Math 490 - Homework 11

Due December 1, 2005

#1. Prove that the set $\mathbb{R}$ of real numbers is **not** compact by finding an open covering of $\mathbb{R}$ which does not have a finite subcover.

#2. Suppose that X is a compact topological space. Let A be a subset of X which has finitely many points. Prove that A is a compact subset of X.

#3. Let X be a set and let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be two different topologies on X. So we have two topological spaces $X_1 = (X, \mathfrak{I}_1)$ and $X_2 = (X, \mathfrak{I}_2)$. Suppose now that we know that $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$. Prove that if X_2 is a compact space, then X_1 is also a compact space.

#4. Let X be a topological space. A family $\{F_\alpha\}_{\alpha \in I}$ of subsets of X has the **finite intersection property** if every finite subset of the family has a nonempty intersection. For example, if the family has only the three sets $F_1 = \{1, 2, 3\}$, $F_2 = \{3, 4, 5\}$, $F_3 = \{1, 3, 5\}$, then this set has the finite intersection property because each of the sets

$$F_1 \cap F_2, \quad F_1 \cap F_3, \quad F_2 \cap F_3, \quad \text{and} \quad F_1 \cap F_2 \cap F_3$$

is nonempty.

As another example, if $F_\alpha = (0, \alpha)$ for every $\alpha \in (0, 1)$, then this set also has the finite intersection property because if you look at any finite number of these sets, their intersection is not the empty set. For example, we have

$$F_{1/2} \cap F_{1/10} \cap F_{1/100} = (0, 1/2) \cap (0, 1/10) \cap (0, 1/100) = (0, 1/100) \neq \emptyset.$$

Prove that the space X is compact if and only if whenever a family $\{F_\alpha\}_{\alpha \in I}$ of **closed** subsets of X has the finite intersection property, we also have $\bigcap_{\alpha \in I} F_\alpha \neq \emptyset$.

Hint: Prove each direction by contradiction.