

Homework 1 Solutions
Math 162Q - Fall 2002

Section 5.1:

#1.

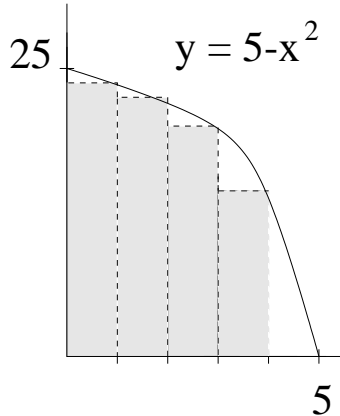
a. The lower and upper estimates you should obtain are 40 and 52, respectively. You can find the sketches you are asked to draw in the back of the book. The estimates are equal to the shaded area in each case.

b. The lower and upper estimates you should obtain are approximately 43.2 and 49.2, depending on your interpretation of the height of the graph when $x = 1, \dots, 10$. The sketches should be similar to the ones in part (a), but with 10 rectangles instead of 5.

#4.

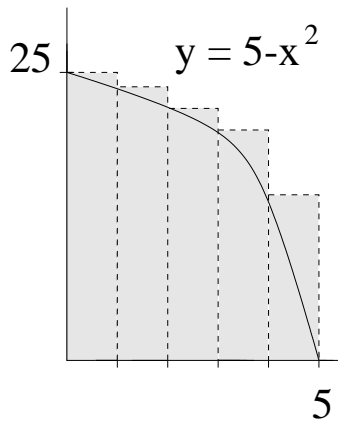
a. The graph and rectangles should be as in the diagram below. Since we're using right endpoints, the final rectangle has height 0, which is why it looks like there are only four rectangles in the picture. Your estimate should then be the total area inside the rectangles, which is

$$\begin{aligned} & f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x + f(5)\Delta x \\ &= (24)(1) + (21)(1) + (16)(1) + (9)(1) + (0)(1) \\ &= 70. \end{aligned}$$



b. In this case, the graph and rectangles should look like the ones at the top of the next page. Your estimate should then be

$$\begin{aligned} & f(0)\Delta x + f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x \\ &= (25)(1) + (24)(1) + (21)(1) + (16)(1) + (9)(1) \\ &= 95. \end{aligned}$$



#12. I can think of three reasonable ways to estimate the height.

1) Left-hand sum:

$$\begin{aligned} \text{Height} &\approx 0(10 - 0) + 185(15 - 10) + 319(20 - 15) + 447(32 - 20) \\ &\quad + 742(59 - 32) + 1325(62 - 59) \\ &= 31893 \text{ ft.} \end{aligned}$$

2) Right-hand sum:

$$\begin{aligned} \text{Height} &\approx 185(10 - 0) + 319(15 - 10) + 447(20 - 15) + 742(32 - 20) \\ &\quad + 1325(59 - 32) + 1445(62 - 59) \\ &= 54694 \text{ ft.} \end{aligned}$$

3) Average the LHS and RHS:

$$(31893 + 54694)/2 = 43293.5 \text{ ft.}$$

#14. Using a Right-hand sum with 6 subintervals of length 5 seconds, I get

$$\text{Distance} \approx (50 + 80 + 96 + 108 + 116 + 120) * 5 = 2850 \text{ ft.}$$

With a left-hand sum, I get

$$(0 + 50 + 80 + 96 + 108 + 116) * 5 = 2250 \text{ ft.}$$

Averaging these gives 2550 feet. I would consider any of these three answers acceptable, and also answers that are slightly different due to reading the graph slightly differently.

#16. If we divide the interval $[1, 8]$ into N subintervals, we get

$$\Delta x = \frac{7}{N} \quad \text{and} \quad x_i = 1 + \frac{7i}{N}.$$

Hence the desired limit is

$$\lim_{N \rightarrow \infty} \sum_{i=1}^N \left(5 + \sqrt[3]{1 + \frac{7i}{N}} \right) \frac{7}{N}.$$

Section 5.2

#20. If we split the interval $[1, 5]$ into N subintervals, then we get

$$\Delta x = \frac{4}{N} \quad \text{and} \quad x_i = 1 + \frac{4i}{N}.$$

Then using equations 4-6 of section 5.2, we have

$$\begin{aligned} \int_1^5 (2 + 3x - x^2) dx &= \lim_{N \rightarrow \infty} \sum_{i=1}^N f(x_i) \Delta x \\ &= \lim_{N \rightarrow \infty} \sum_{i=1}^N \left(2 + 3 \left(1 + \frac{4i}{N} \right) - \left(1 + \frac{4i}{N} \right)^2 \right) \frac{4}{N} \\ &= \lim_{N \rightarrow \infty} \sum_{i=1}^N \frac{16}{N} + \frac{16i}{N^2} - \frac{64i^2}{N^3} \\ &= \lim_{N \rightarrow \infty} \frac{16}{N} \sum_{i=1}^N 1 + \lim_{N \rightarrow \infty} \frac{16}{N^2} \sum_{i=1}^N i - \lim_{N \rightarrow \infty} \frac{64}{N^3} \sum_{i=1}^N i^2 \\ &= \lim_{N \rightarrow \infty} \frac{16}{N} \cdot N + \lim_{N \rightarrow \infty} \frac{16}{N^2} \cdot \frac{N^2 + N}{2} - \lim_{N \rightarrow \infty} \frac{64}{N^3} \cdot \frac{2N^3 + 3N^2 + N}{6} \\ &= \lim_{N \rightarrow \infty} 16 + \lim_{N \rightarrow \infty} 8 \frac{N^2 + N}{N^2} - \lim_{N \rightarrow \infty} \frac{64}{6} \cdot \frac{2N^3 + 3N^2 + N}{N^3} \\ &= 16 + 8 - \frac{64}{6} \cdot 2 \\ &= \frac{8}{3} \end{aligned}$$

#30. First note that the area under the graph from 0 to 2 is the area of a triangle, and is equal to 4. Also, the area above the graph from 2 to 6 is the area under a semicircle of radius 2, and so is 2π . Finally, the area under the graph from 6 to 7 is the area of a triangle, which is $1/2$. This gives us

$$\int_0^2 g(x) dx = 4, \quad \int_2^6 g(x) dx = -2\pi, \quad \int_6^7 g(x) dx = 1/2.$$

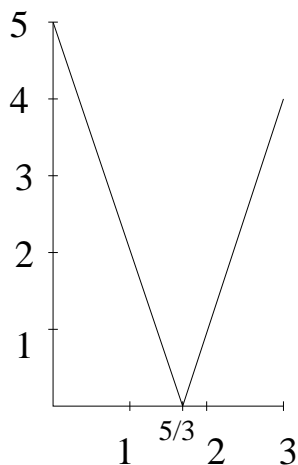
Note that the middle integral is negative because the area is below the x -axis. So we get:

a) $\int_0^2 g(x) dx = 4.$

$$\text{b) } \int_2^6 g(x) dx = -2\pi.$$

$$\text{c) } \int_0^7 g(x) dx = \int_0^2 g(x) dx + \int_2^6 g(x) dx + \int_6^7 g(x) dx = 4 - 2\pi + 1/2.$$

#36. As shown in the graph below, the area in question consists of two triangles. The left one has height 5 and base $5/3$, and so its area is $25/6$. The triangle on the right has height 4 and base $3 - 5/3 = 4/3$, so its area is $8/3$. So the integral is equal to the total area under the curve, which is $25/6 + 8/3 = 41/6$.



#39. Using properties 1,2 and 3 of integrals from Section 5.2, we get

$$\begin{aligned} \int_0^1 (5 - 6x^2) dx &= \int_0^1 5 dx - \int_0^1 6x^2 dx \\ &= 5(1 - 0) - 6 \int_0^1 x^2 dx \\ &= 5 - 6(1/3) \\ &= 3. \end{aligned}$$

#44. Since we know that

$$\int_2^7 f(x) dx + \int_7^{10} f(x) dx = \int_2^{10} f(x) dx,$$

we find that

$$\int_2^{10} f(x) dx - \int_2^7 f(x) dx = \int_7^{10} f(x) dx.$$

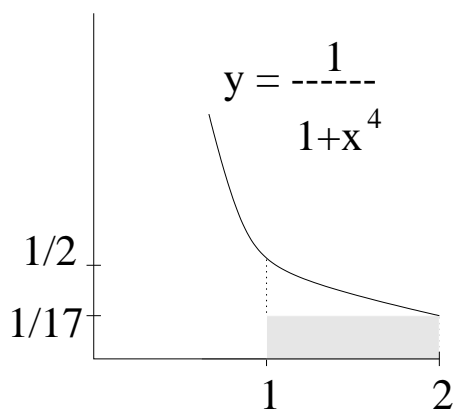
#46. First, note that we have

$$\begin{aligned}\int_0^3 f(x) dx + \int_3^4 f(x) dx &= \int_0^4 f(x) dx \\ \int_0^3 f(x) dx + 1 &= -6 \\ \int_0^3 f(x) dx &= -7.\end{aligned}$$

Using this, we find that

$$\begin{aligned}\int_0^1 f(x) dx + \int_1^3 f(x) dx &= \int_0^3 f(x) dx \\ 2 + \int_1^3 f(x) dx &= -7 \\ \int_1^3 f(x) dx &= -9.\end{aligned}$$

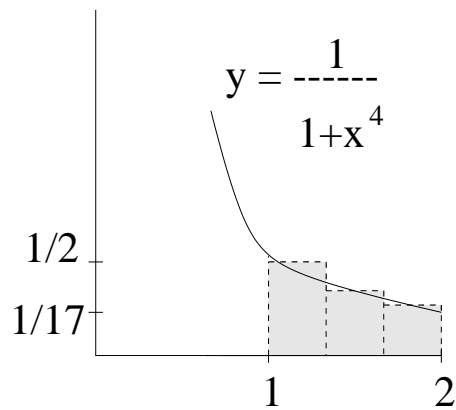
p. 430 #3: Note that the function we are given is decreasing, and that therefore a right-hand sum gives a lower bound for the integral. In fact, if we do a right-hand sum with only one interval, as in the (not to scale) diagram below, we find that



$$\int_1^2 \frac{1}{1+x^4} dx \geq f(2)\Delta x = \left(\frac{1}{17}\right)(1) = \frac{1}{17}.$$

I can think of two ways to to the other part of the problem. The first is to realize that a left-hand sum will give an overestimate of the integral. Doing a left-hand sum with 3

subintervals, as in the next picture, we get



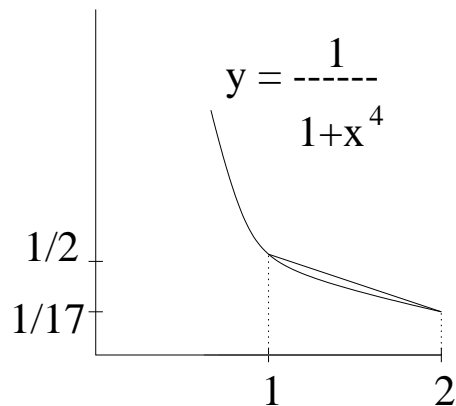
$$\begin{aligned}
 \int_1^2 \frac{1}{1+x^4} dx &\leq f(1)\Delta x + f(4/3)\Delta x + f(5/3)\Delta x \\
 &= \left(\frac{1}{2}\right) \left(\frac{1}{3}\right) + \left(\frac{81}{337}\right) \left(\frac{1}{3}\right) + \left(\frac{81}{706}\right) \left(\frac{1}{3}\right) \\
 &\approx 0.285\dots \\
 &< 0.286.
 \end{aligned}$$

But we have

$$7/24 \approx 0.29167 > 0.29.$$

Since the integral is less than 0.286, it is certainly less than 7/24.

You can also do this problem another way, which gives a slightly better answer. If you look at the graph below, where I have drawn a line connecting the points $(1, 1/2)$ and $(2, 1/17)$, you can see that the area under the curve is less than the area under the trapezoid with corners at the points $(1, 0)$, $(1, 1/2)$, $(2, 1/17)$ and $(2, 0)$. This trapezoid has height 1



and bases of lengths $1/2$ and $1/17$. Thus the trapezoid has area

$$\frac{1}{2}h(b_1 + b_2) = \frac{1}{2}(1) \left(\frac{1}{2} + \frac{1}{17} \right) = \frac{19}{68} \approx 0.2974 \dots$$

So we find that the value of the integral is less than 0.298, which is again less than $7/24$.

Also:

a. Here, we set

$$x_i = \frac{i\pi}{4n} \quad \text{and} \quad \Delta x = \frac{\pi}{4n}.$$

Also, we then get $f(x) = \tan x$. If we plug in $i = 0$, we find that the lower limit of the integral is 0, and if we plug in $i = n$, we find that the upper limit of the integral is $\pi/4$. So the integral we want is

$$\int_0^{\pi/4} \tan x \, dx.$$

b. Here it looks like we should have $x_i = 2 + \frac{2i}{n}$. This gives us $\Delta x = 2/n$, and so we get $f(x) = 4 \ln x$. If we plug in $i = 0$, we find that the lower limit of the integral is 2, and by plugging in $i = n$, we find that the upper limit is 4. So the integral is

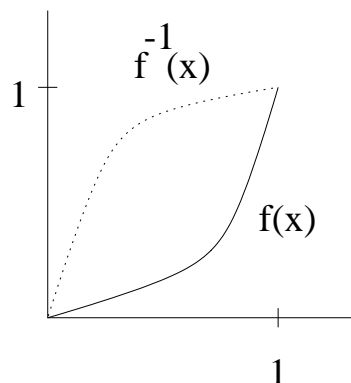
$$\int_2^4 4 \ln x \, dx.$$

Note: There are other possible answers to this problem. For example, you could set $x_i = 2i/n$ and $f(x) = \ln(2 + x)$. This would give you the (also correct) answer of

$$\int_0^2 4 \ln(2 + x) \, dx.$$

Quest Problems:

#1. The function looks something like the one in the diagram on the next page, and I have also sketched the graph of the inverse function obtained by reflecting the graph of $f(x)$ over the line $y = x$. We are given that the area under the graph of $f(x)$ between 0 and 1 is $1/3$. By symmetry, the area to the left of the graph of $f^{-1}(x)$ is also $1/3$. Since the total area of the square with corners at $(0, 0)$, $(0, 1)$, $(1, 1)$ and $(1, 0)$ is 1, we find that the area between the two graphs must also be $1/3$. Therefore the area below the graph of $f^{-1}(x)$ between 0 and 1 is $2/3$.



#2. Student 1 is wearing a black hat. To figure this out, she reasons as follows. “If student 3 saw two white hats in front of him, he would know immediately that his hat was black, since there are only two white hats in total. Since he doesn’t know what color his hat is, either student 2 or I must be wearing a black hat. Now, student 2 knows this too. If I was wearing a white hat, then student 2 would know that she had to be wearing a black hat, and would call out her answer. Since she hasn’t done that, she must not see a white hat. Because she sees my hat and doesn’t see a white hat, my hat must be black.”

#3. a. If there are n subintervals, then you should have $x_i = i/n$ and $\Delta x = 1/n$. Then we get

$$\begin{aligned} \int_0^1 2^x dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n 2^{x_i} \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n 2^{i/n} \cdot \frac{1}{n}. \end{aligned}$$

b. Let $a = 2^{1/n}$. Then we get

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \sum_{i=1}^n 2^{i/n} \cdot \frac{1}{n} &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n 2^{i/n} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n 2^{i/n} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n a^n \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} (a + a^2 + \cdots + a^n) \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot a \cdot \frac{1 - a^n}{1 - a} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot 2^{1/n} \cdot \frac{1 - (2^{1/n})^n}{1 - 2^{1/n}} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot 2^{1/n} \cdot \frac{1 - 2}{1 - 2^{1/n}} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{-2^{1/n}}{1 - 2^{1/n}}.
 \end{aligned}$$

c. If we let $x = 1/n$, then x approaches 0 as n approaches infinity, and we immediately get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{-2^{1/n}}{1 - 2^{1/n}} = \lim_{x \rightarrow 0} x \cdot \frac{-2^x}{1 - 2^x} = \lim_{x \rightarrow 0} \frac{-x \cdot 2^x}{1 - 2^x}.$$

d. Note that as $x \rightarrow 0$, both the top and bottom of the fraction also approach 0. Therefore we can use L'Hospital's rule, which gives us (recall that the derivative of 2^x is $2^x \ln 2$)

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{-x \cdot 2^x}{1 - 2^x} &= \lim_{x \rightarrow 0} \frac{-1 \cdot 2^x - x \cdot 2^x \ln 2}{-2^x \ln 2} \\
 &= \frac{-1 \cdot 2^0 - 0 \cdot 2^0 \ln 2}{-2^0 \ln 2} \\
 &= \frac{1}{\ln 2}.
 \end{aligned}$$