Math 200 - Spring 2002

How to Prove Theorems - Part II

9) To Prove that $f : A \to B$ is a Function.

There are two methods to do this.

<u>Method 1:</u>

If you have a formula for f:

Proof. Let $x \in A$.

Work.

Therefore there is exactly one value of $f(x)$. Hence f is a function.

<u>Method 2:</u>

If you are given f as a set.

Proof. Let $x \in A$.

Work.

Therefore there is exactly one ordered pair $(a, b) \in f$ with $a = x$. Hence f is a function.

10) To Prove Two Functions f and g are Equal.

Again, there are two methods.

<u>Method 1:</u>

If you are given formulas for f and g.

Proof. First we need to prove that f and g have the same domain.

Work needed to do this.

Next, suppose $x \in A$.

Work.

Therefore we have $f(x) = g(x)$. Since f and g have the same domain and $f(x) = g(x)$ for any x in the domain, the functions are equal.

<u>Method 2:</u>

If you are given the functions as sets.

Proof. We need to show that the sets defining the functions f and g are equal.

Work to do this.

Therefore the functions are equal.

11) To Prove $f : A \to B$ is Surjective.

Method 1:

If you have a formula for f.

Proof. Suppose that $y \in B$.

Work.

Therefore there is an element $x \in A$ such that $f(x) = y$. Since this is true for any $y \in B$, the function f is surjective.

Note: The work in the boxed step almost always involves writing down x explicitly. If this is the case, then the work often looks something like this.

Define $x \in A$ by

Write down a definition of x in terms of y.

Then we have

$$f(x) = \boxed{\text{Work.} \; \vdots} = y.$$

Method 2:

If you are given f as a set.

Proof. Suppose $y \in B$.

Work.

Therefore there is an element $(a, b) \in f$ with $b = y$. Therefore the function f is surjective.

12) To Prove $f : A \to B$ is Injective.

If you have a formula for f.

Method 1:

Proof. Suppose that $x_1, x_2 \in A$, and that $f(x_1) = f(x_2)$.

Work.

Therefore $x_1 = x_2$. Hence f is injective.

Method 2:

Proof. Suppose that $x_1, x_2 \in A$, and that $x_1 \neq x_2$.

Work.

Therefore $f(x_1) \neq f(x_2)$. Hence f is injective.

If you are given f as a set.

Method 3:

Proof. Suppose that $x_1, x_2 \in A$.

Work.

Hence the ordered pairs in f with x_1 and x_2 as their first coordinate have equal second coordinates. Therefore f is injective.

Method 4:

Proof. Suppose that $y \in B$, and that the ordered pairs (x_1, y) and (x_2, y) are both elements of f.

Work.

Therefore we have $x_1 = x_2$. Hence f is surjective.

13) To Prove $f : A \to B$ is a Bijection.

Proof. First, we show that f is an injective function.

Work to do this.

Next we show that the function f is surjective.

Work to do this.

Since the function f is both injective and surjective, it is a bijection.