**Theorem (Chain Rule).** Let $A \subseteq \mathbb{C}$ and $B \subseteq \mathbb{C}$ be open sets. Let $f : A \to \mathbb{C}$ and $g : B \to \mathbb{C}$ be analytic and let $f(A) \subseteq B$. Then $g \circ f : A \to \mathbb{C}$ is analytic and

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z).$$

**Proof.** Let $z_0 \in A$ be arbitrary and let $w_0 = f(z_0)$. For $w \in B$ define $h : B \to \mathbb{C}$ by

$$h(w) = \begin{cases} 
\frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) & w \neq w_0 \\
0 & w = w_0 
\end{cases}$$

Since $g$ is analytic, $g$ is continuous on $B$ making $\frac{g(w) - g(w_0)}{w - w_0} - g'(w_0)$ continuous on open sets $U \subseteq B$ with $w_0 \neq U$. So we just need to check continuity of $h$ at $w_0$.

$$\lim_{w \to w_0} h(w) = \lim_{w \to w_0} \left( \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) \right)$$

$$= \left( \lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} \right) - g'(w_0) \quad \text{(using Limit Laws)}$$

$$= g'(w_0) - g'(w_0)$$

$$= 0$$

$$= h(w_0)$$

Thus $h$ is continuous on $B$. Since both $f$ and $g$ are analytic, they are both continuous so $g \circ f$ is continuous. Thus by the definition of $h$ we get

$$\lim_{z \to z_0} h(f(z)) = h(w_0) = 0. \quad (1)$$

By setting $f(z) = w$ for $z \neq z_0$, we have

$$h(f(z)) = \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} - g'(f(z_0))$$

$$h(f(z)) + g'(f(z_0)) = \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)}$$

$$[h(f(z)) + g'(f(z_0))] [f(z) - f(z_0)] = g(f(z)) - g(f(z_0)) \quad (2)$$

Note that (2) also holds for $z = z_0$ since we get $0 = 0$. Now divide (2) by $z - z_0$ and take the limit of both sides to get:

$$\lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} = \lim_{z \to z_0} \left[ h(f(z)) + g'(f(z_0)) \right] \left[ \frac{f(z) - f(z_0)}{z - z_0} \right]$$

$$= [0 + g'(f(z_0))] \left[ \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \right] \quad \text{(by Limit Laws and (1))}$$

$$= g'(f(z_0))f'(z_0)$$

(since $f$ is analytic)

This limit exists for all $z_0 \in A$ and so we have $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$ for all $z_0 \in A$. \qed