Homework 4 Solutions
Math 162Q - Fall 2002

Section 6.4:

#2. Since we're using metric units, the force on the barbell is $F = mg$, where $g = 9.80\, m/s^2$ is the acceleration due to gravity. So the force is 588 Newtons. Since the force is constant, we have

$$\text{Work} \;=\; Fd \;=\; (588)(2) \;=\; 1176\,\text{Joules}.$$

#3. Since we're using English units now, gravity is already factored in. We have

$$\text{Work} = \int_0^9 \frac{10}{(1+x)^2}\,dx \;=\; \left[\frac{-10}{1+x}\right]_0^9 \;=\; 9\,\text{foot-pounds}.$$

#6. Since we know $F = kx$, we have

$$25 \;=\; k(0.1),$$

where we are multiplying by 0.1 because when the spring has length $30\,cm$, it is $0.1\,m$ away from its natural length. Solving this gives $k = 250$. We then have

$$\text{Work} = \int_0^{0.05} 250x\,dx \;=\; 0.3125\,\text{J}.$$

#8. The first part of the problem tells us that

$$12 \;=\; \int_0^1 kx\,dx \;=\; \frac{1}{2}k.$$

This gives $k = 24$. Then the work we want is (note that 9 inches is 3/4 foot)

$$\text{Work} \;=\; \int_0^{3/4} 24x\,dx \;=\; \frac{27}{4}\,\text{ft-lbs}.$$

#15. If you slice the tank at a height of y_i meters above the ground (let $y = 0$ at the ground), then the slice is a rectangle with a little bit of height. The length of the rectangle is 2 meters, the width is 1 meter, and the height is Δy. Thus the volume of the slice is $2\Delta y$. The force on the slice is then $\rho V g$, where ρ is the density of water ($1000\,kg/m^3$), V is the volume of the slice, and g is the acceleration due to gravity. So the force on the slice is

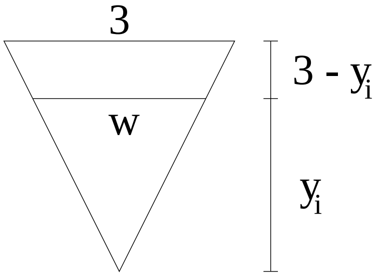
$19600\Delta y$. Now, this slice has to move upwards a distance of $1 - y_i$ meters to the top of the tank. So the work done on the slice is

$$W = Fd = 19600(1 - y_i)\Delta y.$$

Therefore, the total work done is (note that the integral starts at 1/2 since we're only pumping out the top half of the water)

$$W = \int_{1/2}^{1} 19600(1 - y)\, dy = 2450\, J.$$

#17. For this problem, we slice again horizontally. This time, the slice at height y_i has height Δy, length 8, and a width w that we have to figure out. To find w, we look at a horizontal cross-section as in the diagram below.



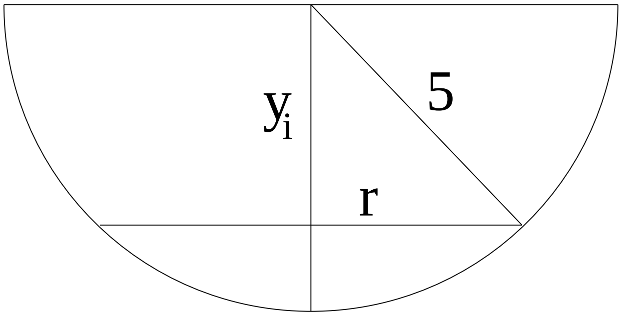
Since the two triangles are similar, we can make a proportion, getting

$$\begin{aligned}\frac{\text{ht. large }\Delta}{\text{ht. small }\Delta} &= \frac{\text{width large }\Delta}{\text{width small }\Delta}\\ \frac{3}{y_i} &= \frac{3}{w}\\ w &= y_i.\end{aligned}$$

As above, we then find that the force on the slice is $78400y_i\,\Delta y$. This slice as to move a distance of $5 - y_i$ since it needs to move to the top of the spout. When we write down an integral for the work, we only integrate from 0 to 3, since the water only goes to a height of 3 meters. The work done is then

$$\text{Work} = \int_0^3 78400y(5 - y)\, dy = 1058400J.$$

#20. For this problem, we set up coordinates differently than in the last two. This time, let the bottom of the hemisphere be at height $y = -5$. Then if we slice horizontally, the slices are disks with height Δy and a radius r which we need to find. We can do this by looking at the diagram below, which is a vertical cross-section through the center of the hemisphere.



The triangle in the figure is a right triangle, so we can use the Pythagorean theorem to find that

$$r = \sqrt{25 - y_i^2}.$$

Therefore the volume of the slice is

$$V = \pi r^2 \Delta y = \pi(25 - y_i^2)\Delta y.$$

Hence the force on the slice is

$$F = \rho V = 62.5\pi(25 - y_i^2)\Delta y.$$

Now, the distance the slice has to move is $-y_i$ (it's negative since y_i is negative), and so the work done on the slice is

$$W = -62.5\pi y_i(25 - y_i^2)\Delta y.$$

Hence the total work done is

$$\text{Work} \;=\; \int_{-5}^{0} -62.5\pi y(25 - y^2)\,dy \;=\; 9765.625\pi \text{ ft-lbs.}$$

Section 7.1:
#6. Let

$$\begin{array}{cc} u = \sin^{-1} x & dv = dx \\ du = \frac{1}{\sqrt{1-x^2}}\,dx & v = x. \end{array}$$

Then we get

$$\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x\,dx}{\sqrt{1-x^2}}.$$

We can evaluate the second integral on the right by making the substitution $w = 1 - x^2$, obtaining the answer

$$\int \sin^{-1} x\, dx \;=\; x \sin^{-1} x + \sqrt{1 - x^2} + C.$$

#7. Let

$$\begin{array}{cc} u = x^2 & dv = \cos(3x)\, dx \\ du = 2x\, dx & v = \frac{1}{3}\sin(3x). \end{array}$$

Then we get

$$\int x^2 \cos(3x)\, dx \;=\; \frac{1}{3} x^2 \sin(3x) - \frac{2}{3} \int x \sin(3x)\, dx.$$

This new integral also needs to be done by parts. Let

$$\begin{array}{cc} u = x & dv = \sin(3x)\, dx \\ du = dx & v = -\frac{1}{3}\cos(3x). \end{array}$$

We then obtain

$$\int x^2 \cos(3x)\, dx = \frac{1}{3} x^2 \sin(3x) - \frac{2}{3}\left(-\frac{1}{3} x \cos(3x) - \int -\frac{1}{3}\cos(3x)\, dx\right).$$

After evaluating the last integral (by substitution) and multiplying out, we get

$$\int x^2 \cos(3x)\, dx \;=\; \frac{1}{3} x^2 \sin(3x) + \frac{2}{9} x \cos(3x) - \frac{2}{27}\sin(3x) + C.$$

#12. Let

$$\begin{array}{cc} u = e^{-\theta} & dv = \cos(2\theta)\, d\theta \\ du = -e^{-\theta}\, d\theta & v = \frac{1}{2}\sin(2\theta). \end{array}$$

Then we get

$$\int e^{-\theta} \cos(2\theta)\, d\theta \;=\; \frac{1}{2} e^{-\theta} \sin(2\theta) + \frac{1}{2} \int e^{-\theta} \sin(2\theta)\, d\theta.$$

To evaluate the last integral, we use integration by parts again. Let

$$\begin{array}{cc} u = e^{-\theta} & dv = \sin(2\theta)\, d\theta \\ du = -e^{-\theta}\, d\theta & v = -\frac{1}{2}\cos(2\theta). \end{array}$$

This yields

$$\int e^{-\theta} \cos(2\theta)\, d\theta \;=\; \frac{1}{2} e^{-\theta} \sin(2\theta) + \frac{1}{2}\left(-\frac{1}{2} e^{-\theta} \cos(2\theta) - \frac{1}{2} \int e^{-\theta} \cos(2\theta)\, d\theta\right),$$

which gives

$$\begin{aligned}\int e^{-\theta}\cos(2\theta)\,d\theta &= \frac{1}{2}e^{-\theta}\sin(2\theta)-\frac{1}{4}e^{-\theta}\cos(2\theta)-\frac{1}{4}\int e^{-\theta}\cos(2\theta)\\ \frac{5}{4}\int e^{-\theta}\cos(2\theta)\,d\theta &= \frac{1}{2}e^{-\theta}\sin(2\theta)-\frac{1}{4}e^{-\theta}\cos(2\theta)\\ \int e^{-\theta}\cos(2\theta)\,d\theta &= \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin(2\theta)-\frac{1}{4}e^{-\theta}\cos(2\theta)\right)+C.\end{aligned}$$

#18. Let

$$\begin{array}{cc} u=\ln t & dv=\sqrt{t}\,dt\\ du=\frac{dt}{t} & v=\frac{2}{3}t^{3/2}.\end{array}$$

Then we get

$$\begin{aligned}\int_1^4 \sqrt{t}\ln t\,dt &= \left[\frac{2}{3}t^{3/2}\ln t\right]_1^4-\int_1^4\frac{2}{3}t^{1/2}\,dt\\ &= \left[\frac{2}{3}t^{3/2}\ln t\right]_1^4-\left[\frac{4}{9}t^{3/2}\right]_1^4\\ &= \frac{16}{3}\ln(4)-\frac{28}{9}.\end{aligned}$$

#19. Note that we have

$$\ln\sqrt{x} = \ln(x^{1/2}) = \frac{1}{2}\ln x.$$

Therefore we have

$$\int_1^4 \ln\sqrt{x}\,dx = \frac{1}{2}\int_1^4\ln x\,dx = \frac{1}{2}\left[x\ln x-x\right]_1^4 = \frac{1}{2}(4\ln(4)-3).$$

#30. First make the substitution $w=\sqrt{x}, dw=dx/(2\sqrt{x})$. This gives

$$\int_1^4 e^{\sqrt{x}}\,dx = \int_1^2 2we^w\,dw.$$

Now, let

$$\begin{array}{cc} u=2w & dv=e^w\,dw\\ du=2\,dw & v=e^w.\end{array}$$

This gives

$$\int_1^2 2we^w\,dw = \left[2we^w\right]_1^2-\int_1^2 2e^w\,dw = 2e^2.$$

#39.

a) The reduction formula tells us that we have

$$\int_0^{\pi/2} \sin^n x\,dx = \left[-\cos x \sin^{n-1} x\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx.$$

b) By the formula, we have

$$\int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \frac{2}{3}(1) = \frac{2}{3}.$$

Also, we have

$$\int_0^{\pi/2} \sin^5 x\,dx = \frac{4}{5}\int_0^{\pi/2} \sin^3 x\,dx = \frac{2\cdot 4}{3\cdot 5}.$$

c) Using the formula, we get

$$\begin{aligned}
\int_0^{\pi/2} \sin^{2n+1} x\,dx &= \frac{2n}{2n+1}\int_0^{\pi/2} \sin^{2n-1} x\,dx \\
&= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\int_0^{\pi/2} \sin^{2n-3} x\,dx \\
&= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\int_0^{\pi/2} \sin^{2n-5} x\,dx \\
&\vdots\ \vdots \\
&= \frac{(2n)\cdot(2n-2)\cdot(2n-4)\cdots 2}{(2n+1)\cdot(2n-1)\cdot(2n-3)\cdots 3}\int_0^{\pi/2} \sin x\,dx \\
&= \frac{(2n)\cdot(2n-2)\cdot(2n-4)\cdots 2}{(2n+1)\cdot(2n-1)\cdot(2n-3)\cdots 3}.
\end{aligned}$$

(Note that $\int_0^{\pi/2} \sin x\,dx = 1$).

#40. To do this, we use the formula from problem 39 part a), and get

$$\begin{aligned}
\int_0^{\pi/2} \sin^{2n} x\,dx &= \frac{2n-1}{2n}\int_0^{\pi/2} \sin^{2n-2} x\,dx \\
&= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\int_0^{\pi/2} \sin^{2n-4} x\,dx \\
&= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\int_0^{\pi/2} \sin^{2n-6} x\,dx \\
&\vdots\ \ \vdots \\
&= \frac{(2n-1)\cdot(2n-3)\cdot(2n-5)\cdots 3}{(2n)\cdot(2n-2)\cdot(2n-4)\cdots 4}\int_0^{\pi/2} \sin^{2} x\,dx \\
&= \frac{(2n-1)\cdot(2n-3)\cdot(2n-5)\cdots 3\cdot 1}{(2n)\cdot(2n-2)\cdot(2n-4)\cdots 4\cdot 2}\int_0^{\pi/2} \sin^{0} x\,dx \\
&= \frac{(2n-1)\cdot(2n-3)\cdot(2n-5)\cdots 3\cdot 1}{(2n)\cdot(2n-2)\cdot(2n-4)\cdots 4\cdot 2}\int_0^{\pi/2} 1\,dx \\
&= \frac{(2n-1)\cdot(2n-3)\cdot(2n-5)\cdots 3\cdot 1}{(2n)\cdot(2n-2)\cdot(2n-4)\cdots 4\cdot 2}\cdot\frac{\pi}{2}
\end{aligned}$$

Section 7.2:

#2. We have

$$\int \sin^6 x\cos^3 x\,dx = \int \sin^6 x\cos^2 x\cos x\,dx = \int (\sin^6 x)(1-\sin^2 x)\cos x\,dx.$$

Now make the substitution $u = \sin x, du = \cos x\,dx$. This gives us

$$\int \sin^6 x\cos^3 x\,dx = \int u^6(1-u^2)\,du = \frac{1}{7}u^7 - \frac{1}{9}u^9 + C = \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C.$$

#4. We have

$$\int_0^{\pi/2} \cos^5 x\,dx = \int_0^{\pi/2} (\cos^2 x)^2\cos x\,dx = \int_0^{\pi/2} (1-\sin^2 x)^2\cos x\,dx.$$

Now make the substitution $u = \sin x, du = \cos x\,dx$. This gives us

$$\int_0^{\pi/2} \cos^5 x\,dx = \int_0^1 (1-u^2)^2\,du = \int_0^1 1-2u^2+u^4\,du = \frac{8}{15}.$$

#10. We have

$$\begin{aligned}
\int \sin^6(\pi x)\,dx &= \int (\sin^2 \pi x)^3\,dx \\
&= \int \left(\frac{1-\cos(2\pi x)}{2}\right)^3 dx \\
&= \frac{1}{8}\int 1 - 3\cos(2\pi x) + 3\cos^2(2\pi x) - \cos^3(2\pi x)\,dx \\
&= \frac{1}{8}\int 1\,dx - \frac{3}{8}\int \cos(2\pi x)\,dx + \frac{3}{8}\int \cos^2(2\pi x)\,dx - \frac{1}{8}\int \cos^3(2\pi x)\,dx.
\end{aligned}$$

The first two integrals in the last equality are easy. We have

$$\frac{1}{8}\int 1\,dx = \frac{1}{8}x + C \qquad \text{and} \qquad \frac{3}{8}\int \cos(2\pi x)\,dx = \frac{3}{16\pi}\sin(2\pi x) + C.$$

For the third integral, we use a trig identity, getting

$$\frac{3}{8}\int \cos^2(2\pi x)\,dx = \frac{3}{8}\int \frac{1+\cos(4\pi x)}{2}\,dx = \frac{3}{16}x + \frac{3}{64\pi}\sin(4\pi x) + C.$$

Finally, for the last integral, we first make the substitution $u = 2\pi x, du = 2\pi\,dx$, getting

$$\frac{1}{8}\int \cos^3(2\pi x)\,dx = \frac{1}{16\pi}\int \cos^3 u\,du.$$

Then we have (making along the way the substitution $w = \sin u, dw = \cos u\,du$),

$$\frac{1}{16\pi}\int \cos^3(u)\,du = \frac{1}{16\pi}\int (1-\sin^2 u)\cos u\,du = \frac{1}{16\pi}\int 1 - w^2\,dw = \frac{1}{16\pi}w - \frac{1}{48\pi}w^3 + C.$$

Putting this back in terms of x gives

$$\frac{1}{8}\int \cos^3(2\pi x)\,dx = \frac{1}{16\pi}\sin(2\pi x) - \frac{1}{48\pi}\sin^3(2\pi x) + C.$$

If we now add all these integrals, we find that

$$\int \sin^6 x\,dx = \frac{5}{16}x - \frac{1}{4\pi}\sin(2\pi x) + \frac{3}{64\pi}\sin(4\pi x) + \frac{1}{48\pi}\sin^3(2\pi x) + C.$$

#18. If we put everything in terms of sine and cosine, we find that

$$\int \cot^5 x \sin^4 x\,dx = \int \frac{\cos^5 x}{\sin^5 x}\sin^4 x\,dx = \int \frac{\cos^5 x}{\sin x}\,dx.$$

Now we use trig identities and the substitution $u = \sin x, du = \cos x\,dx$ to get

$$\begin{aligned}
\int \frac{\cos^5 x}{\sin x}\,dx &= \int \frac{(1-\sin^2 x)^2}{\sin x}\cos x\,dx \\
&= \int \frac{(1-u^2)^2}{u}\,du \\
&= \int \frac{1}{u} - 2u + u^3\,du \\
&= \ln|u| - u^2 + \frac{1}{4}u^4 + C \\
&= \ln|\sin x| - \sin^2 x + \frac{1}{4}\sin^4 x + C.
\end{aligned}$$

#24. We have

$$\begin{aligned}
\int_0^{\pi/4} \sec^6 x\,dx &= \int_0^{\pi/4} (\sec^2 x)^2 \sec^2 x\,dx \\
&= \int_0^{\pi/4} (\tan^2 x + 1)^2 \sec^2 x\,dx \\
&= \int_0^{\pi/4} (\tan^4 x + 2\tan^2 x + 1)\sec^2 x\,dx.
\end{aligned}$$

Now make the substitution $u = \tan x, du = \sec^2 x\,dx$, to get

$$\int_0^{\pi/4} (\tan^4 x + 2\tan^2 x + 1)\sec^2 x\,dx = \int_0^1 u^4 + 2u^2 + 1\,du = \frac{28}{15}.$$

#28. We have

$$\int \tan^3 x \sec^3 x\,dx = \int \tan^2 x \sec^2 x \tan x \sec x\,dx = \int (\sec^2 x - 1)\sec^2 x \tan x \sec x\,dx.$$

Now make the substitution $u = \sec x, du = \tan x \sec x\,dx$ to get

$$\begin{aligned}
\int (\sec^2 x - 1)\sec^2 x \tan x \sec x\,dx &= \int (u^2-1)u^2\,du \\
&= \int u^4 - u^2\,du \\
&= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\
&= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.
\end{aligned}$$

Quest Problems:
#1.
a) If $0 \leq x \leq \pi/2$, then we have $\sin x \leq 1$, and so

$$\sin^{2n+2} x \leq \sin^{2n+1} x \leq \sin^{2n} x.$$

Therefore, by one of the integral properties, we have

$$\int_0^{\pi/2} \sin^{2n+2} x\, dx \leq \int_0^{\pi/2} \sin^{2n+1} x\, dx \leq \int_0^{\pi/2} \sin^{2n} x\, dx.$$

b) By exercise 40 of section 7.1, we have

$$\begin{aligned}\frac{I_{2n+2}}{I_{2n}} &= \frac{\int_0^{\pi/2} \sin^{2n+2} x\, dx}{\int_0^{\pi/2} \sin^{2n} x\, dx} \\ &= \frac{\left(\frac{(2n+1)\cdot(2n-1)\cdot(2n-3)\cdots 3\cdot 1}{(2n+2)\cdot(2n)\cdot(2n-2)\cdots 4\cdot 2}\cdot\frac{\pi}{2}\right)}{\left(\frac{(2n-1)\cdot(2n-3)\cdot(2n-5)\cdots 3\cdot 1}{(2n)\cdot(2n-2)\cdot(2n-4)\cdots 4\cdot 2}\cdot\frac{\pi}{2}\right)} \\ &= \frac{2n+1}{2n+2}.\end{aligned}$$

c) Using parts a) and b), we have

$$\frac{2n+1}{2n+2} = \frac{I_{2n+2}}{I_{2n}} \leq \frac{I_{2n+1}}{I_{2n}} \leq \frac{I_{2n}}{I_{2n}} = 1.$$

Now, by limit laws, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{2n+1}{2n+2} \leq &\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} \leq \lim_{n\to\infty} 1 \\ 1 \leq &\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} \leq 1.\end{aligned}$$

Therefore, by the Squeeze Theorem, we have

$$\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1.$$

d) By part c) and problems 39 and 40 of section 7.1, we have

$$\begin{aligned}
1 &= \lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} \\
&= \lim_{n\to\infty} \frac{\left(\frac{(2n)\cdot(2n-2)\cdot(2n-4)\cdots 2}{(2n+1)\cdot(2n-1)\cdot(2n-3)\cdots 3}\right)}{\left(\frac{(2n-1)\cdot(2n-3)\cdot(2n-5)\cdots 3\cdot 1}{(2n)\cdot(2n-2)\cdot(2n-4)\cdots 4\cdot 2}\cdot\frac{\pi}{2}\right)} \\
&= \lim_{n\to\infty} \frac{(2n)(2n)(2n-2)(2n-2)\cdots(4)(4)(2)(2)}{(2n+1)(2n-1)(2n-1)\cdots(5)(3)(3)(1)}\cdot\frac{2}{\pi} \\
&= \frac{2}{\pi}\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}.
\end{aligned}$$

To finish the problem, multiply both sides of this series of equalities by $\pi/2$.

#2. The method is as follows. From the tray with thousands of chips, pick any 20, and move them to the other tray. Now, on the tray that now holds 20 chips, turn all of the chips upside-down. That's it! Here's why it works. When you pick up the 20 chips, suppose you take N that have the white side up. Then the first tray still contains $20-N$ chips with white showing. Now, you have picked N chips with white up and $20-N$ chips with black up. When you turn these chips upside down, you get N chips with black facing up and $20-N$ with white facing up. So both trays have $20-N$ chips facing up.

Note: A feature of this problem that I think that is really interesting is that even though you know that both trays have the same number of chips with the white side up, you **do not** know what that number is!