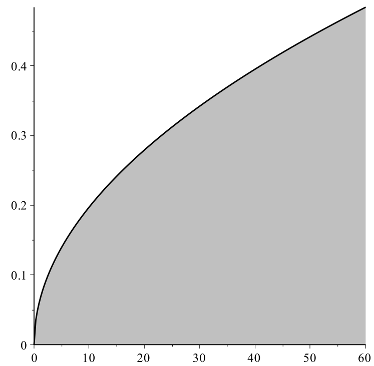


50 rectangles



100 rectangles

Hopefully you can see what's happening as we use more rectangles: the area of the rectangles is getting closer and closer to the area under the curve:



Putting all of this together we have:

$$\begin{aligned}
\text{Approximate distance} &= \text{6 terms added together} \\
&= \text{ area of 6 rectangles} \\
\text{Better approximate distance} &= \text{12 terms added together} \\
&= \text{ area of 12 rectangles} \\
\text{Even better approxmiate distance} &= \text{100 terms added together} \\
&= \text{area of 100 rectangles} \\
\underline{\text{Exact}}\text{ distance} &= \text{exact area under curve}
\end{aligned}$$

This is where we ended on Wednesday, April 16

5.2 The Definite Integral

Comments. In the previous section, we saw the importance of sums like this:

$$v(10) \times 10 + v(20) \times 10 + v(30) \times 10 + v(40) \times 10 + v(50) \times 10 + v(60) \times 10.$$

This calculation gave us an estimate for the distance travelled by a car. We also saw how to make the estimate more accurate: use smaller time intervals. In short,

$$\text{distance} = \text{sum of infinite number of terms like } v(t) \times \Delta t.$$

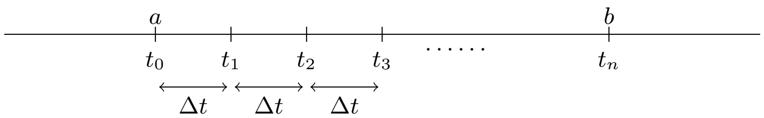
Now we formalize what we've just described, and apply it to other functions.

Definition. If $v(t)$ is velocity, and positive, then

$$\boxed{\begin{array}{l}\text{Approxiate distance}\\ \text{travelled from}\\ t = a \text{ to } t = b\end{array}} = v(t_0)\Delta t + v(t_1)\Delta t + \cdots + v(t_{n-1})\Delta t, \tag{5.1}$$

$$\boxed{\begin{array}{l}\text{Approxiate distance}\\ \text{travelled from}\\ t = a \text{ to } t = b\end{array}} = v(t_1)\Delta t + v(t_2)\Delta t + \cdots + v(t_n)\Delta t. \tag{5.2}$$

We call the formula in Equation 5.1 a **Left Hand Riemann Sum** and the formula in Equation 5.2 a **Right Hand Riemann Sum**. The interval $[a, b]$ is broken up into pieces of width Δt, and t_0, t_1, ..., t_n are the endpoints of these pieces, and n is the number of pieces:



We define the *exact* distance to be the number obtained (as a limit) by using smaller and smaller Δt, which means more and more pieces:

Exact distance travelled from $t = a$ to $t = b$	=	sum of an infinite number of terms like $v(t_i) \times \Delta t$ in a Riemann Sum

Finally, we invent a symbol that stands for the number you get by adding more and more terms as just described:

$$\int_a^b v(t)\,dt = \text{sum of infinite number of terms like } v(t_i) \times \Delta t \text{ in a Riemann Sum}$$

We call $\int_a^b v(t)\,dt$ the **definite integral of $v(t)$ from a to b.**

Example 1. Suppose that $v(t) = \dfrac{\sqrt{t}}{16}$ is velocity in miles per minute (this is the function used to make Example 2 (Section 5.1).

(a) Estimate the distance travelled from 0 to 60 using a Left Hand Riemann Sum with $\Delta t = 5$.

(b) Write a definite integral that equals the exact distance travelled and then use your calculator to calculate this integral.

Solution. (a) Since $\Delta t = 5$ we have $t_0 = 0$, $t_1 = 5$, $t_2 = 10$, $t_3 = 15$, $t_4 = 20, \ldots$. Since we are using the left hand rule we will start with $t_0 = 0$ and finish with $t_{20} = 55$. Now we write down the formulas:

$$\boxed{\text{Approximate distance travelled from } t = 0 \text{ to } t = 60} = v(0) \times 5 + v(5) \times 5 + v(10) \times 5 + v(15) \times 5 + v(20) \times 5$$
$$+ v(25) \times 5 + v(30) \times 5 + v(35) \times 5 + v(40) \times 5 + v(45) \times 5$$
$$+ v(50) \times 5 + v(55)\times$$
$$= 5(0 + 0.14 + 0.20 + 0.24 + 0.28 + 0.31$$
$$+ 0.34 + 0.37 + 0.40 + 0.42 + 0.44 + 0.46)$$
$$= 18.01$$

(b)

$$\boxed{\text{Exact distance travelled from } t = 0 \text{ to } t = 60} = \int_0^{60} \frac{\sqrt{t}}{16}\,dt.$$

There are two ways to use a built-in function from your calculator to find this number[1]. Here's the first way. Hit `MATH`, then `9: fnInt(`, then enter `√(X)/16,X,0,60)` and then `Enter`.

Here's the other way. It's less work if you've already entered the function on the `Y=` menu, but probably more work otherwise. On the `Y=` menu, enter `Y1=√(X)/16`. Then hit `2cnd`, `CALC`, then choose `7: ∫f(x)dx`. When it asks you for the lower bound, enter 0. When it asks for the upper bound, enter 60. (Note: for this to work, you need to have `xmin` and `xmax` set to include 0 and 60. In other words, you need `xmin`≤ 0 and `xmax`≥ 60.)

In any case, from our calculation, we find out that

$$\int_0^{60} \frac{\sqrt{t}}{16}\, dt \approx 19.365$$

and this final answer is a very good approximation indeed.

This is where we ended on Wednesday, April 23

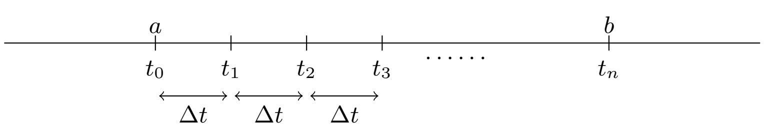
Comments. It turns out the type of calculation we've been doing to obtain distance travelled has a lot of applications to other kinds of problems: areas, surface areas, volumes, lengths, probabilities, populations, mass, center of mass, total change of anything, magnetic flux, etc. In light of this importance, we extend the previous definition to all functions.

Definition. If $f(t)$ is any function we define

$$\int_a^b f(t)\, dt \approx f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t, \tag{5.3}$$

$$\int_a^b f(t)\, dt \approx f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t. \tag{5.4}$$

We call the formula in Equation 5.3 a **Left Hand Riemann Sum** and the formula in Equation 5.4 a **Right Hand Riemann Sum**. The interval $[a, b]$ is broken up into pieces of width Δt, and t_0, t_1, ..., t_n are the endpoints of these pieces, and n is the number of pieces:



The approximation in the previous equations is made exact (as a limit) by using smaller and smaller Δt, which means more and more pieces:

$$\int_a^b f(t)\, dt = \text{sum of an infinite number of terms like } f(t_i) \times \Delta t \text{ in a Riemann Sum}$$

We call $\int_a^b f(t)\, dt$ the **definite integral of $f(t)$ from a to b.**

[1]If you have a Macintosh, you can do this on your computer. Go to Applications/Utilities/Grapher. Start the program and choose "default" as the graphing method. Enter the function in the "$y =$" field above the graph. Then go to Equation → Integration, and enter your limits.

Example 2. (a) Approximate $\int_1^3 \ln(x)\,dx$ using a right hand Riemann Sum with $n = 6$.

(b) Represent your answer on a graph.

(c) Use your calculator to find a more accurate numerical approximation.

Solution. (a) Since $n = 6$ we are supposed to divide the interval $[1, 3]$ into 6 equal pieces. Thus, each piece should have a width of

$$\Delta x = \frac{3-1}{6} = \frac{1}{3} \approx 0.333.$$

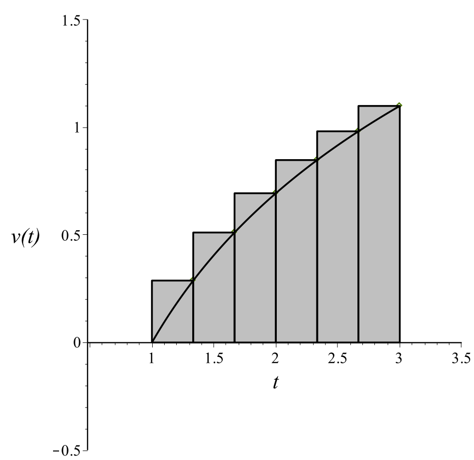
So the endpoints of the intervals are:

$$\begin{aligned}
x_0 &= 1 \\
x_1 &= 1 + 0.333 = 1.333 \\
x_2 &= 1.333 + 0.333 = 1.666 \\
x_3 &= 2 \\
x_4 &= 2.333 \\
x_5 &= 2.666 \\
x_6 &= 3
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_1^3 \ln(x)\,dx &\approx \ln(1.333)(0.333) + \ln(1.666)(0.333) + \ln(2)(0.333) \\
&\qquad + \ln(2.333)(0.333) + \ln(2.666)(0.333) + \ln(3)(0.333) \\
&\approx 1.47
\end{aligned}$$

(b) The terms we have just added together equal the areas of the 6 rectangles shown below:



(c) We enter $\int_1^3 \ln(x)\,dx$ into our calculator (see Section 5.2, Example 1) and get

$$\int_1^3 \ln(x)\,dx \approx 1.2958$$

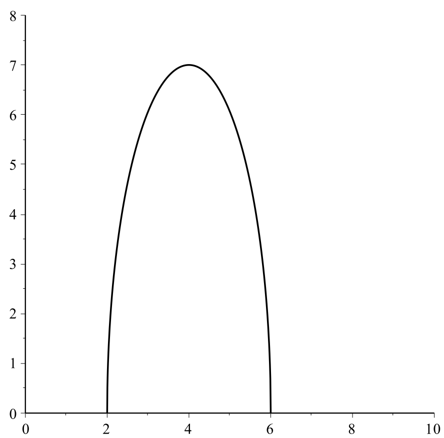
It shouldn't be suprising that the answer we just got is less than the answer we got in part (a). The answer in part (a) equals area of the rectangles in (b), and these rectangles obviously contain a little bit of area that is not part of the area under the curve.

5.3 The Definite Integral as Area

Fact.

$$\int_a^b f(x)\,dx = \boxed{\begin{array}{l}\text{Area under } f(x)\\ \text{from } x=a \text{ to } x=b\end{array}} \qquad (\text{when } f(x)>0 \text{ and } a<b)$$

Example 1. The function $f(x) = \dfrac{7}{2}\sqrt{-x^2+8x-12}$ does a reasonably good job modelling the shape of one McDonald's Golden Arch



Find the area under one arch by using the function $f(x)$, an integral, and your calculator.

Solution.

$$\begin{aligned}\text{area} &= \int_2^6 f(x)\,dx\\ &= \int_2^6 \frac{7}{2}\sqrt{-x^2+8x-12}\;dx\end{aligned}$$

We enter this formula in our calculators (see Section 5.2, Example 1) and get

$$\text{area } = 21.99$$

(Actually, it turns out this area is exactly 7π!)

Fact. If $f(x)$ is sometimes positive, and sometimes negative, then

$$\int_a^b f(x)\,dx = \boxed{\begin{array}{l}\text{Area above the } x\text{-axis}\\ \text{minus area below } x\text{-axis}\\ \text{(between graph of } f(x) \text{ and the } x\text{-axis}\\ \text{ and from } a \text{ to } b)\end{array}} \qquad (a<b)$$

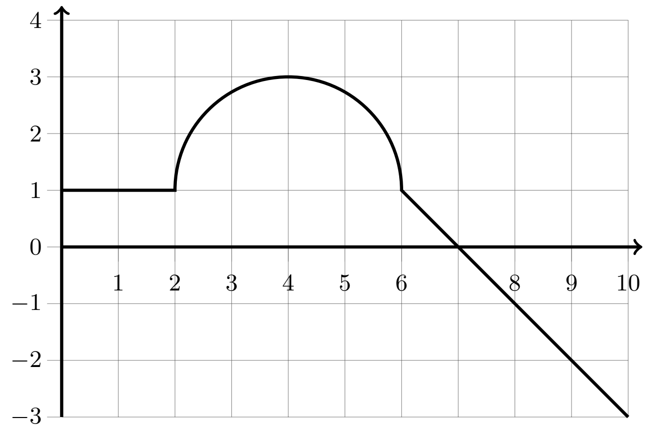
Example 2. The graph of $f(x)$ consists of straight lines and a semicircle, shown below.

(a) Find $\int_0^2 f(x)\,dx$ exactly.

(b) Find $\int_2^6 f(x)\,dx$ exactly.

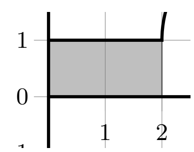
(c) Find $\int_0^7 f(x)\,dx$ exactly.

(d) Find $\int_0^{10} f(x)\,dx$ exactly.



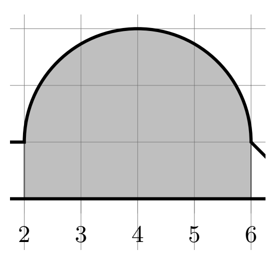
Solution. (a) Since the word "exact" appears in the problem we cannot use Riemann sums. Instead, we interpret $\int_0^2 f(x)\,dx$ as the area under $f(x)$ (and above the x-axis, between $x = 0$ and $x = 2$). This is the area of a rectangle, and we can easily find it:

$$\int_0^2 f(x)\,dx = \text{ area of rectangle}$$



$$= 2 \cdot 1 = 1$$

(b) We interpret $\int_2^6 f(x)\,dx$ as the area under $f(x)$ (and above the x-axis, between $x = 2$ and $x = 6$). This is the area of a rectangle (under the circle), and a half circle:

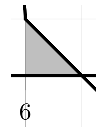


$$\int_0^2 f(x)\,dx = \text{ area of rectangle and a half circle}$$

$$= 4 \cdot 1 + \frac{1}{2}(\pi r^2)$$
$$= 4 + \frac{1}{2}(\pi 2^2)$$
$$= 4 + 2\pi$$

(c) We interpret $\int_0^7 f(x)\,dx$ as the area under $f(x)$ (and above the x-axis, between $x = 0$ and $x = 7$). This is the area from part (a), plus the area from part (b), plus the area of a triangle:

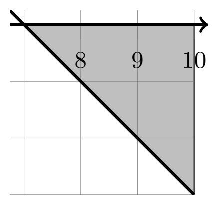
$$\int_0^7 f(x)\,dx = \text{ area from (a) and (b) plus area of triangle}$$



$$= 2 + 4 + 2\pi + \frac{1}{2}bh$$
$$= 6 + 2\pi + \frac{1}{2}1 \cdot 1$$
$$= 6.5 + 2\pi$$

(d) We interpret $\int_0^{10} f(x)\,dx$ as the area above the x-axis, minus the area below the x-axis. In other words, it's the area from part (c), minus the area of the triangle from $x = 7$ to $x = 10$:

$$\int_0^{10} f(x)\,dx = \text{ area from (c) minus area of triangle}$$



$$= 6.5 + 2\pi - \frac{1}{2}bh$$
$$= 6.5 + 2\pi - \frac{1}{2}3 \cdot 3$$
$$= 6.5 + 2\pi - 4.5$$
$$= 2 + 2\pi$$

This is where we ended on Friday, April 25.