BE SURE TO SHOW YOUR WORK FOR FULL CREDIT!

NAME:

Scores: (for grader’s use only).

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1. Evaluate the following integral.

\[ \int \frac{x^3 + 4x^2 - x}{x^3 - x^2 + x - 1} \, dx \]

**Note:** \( x^3 - x^2 + x - 1 = (x - 1)(x^2 + 1) \).

Since the degrees of the polynomials on the top and bottom are the same, you have to divide. After doing the long division of polynomials, you should find that

\[ \frac{x^3 + 4x^2 - x}{x^3 - x^2 + x - 1} = 1 + \frac{5x^2 - 2x + 1}{x^3 - x^2 + x - 1}. \]

We need to take the fraction on the right and find a partial fraction decomposition. Note that we are given the factorization of the denominator. Therefore we have

\[ \frac{5x^2 - 2x + 1}{x^3 - x^2 + x - 1} = \frac{A}{x - 1} + \frac{Bx + D}{x^2 + 1} = \frac{A(x^2 + 1)}{(x - 1)(x^2 + 1)} + \frac{(Bx + D)(x - 1)}{(x - 1)(x^2 + 1)}. \]

Since the denominators are equal, we need to set the numerators equal to each other. In other words, we need to have

\[ 5x^2 - 2x + 1 = A(x^2 + 1) + (Bx + D)(x - 1). \]

Now, if we set \( x = 1 \), then we find that \( A = 2 \). Next, setting \( x = 0 \) shows us that \( D = 1 \). Finally, if we set \( x = 2 \), we can solve and find that \( B = 3 \). Hence we can evaluate the integral as follows:

\[ \int \frac{x^3 + 4x^2 - x}{x^3 - x^2 + x - 1} \, dx = \int 1 + \frac{5x^2 - 2x + 1}{x^3 - x^2 + x - 1} \, dx = \int 1 + \frac{2}{x - 1} + \frac{3x + 1}{x^2 + 1} \, dx = x + 2\ln |x - 1| + \int \frac{3x}{x^2 + 1} \, dx + \int \frac{1}{x^2 + 1} \, dx = x + 2\ln |x - 1| + \frac{3}{2} \ln (x^2 + 1) + \tan^{-1} x + C. \]
2. Evaluate the following integral.

\[ \int_{5\sqrt{2}}^{10} \frac{dx}{x^2\sqrt{x^2 - 25}} \]

If we make the trigonometric substitution \( x = 5 \sec \theta \), \( dx = 5 \sec \theta \tan \theta \, d\theta \), we find that

\[
\int_{5\sqrt{2}}^{10} \frac{dx}{x^2\sqrt{x^2 - 25}} = \int_{\pi/4}^{\pi/3} \frac{5 \sec \theta \tan \theta \, d\theta}{25 \sec^2 \theta \sqrt{25 \sec^2 \theta - 25}}
\]

\[
= \int_{\pi/4}^{\pi/3} \frac{\tan \theta \, d\theta}{5 \sec \theta \sqrt{5 \sec^2 \theta - 5}}
\]

\[
= \int_{\pi/4}^{\pi/3} \frac{\tan \theta \, d\theta}{5 \sec \theta \tan \frac{\pi}{3}}
\]

\[
= \frac{1}{25} \int_{\pi/4}^{\pi/3} \sec \theta \, d\theta
\]

\[
= \frac{1}{25} \int_{\pi/4}^{\pi/3} \cos \theta \, d\theta
\]

\[
= \frac{1}{25} \left[ \sin \theta \right]_{\pi/4}^{\pi/3}
\]

\[
= \frac{1}{25} \left( \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \right)
\]

\[
= \frac{1}{50} (\sqrt{3} - \sqrt{2}).
\]
3. Evaluate the following integral.

\[ \int e^{x+e^x} \, dx \]

We have, making the substitution \( u = e^x \), \( du = e^x \, dx \),

\[
\int e^{x+e^x} \, dx = \int e^x e^u \, dx \\
= \int e^u \, du \\
= e^u + C \\
= e^{e^x} + C.
\]
4. Evaluate the following integral.

\[\int_0^3 \frac{dx}{(2x - 5)^2}\]

Note that this integral is improper since the integrand is undefined at \(x = 5/2\), which is inside the interval we're integrating over. So we have to break this integral up at 5/2, getting

\[\int_0^3 \frac{dx}{(2x - 5)^2} = \int_0^{5/2} \frac{dx}{(2x - 5)^2} + \int_{5/2}^3 \frac{dx}{(2x - 5)^2}.
\]

Now look at the second integral on the right. If we make the substitution \(u = 2x - 5\), \(du = 2\,dx\), we get

\[\int_{5/2}^3 \frac{dx}{(2x - 5)^2} = \frac{1}{2} \int_0^1 \frac{du}{u^2}.
\]

We know from class that this new integral is undefined. Since one of the integrals that we broke up \(\int_0^3 \frac{dx}{(2x - 5)^2}\) is undefined, the entire thing is undefined.
5. a) Find the values of $t$ at which the curve traced out by

$$x = t^3 - t^2 - 2t + 4, \quad y = t^4 + 3t^2 - 14t - 1$$

goes through the point $(4, -1)$.

First, we’ll solve the equation $t^3 - t^2 - 2t + 4 = 4$. We have

$$t^3 - t^2 - 2t = 0$$
$$t(t^2 - t - 2) = 0$$
$$t(t - 2)(t + 1) = 0.$$

So the only values of $t$ that give you the right $x$-value are $t = 0$, $t = 2$, and $t = -1$. If we plug these values into the equation for $y$, we find that only 0 and 2 give $y = -1$. So the values of $t$ we’re looking for are $t = 0$ and $t = 2$.

b) Find the equations of the tangent lines to the curve above at the point $(4, -1)$.

We have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3 + 6t - 14}{3t^2 - 2t - 2}.$$

So if $t = 0$, then the slope is $\frac{dy}{dx} = 7$. Since a point on the tangent line is $(4, -1)$, the equation of the tangent line is

$$y + 1 = 7(x - 4) \quad \text{or} \quad y = 7x - 29.$$

If we have $t = 2$, then the slope is $\frac{dy}{dx} = 5$, and the tangent line has the equation

$$y + 1 = 5(x - 4) \quad \text{or} \quad y = 5x - 21.$$
6. Evaluate the following integral.

\[ \int \frac{1}{1 + 3 \sin \theta + \cos \theta} \, d\theta. \]

To do this, we make the Weierstrass substitution \( t = \tan(\theta/2) \). Then we have

\[ \sin \theta = \frac{2t}{1 + t^2}, \quad \cos \theta = \frac{1 - t^2}{1 + t^2}, \quad d\theta = \frac{2}{1 + t^2} \, dt. \]

Then the integral becomes

\[
\int \frac{1}{1 + 3 \sin \theta + \cos \theta} \, d\theta = \int \frac{1}{1 + \frac{6t}{1 + t^2} + \frac{1 - t^2}{1 + t^2}} \cdot \frac{2}{1 + t^2} \, dt
\]

\[
= \int \frac{2 \, dt}{1 + t^2 + 6t + 1 - t^2}
\]

\[
= \int \frac{dt}{1 + 3t}
\]

\[
= \frac{1}{3} \ln |1 + 3t| + C
\]

\[
= \frac{1}{3} \ln \left| 1 + 3 \tan \left( \frac{\theta}{2} \right) \right| + C.
\]
7. Find the area under the graph traced out by

\[ x = 5 + 4t, \quad y = 1 + 6e^{2t} \]

between \( t = 0 \) and \( t = 17 \).

Note that since \( x' > 0 \) for all values of \( t \), the graph is traced out exactly once as \( t \) goes from \( 5 \) to \( 17 \). Also, if \( t = 0 \), we see that \( x = 5 \), and if \( t = 17 \), we have \( x = 73 \). Hence in order to integrate from left to right, we must start at \( t = 0 \) and end at \( t = 17 \). We then have

\[
\text{Area} = \int_0^{17} y \, dx \\
= \int_0^{17} (1 + 6e^{2t})(4) \, dt \\
= \int_0^{17} 4 + 24e^{2t} \, dt \\
= [4t + 12e^{2t}]_0^{17} \\
= (4)(17) + 12e^{34} - (4)(0) - 12e^0 \\
= 56 + 12e^{34}.
\]
8. Find the length of the curve traced out by

\[ x = \frac{1}{3} t^3 - t, \quad y = t^2 \]

between \( t = -3 \) and \( t = 5 \).

Note that since \( y' = 2t^2 > 0 \) for every value of \( t \), the curve is traced out exactly once as \( t \) goes from \(-3\) to \( 5 \). Then we have

\[
\text{Length} = \int_{-3}^{5} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \\
= \int_{-3}^{5} \sqrt{(t^2 - 1)^2 + (2t)^2} \, dt \\
= \int_{-3}^{5} \sqrt{t^4 - 2t^2 + 1 + 4t^2} \, dt \\
= \int_{-3}^{5} \sqrt{t^4 + 2t^2 + 1} \, dt \\
= \int_{-3}^{5} \sqrt{(t^2 + 1)^2} \, dt \\
= \int_{-3}^{5} t^2 + 1 \, dt \\
= \frac{125}{3} + 17.
\]
9. Evaluate the following integral.

\[ \int_1^\infty x e^{-x^2} \, dx \]

We have, making along the way the substitution \( u = -x^2, \, du = -2x \, dx \),

\[ \int_1^\infty x e^{-x^2} \, dx = \lim_{t \to \infty} \int_1^t x e^{-x^2} \, dx \]
\[ = \lim_{t \to \infty} \int_{-1}^{-t^2} \frac{-1}{2} e^u \, du \]
\[ = \frac{-1}{2} \lim_{t \to \infty} \int_{-1}^{-t^2} \frac{-1}{2} e^u \, du \]
\[ = \frac{-1}{2} \left[ \lim_{t \to \infty} [e^u]_{-1}^{-t^2} \right] \]
\[ = \frac{-1}{2} \lim_{t \to \infty} (e^{-t^2} - e^{-1}) \]
\[ = \left( \frac{-1}{2} \right) \left( \frac{-1}{e} \right) \]
\[ = \frac{1}{2e} \]