Homework 5 Solutions
Math 162Q - Fall 2002

Section 7.3:
#3. If we make the substitution $x = 3\tan\theta, dx = 3\sec^2\theta\,d\theta$, and substitute in, we find

$$\begin{aligned}
\int \frac{x^3}{\sqrt{x^2+9}}\,dx &= \int \frac{81\tan^3\theta\sec^2\theta\,d\theta}{\sqrt{9\tan^2\theta+9}} \\
&= \int \frac{81\tan^3\theta\sec^2\theta\,d\theta}{\sqrt{9\sec^2\theta}} \\
&= \int \frac{81\tan^3\theta\sec^2\theta\,d\theta}{3\sec\theta} \\
&= \int 27\tan^3\theta\sec\theta\,d\theta \\
&= \int 27\tan^2\theta\sec\theta\tan\theta d\theta \\
&= \int 27\left(\sec^2\theta-1\right)\sec\theta\tan\theta d\theta
\end{aligned}$$

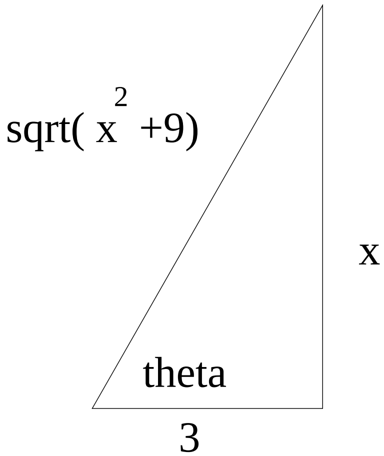
Now we make the substitution $u = \sec\theta, du = \sec\theta\tan\theta\,d\theta$, giving us

$$\begin{aligned}
\int \frac{x^3}{\sqrt{x^2+9}}\,dx &= 27\int u^2 - 1\,du \\
&= 9u^3 - 27u + C \\
&= 9\sec^3\theta - 27\sec\theta + C \\
&= 9\sec^3(\tan^{-1}(x/3)) - 27\sec(\tan^{-1}(x/3)) + C.
\end{aligned}$$

Now, to get rid of the trig functions, we use a right triangle like the one below. If we let

$$\tan\theta = x/3,$$

then we let the side of the triangle opposite θ be x and the adjacent side be 3, as in the diagram on the next page.



Then by the Pythagorean theorem, we find that the hypoteneuse has length $\sqrt{x^2+9}$. Therefore, we find that

$$\cos\theta = \frac{3}{\sqrt{x^2+9}},$$

and hence that

$$\sec(\tan^{-1}(x/3)) = \sec\theta = \frac{\sqrt{x^2+9}}{3}.$$

Therefore, we find that

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = 9\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 9\sqrt{x^2+9} + C.$$

#4. If we use the substitution $x = 4\sin\theta, dx = 4\cos\theta\,d\theta$, and change the limits as we go, we obtain

$$\begin{aligned}
\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx &= \int_0^{\pi/3} \frac{64\sin^3\theta \cdot 4\cos\theta\,d\theta}{\sqrt{16-16\sin^2\theta}} \\
&= \int_0^{\pi/3} \frac{256\sin^3\theta\cos\theta\,d\theta}{\sqrt{16\cos^2\theta}} \\
&= \int_0^{\pi/3} \frac{256\sin^3\theta\cos\theta\,d\theta}{4\cos\theta} \\
&= \int_0^{\pi/3} 64\sin^3\theta\,d\theta \\
&= \int_0^{\pi/3} 64(1-\cos^2\theta)\sin\theta\,d\theta.
\end{aligned}$$

If we make the substitution $u = \cos\theta, du = -\sin\theta\, d\theta$, we obtain

$$\begin{aligned}\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx &= \int_0^{\pi/3} 64(1-\cos^2\theta)\sin\theta\,d\theta \\ &= \int_1^{1/2} -64(1-u^2)\,du \\ &= \frac{40}{3}.\end{aligned}$$

#5. For this one, we make the substitution $t = \sec\theta, dt = \sec\theta\tan\theta\,d\theta$. If we change the limits as we go, we get

$$\begin{aligned}\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt &= \int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta\,d\theta}{\sec^3\theta\sqrt{\sec^2\theta-1}} \\ &= \int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta\,d\theta}{\sec^3\theta\sqrt{\tan^2\theta}} \\ &= \int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta\,d\theta}{\sec^3\theta\tan\theta} \\ &= \int_{\pi/4}^{\pi/3} \frac{d\theta}{\sec^2\theta} \\ &= \int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta \\ &= \frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}.\end{aligned}$$

#6. This time, we substitute $x = 2\tan\theta, dx = 2\sec^2\theta\,d\theta$. This gives us

$$\begin{aligned}\int_0^2 x^3\sqrt{x^2+4}\,dx &= \int_0^{\pi/4} 8\tan^3\theta\sqrt{4\tan^2\theta+4}\,\cdot 2\sec^2\theta\,d\theta \\ &= \int_0^{\pi/4} 8\tan^3\theta\sqrt{4\sec^2\theta}\,\cdot 2\sec^2\theta\,d\theta \\ &= \int_0^{\pi/4} 8\tan^3\theta\cdot 2\sec\theta\,\cdot 2\sec^2\theta\,d\theta \\ &= \int_0^{\pi/4} 32\tan^3\theta\sec^3\theta\,d\theta \\ &= \int_0^{\pi/4} 32(\sec^2\theta-1)\sec^2\theta\sec\theta\tan\theta\,d\theta\end{aligned}$$

If we now make the substitution $w = \sec\theta\, dw = \sec\theta\tan\theta\, d\theta$, we get

$$\begin{aligned}
\int_0^2 x^3\sqrt{x^2+4}\,dx &= \int_0^{\pi/4} 32(\sec^2\theta - 1)\sec^2\theta\sec\theta\tan\theta\,d\theta \\
&= \int_1^{\sqrt{2}} 32(w^2-1)w^2\,dw \\
&= \frac{128\sqrt{2}}{5} - \frac{64\sqrt{2}}{3} + \frac{64}{15}.
\end{aligned}$$

#14. Letting $u = \sqrt{5}\sin\theta, du = \sqrt{5}\cos\theta\, d\theta$, we get

$$\begin{aligned}
\int \frac{du}{u\sqrt{5-u^2}} &= \frac{\sqrt{5}\cos\theta\,d\theta}{\sqrt{5}\sin\theta\sqrt{5-5\sin^2\theta}} \\
&= \int \frac{\cos\theta\,d\theta}{\sin\theta\sqrt{5\cos^2\theta}} \\
&= \int \frac{\cos\theta\,d\theta}{\sqrt{5}\sin\theta\cos\theta} \\
&= \frac{1}{\sqrt{5}}\int \frac{d\theta}{\sin\theta} \\
&= \frac{1}{\sqrt{5}}\int \csc\theta\,d\theta \\
&= \frac{1}{\sqrt{5}}\ln|\csc\theta - \cot\theta| + C \\
&= \frac{1}{\sqrt{5}}\ln\left|\csc\left(\sin^{-1}\left(u/\sqrt{5}\right)\right) - \cot\left(\sin^{-1}\left(u/\sqrt{5}\right)\right)\right| + C.
\end{aligned}$$

If you use a right triangle to get rid of the trig functions, you get an answer of

$$\int \frac{du}{u\sqrt{5-u^2}} = \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}}{u} - \frac{\sqrt{5-u^2}}{u}\right| + C.$$

#16. First, we make the preliminary substitution $u = 4x, du = 4\,dx$ to get

$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{4\,du}{u^2\sqrt{u^2-9}}.$$

Then we make the substitution $u = 3\sec\theta, du = 3\sec\theta\tan\theta\, d\theta$ and get

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{16x^2-9}} &= \int \frac{4\,du}{u^2\sqrt{u^2-9}} \\
&= \int \frac{4\cdot 3\sec\theta\tan\theta\, d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta-9}} \\
&= \int \frac{4\cdot 3\sec\theta\tan\theta\, d\theta}{9\sec^2\theta\sqrt{9\tan^2\theta}} \\
&= \int \frac{4\cdot 3\sec\theta\tan\theta\, d\theta}{9\sec^2\theta\cdot 3\tan\theta} \\
&= \frac{4}{9}\int \frac{d\theta}{\sec\theta} \\
&= \frac{4}{9}\int \cos\theta\, d\theta \\
&= \frac{4}{9}\sin\theta + C \\
&= \frac{4}{9}\sin\left(\sec^{-1}(u/3)\right) + C \\
&= \frac{4}{9}\sin\left(\sec^{-1}(4x/3)\right) + C.
\end{aligned}$$

If you use a right triangle to get rid of the trig functions, you get an answer of

$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \frac{4}{9}\cdot\frac{\sqrt{16x^2-9}}{4x} + C.$$

#17. For this problem, we do not need trigonometric substitution. Simply make the substitution $u = x^2 - 7, du = 2x\,dx$. This gives

$$\begin{aligned}
\int \frac{x}{\sqrt{x^2-7}}\,dx &= \int \frac{du}{2\sqrt{u}} \\
&= \sqrt{u} + C \\
&= \sqrt{x^2-7} + C.
\end{aligned}$$

Note: The moral of this problem is that even though you can use a trig substitution on a problem (letting $x = \sqrt{7}\sec\theta$ will work nicely), sometimes there's an easier way.

#27. To do this problem, we first complete the square in the denominator to get

$$\begin{aligned}
\int \frac{dx}{(x^2+2x+2)^2} &= \int \frac{dx}{(x^2+2x+1+1)^2} \\
&= \int \frac{dx}{((x+1)^2+1)^2}.
\end{aligned}$$

Now we make the substitution $u = x + 1, du = dx$, obtaining

$$\int \frac{dx}{(x^2 + 2x + 2)^2} = \int \frac{du}{(u^2 + 1)^2}.$$

Next we make the trig substitution $u = \tan\theta, du = \sec^2\theta\, d\theta$. This yields

$$\begin{aligned}
\int \frac{dx}{(x^2 + 2x + 2)^2} &= \int \frac{du}{(u^2 + 1)^2} \\
&= \int \frac{\sec^2\theta\, d\theta}{(\tan^2\theta + 1)^2} \\
&= \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} \\
&= \int \frac{d\theta}{\sec^2\theta} \\
&= \int \cos^2\theta\, d\theta \\
&= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C \\
&= \frac{1}{2}\tan^{-1} u + \frac{1}{2}\sin\left(\tan^{-1} u\right)\cos\left(\tan^{-1} u\right) + C \\
&= \frac{1}{2}\tan^{-1}(x + 1) + \frac{1}{2}\sin\left(\tan^{-1}(x + 1)\right)\cos\left(\tan^{-1}(x + 1)\right) + C.
\end{aligned}$$

If you use right triangle methods to get rid of the trig functions, you get

$$\int \frac{dx}{(x^2 + 2x + 2)^2} = \frac{1}{2}\tan^{-1}(x + 1) + \frac{1}{2} \cdot \frac{x + 1}{x^2 + 2x + 2}.$$

Quest Problems:

#1. Looking carefully at the expression in the problem, we see that we have

$$x = \frac{1}{1 + \frac{1}{2+x}}.$$

A little algebra reduces this to

$$x = \frac{2 + x}{3 + x}.$$

More algebra leads to the equation

$$x^2 + 2x - 2 = 0,$$

which we can solve with the quadratic equation, giving

$$x = -1 \pm \sqrt{3}.$$

Now, the number $-1-\sqrt{3}$ is negative, while the expression in the problem is clearly positive. Therefore we must have $x = -1 + \sqrt{3}$.

#2. If you place a domino on the checkerboard, it covers two adjacent squares. One of these squares must be white and one must be black. So if we put several dominos on the board without overlapping them, the number of black squares covered must equal the number of white squares covered. After the corners have been cut off, however, the board contains 30 black squares and 32 white squares. Since the numbers of black and white squares are not equal, it is impossible to cover them all without overlapping the dominos.