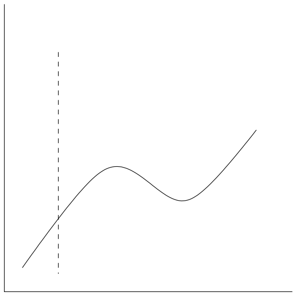
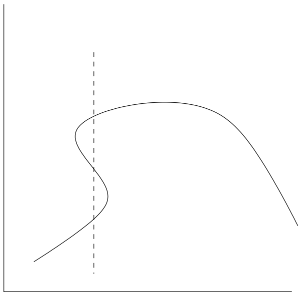
A Primer on Functions
Math 200 - Spring 2002

The basic definitions of functions and their properties can be confusing and hard to keep straight. In this handout, I will try to make them more understandable. I will not talk here about how to prove that something is a function, or that it is surjective or injective. That can be found in the "How to Prove Theorems II" handout.

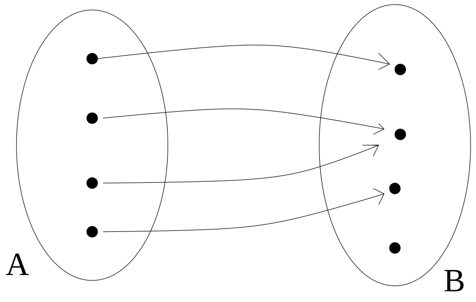
The Basic Idea:

Saying that $f : A \to B$ is a **function from A to B** simply means that f maps every point of A to some point in B. It doesn't matter what elements of B are involved, as long as no element of A is mapped to more than one element of B. The set A is called the **domain** of the function. If $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$, then we can graph $y = f(x)$ on regular axes. Then the definition of a function is just the **vertical line test.**

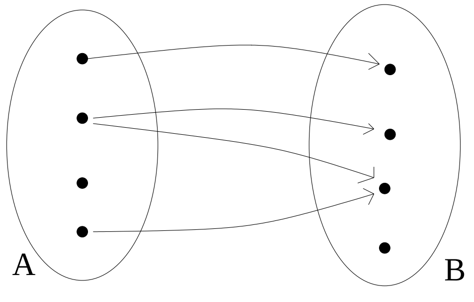

Function


Not a function

As pictures, we can think of the definition like this.



Function



Not a Function

In the picture on the left, it doesn't matter that the bottom element of B doesn't get hit. The picture on the right is not a function because the second element of A gets mapped to two elements of B. Think of the elements $x \in A$ as **input values**, and if $y = f(x)$, think of y as an **output value**.

Examples:

1) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is a function.

2) $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \sqrt{x}$ is **not** a function from $\mathbb{R}$ to $\mathbb{R}$ because you can't plug in $x = -1$. We can turn this into a function by changing the domain to $\mathbb{R}^+ \cup \{0\}$, giving $g : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ instead.

Note: the fact that not every element of $\mathbb{R}$ gets hit doesn't matter.

The Set Definition

We can also think of f as a set of ordered pairs $f \subseteq A \times B$. Each ordered pair is of the form

$$(x, y) = (\text{input value}, \text{output value}).$$

If we think of functions as sets, then the definition means that every element $x \in A$ **must** occur as the first coordinate of **exactly one** ordered pair in f.

Examples: In all of these examples, let $A = \{1, 2, 3\}$ and $B = \{s, t, u\}$.

1) $f = \{(1, s), (2, t), (3, u)\}$ is a function, with $f(1) = s, f(2) = t$, and $f(3) = u$.

2) $g = \{(1, 3), (2, t), (s, u)\}$ is **not** a function from A to B, since it's not a subset of $A \times B$.

3) $h = \{(1, s), (2, t)\}$ is **not** a function since $f(3)$ is undefined. If we change A to be the set $\{1, 2\}$, then h is a function.

4) $k = \{(1, s), (2, s), (3, t)\}$ **is** a function. It does not matter that u is not an output value.

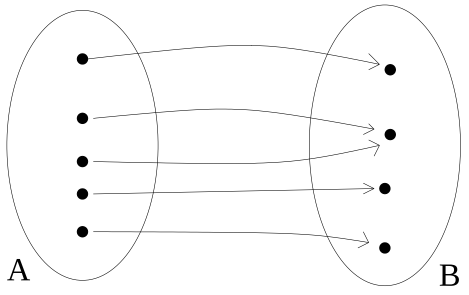
5) $m = \{(1, s), (1, t), (2, t), (3, u)\}$ is **not** a function. The first ordered pair says that $f(1) = s$, and the second says that $f(1) = t$.

Now we're going to talk about surjective and injective functions. When we say that f is surjective or injective, we are **assuming** that f is actually a function. If f is not a function, then it is meaningless to say that it is surjective or injective.

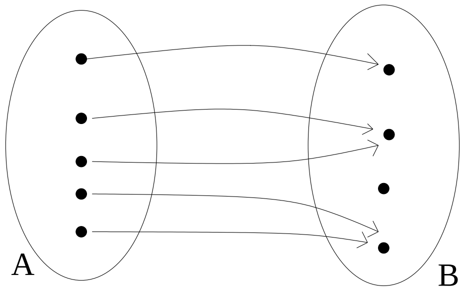
Surjective Functions:

Definition: A function $f : A \to B$ is **surjective** (or **onto**) if for **every** element $y \in B$, there is an element $x \in A$ with $y = f(x)$. In other words, **every element of B gets hit by some element of A.**

At the top of the next page is a picture of a surjective function, and one of a function which is not surjective.



Surjective



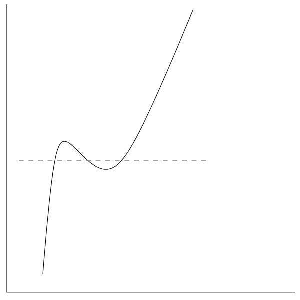
Not Surjective

Examples:

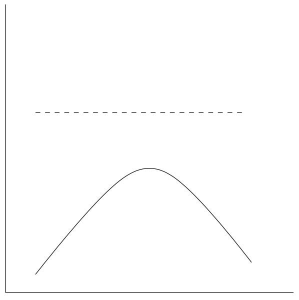
1) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 5x + 1$ is surjective. Given any element $y \in \mathbb{R}$, if we set $x = (y - 1)/5$, then $f(x) = y$.

2) $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = 6 - x^2$ is **not** surjective. For example, if we set $y = 10$, then there is no $x \in \mathbb{R}$ with $f(x) = 10$. If we change B to be the set $B = \{y \in \mathbb{R} : y \leq 6\}$, then $g : \mathbb{R} \to B$ **is** surjective.

If $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$, and we draw a regular graph of $y = f(x)$, then we can test whether the graph is surjective by using a **horizontal line test**. (Be careful, we will see three horizontal line tests). A function is surjective if every horizontal line hits the graph **at least once**. It is OK if some horizontal lines hit the graph more than once, but if even one horizontal line misses the graph, then the function is **not** surjective. Here are a few examples.



Surjective



Not surjective

Being surjective means that every $y \in B$ is an output value. If you use the set definition of a function f, this means that every element of B is the **second** coordinate of some ordered pair in f.

Remember:
The definition of **function** says that the **first** coordinate of every ordered pair has a certain property.
The definition of **surjective** says that the **second** coordinate of every ordered pair has a certain property.

Examples: In these examples, let $A = \{1, 2, 3, 4\}$ and $B = \{s, t, u\}$.
1) $f = \{(1, s), (2, t), (3, u), (4, u)\}$ is surjective, since every element of B appears as the second coordinate of some ordered pair.
2) $g = \{(1, s), (2, t), (3, s), (4, t)\}$ is **not** surjective since u is not the second coordinate of any ordered pair. In other words, there is no $x \in A$ with $f(x) = u$.
3) $h = \{(1, s), (2, s), (2, t), (3, u), (4, u)\}$ is **not** a surjective function. Even though each letter is the second coordinate of some ordered pair, h is not a function.

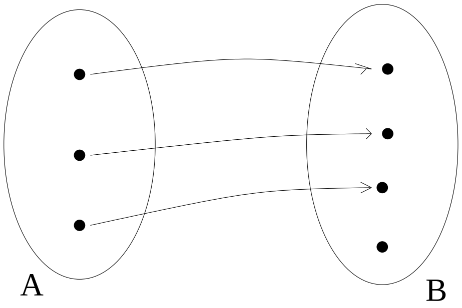
Injective Functions:

The basic idea in saying that a function $f : A \to B$ is injective is that different input values have different output values. We state this formally as follows.

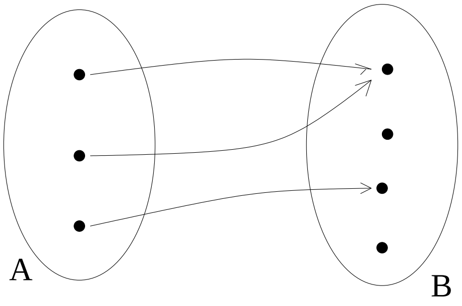
Definition: A function $f : A \to B$ is **injective** (or **one-to-one**) if whenever you have two **different** elements $x_1, x_2 \in A$ (so that $x_1 \neq x_2$), then $f(x_1) \neq f(x_2)$. Another way to say this is that if x_1, x_2 are **any** elements of

A, and $f(x_1) = f(x_2)$, then we must have $x_1 = x_2$.

As a picture, an injective function would look like the one on the left below, and a noninjective function would look like the one on the right.



Injective



Not injective

In the example on the left, is OK that the bottom element of B doesn't get hit. That only matters if we're talking about surjective functions, not injective ones. The function in the picture on the right is not injective because the top element of B gets hit more than once.

Examples:

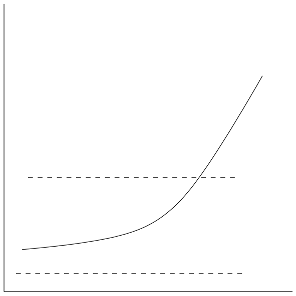
1) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$ is injective. To see this, suppose that $f(x_1) = f(x_2)$. Then $e^{x_1} = e^{x_2}$, and taking logs gives $x_1 = x_2$.

2) $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = 6 - x^2$ is **not** injective, because $g(-1) = g(1) = 5$, and $-1 \neq 1$. If we restrict the domain to $\mathbb{R}^+ \cup \{0\}$, then $g|_{\mathbb{R}^+ \cup \{0\}} : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ **is** an injective function.

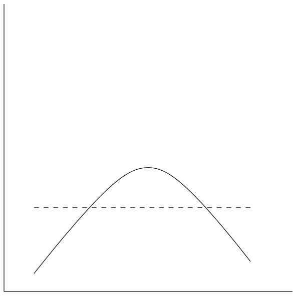
If we have $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$, and make a regular xy-graph of $y = f(x)$, then there is another horizontal line test to see whether f is injective. If every horizontal line hits the graph of $y = f(x)$ **at most once**, then f is an injective function. (You should compare this with the horizontal line test for surjective functions, and make certain you understand the difference).

Note: It is perfectly OK if some horizontal lines never hit the graph. In this test, we only care than none of them hit it more than once.

Here are some examples.



Injective



Not injective

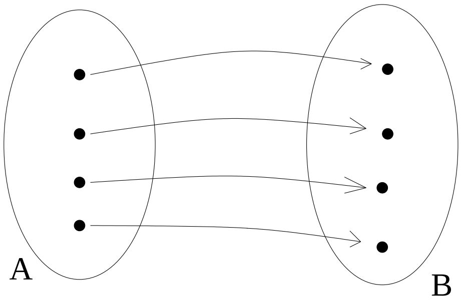
If we use the set definition for a function, then being injective means that in all the ordered pairs, no element of B is the second coordinate of more than one ordered pair.

Examples: Let $A = \{1, 2, 3\}$ and $B = \{s, t, u, v\}$.
1) $f = \{(1, s), (2, t), (3, u)\}$ is injective. It does not matter that v is not in any of the ordered pairs.
2) $g = \{(1, s), (2, t), (3, s)\}$ is **not** injective, since the element s is the second coordinate of two ordered pairs. In other words, because $f(1) = f(3) = s$, but $1 \neq 3$.
3) $h = \{(1, s), (2, t), (3, u), (3, v)\}$ is **not** an injective function. Even though each element of B appears only once, h is not a function because $f(3) = u$ and $f(3) = v$.

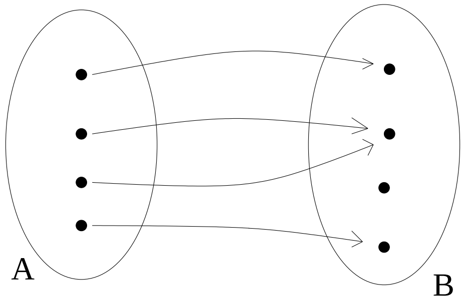
Bijective Functions:

Definition: A function $f : A \to B$ is **bijective** if it is both injective and surjective (in other words, if it is both one-to-one and onto).

Here are pictures of a bijective function and a function that is not bijective.



Bijective



Not bijective

It is important to realize that surjective and injective functions are not at all linked, as the following examples show. Thus, to check if a function is bijective, you really do have to check both conditions.

Examples:

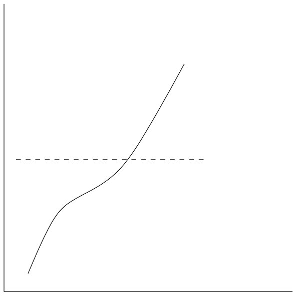
1) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 5x - 3$ is a bijection.

2) $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = e^x$ is **not** a bijection, since it is injective but not surjective.

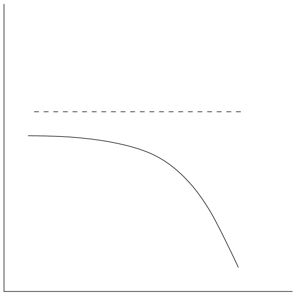
3) $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^2(x - 1)$ is **not** a bijection, since it is surjective but not injective (graph it and see!)

4) $k : \mathbb{R} \to \mathbb{R}$ given by $k(x) = 6 - x^2$ is **not** a bijection since it is neither injective nor surjective.

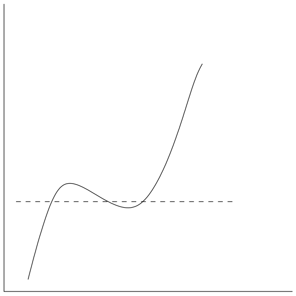
If we have $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$, and make an xy-graph of $y = f(x)$, then we can combine the horizontal line tests for surjective and injective functions to get one for bijections. For surjective functions, each horizontal line has to hit the graph at **least** once. For injective functions, each horizontal line has to hit the graph at **most** once. A bijective function must pass both of these tests. Hence a function is bijective if each horizontal line hits the graph **exactly** once. Here are some examples.



Bijective



Not bijective



Not bijective

Finally, if we use the set definition of a function f, then putting the definitions of surjective and injective together we find that **every** element of B has to be the second coordinate of **exactly one** ordered pair.

Examples:

1) If $A = \{1, 2, 3\}$ and $B = \{s, t, u\}$, then $f = \{(1, s), (2, t), (3, u)\}$ is a

bijection.

2) If $A = \{1, 2, 3\}$ and $B = \{s, t, u, v\}$, then $g = \{(1, s), (2, t), (3, u)\}$ is **not** a bijection since it is injective but not surjective. The element v is not the second coordinate of any ordered pair.

3) If $A = \{1, 2, 3\}$ and $B = \{s, t\}$, then $h = \{(1, s), (2, s), (3, t)\}$ is **not** a bijection since it is surjective but not injective. The element s is the second coordinate of too many ordered pairs.

4) If $A = \{1, 2, 3\}$ and $B = \{s, t, u\}$, then $k = \{(1, s), (2, s), (3, t)\}$ is not a bijection since it is neither injective nor surjective. The element u is not the second coordinate of any ordered pair, and the element s is the second coordinate of too many ordered pairs.