BE SURE TO SHOW YOUR WORK FOR FULL CREDIT!

NAME:

Scores: (for grader’s use only).

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1. Evaluate the following integral.

\[ \int \sin^4 x \cos^3 x \, dx \]

We have

\[ \int \sin^4 x \cos^3 x \, dx = \int \sin^4 x \cos^2 x \cos x \, dx \]
\[ = \int \sin^4 x (1 - \sin^2 x) \cos x \, dx. \]

Now, making the substitution \( u = \sin x, du = \cos x \, dx \), we get

\[ \int \sin^4 x \cos^3 x \, dx = \int \sin^4 x (1 - \sin^2 x) \cos x \, dx \]
\[ = \int u^4 (1 - u^2) \, du \]
\[ = \int u^4 - u^6 \, du \]
\[ = \frac{u^5}{5} - \frac{u^7}{7} + C \]
\[ = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C. \]
2. Evaluate the following integral.

\[ \int_0^{\pi/12} x + 4 \sin(3x) \, dx \]

Note that we have

\[ \int_0^{\pi/12} x + 4 \sin(3x) \, dx = \int_0^{\pi/12} x \, dx + \int_0^{\pi/12} 4 \sin(3x) \, dx. \]

We use the substitution \( u = 3x, \, du = 3 \, dx \) to evaluate the second integral, and we get

\[
\int_0^{\pi/12} x + 4 \sin(3x) \, dx = \int_0^{\pi/12} x \, dx + \int_0^{\pi/12} 4 \sin(3x) \, dx \\
= \left[ \frac{1}{2} x^2 \right]_0^{\pi/12} + \frac{1}{3} \int_0^{\pi/4} 4 \sin(u) \, du \\
= \frac{\pi^2}{288} + \frac{1}{3} \left[ -4 \cos u \right]_0^{\pi/4} \\
= \frac{\pi^2}{288} + \frac{1}{3} \left( -4 \cos(\pi/4) + 4 \cos(0) \right) \\
= \frac{\pi^2}{288} - \frac{4\sqrt{2}}{6} + \frac{4}{3} \\
= \frac{\pi^2}{288} - \frac{2\sqrt{2}}{3} + \frac{4}{3}. \]
3. Evaluate the following integral.

\[ \int_0^1 \tan^3 \theta \sec^3 \theta \, d\theta \]

We have

\[ \int_0^1 \tan^3 \theta \sec^3 \theta \, d\theta = \int_0^1 \tan^2 \theta \sec^2 \theta \tan \theta \sec \theta \, d\theta \]
\[ = \int_0^1 (\sec^2 \theta - 1) \sec^2 \theta \tan \theta \sec \theta \, d\theta. \]

Now we make the substitution \( u = \sec \theta, \, du = \sec \theta \tan \theta \, d\theta \) to get

\[ \int_0^1 \tan^3 \theta \sec^3 \theta \, d\theta = \int_0^1 (\sec^2 \theta - 1) \sec^2 \theta \tan \theta \sec \theta \, d\theta \]
\[ = \int_1^{\sec(1)} (u^2 - 1)u^2 \, du \]
\[ = \int_1^{\sec(1)} u^4 - u^2 \, du \]
\[ = \left[ \frac{u^5}{5} - \frac{u^3}{3} \right]_{\sec(1)}^{1} \]
\[ = \frac{\sec^5(1)}{5} - \frac{\sec^3(1)}{3} - \frac{1}{5} + \frac{1}{3} \]
\[ = \frac{\sec^5(1)}{5} - \frac{\sec^3(1)}{3} + \frac{2}{15}. \]
4. Evaluate the following integral.

\[ \int e^{3x} \sin(4x) \, dx \]

This one needs to be done by parts. Let

\[ u = e^{3x}, \quad du = 3e^{3x} \, dx, \quad dv = \sin(4x), \quad v = \frac{1}{4} \cos(4x). \]

Then integration by parts gives us

\[ I = \int e^{3x} \sin(4x) \, dx = -\frac{1}{4} e^{3x} \cos(4x) + \frac{3}{4} \int e^{3x} \cos(4x) \, dx. \]

This integral also has to be evaluated by parts. Let

\[ u = e^{3x}, \quad du = 3e^{3x} \, dx, \quad dv = \cos(4x), \quad v = \frac{1}{4} \sin(4x). \]

Then we get

\[ I = -\frac{1}{4} e^{3x} \cos(4x) + \frac{3}{4} \int e^{3x} \cos(4x) \, dx = -\frac{1}{4} e^{3x} \cos(4x) + \frac{3}{4} \left( \frac{1}{4} e^{3x} \sin(4x) - \frac{9}{16} \int e^{3x} \sin(4x) \, dx \right) = -\frac{1}{4} e^{3x} \cos(4x) + \frac{3}{16} e^{3x} \sin(4x) - \frac{9}{16} \int e^{3x} \sin(4x) \, dx. \]

Now, we have

\[ \int e^{3x} \sin(4x) \, dx = -\frac{1}{4} e^{3x} \cos(4x) + \frac{3}{16} e^{3x} \sin(4x) - \frac{9}{16} \int e^{3x} \sin(4x) \, dx \]

\[ \frac{25}{16} \int e^{3x} \sin(4x) \, dx = -\frac{1}{4} e^{3x} \cos(4x) + \frac{3}{16} e^{3x} \sin(4x) \]

\[ \int e^{3x} \sin(4x) \, dx = \frac{16}{25} \left( -\frac{1}{4} e^{3x} \cos(4x) + \frac{3}{16} e^{3x} \sin(4x) \right) \]

\[ \int e^{3x} \sin(4x) \, dx = -\frac{4}{25} e^{3x} \cos(4x) + \frac{3}{25} e^{3x} \sin(4x) + C. \]
5. Find the area between the graph of

\[ y = \frac{3e^x}{1 + e^{2x}} \]

and the \( x \)-axis between \( x = 0 \) and \( x = e \).

This area is equal to

\[ \int_0^e \frac{3e^x}{1 + e^{2x}} \, dx. \]

To evaluate this, make the substitution \( w = e^x \), \( dw = e^x \, dx \). Then we get

\[
\text{Area} = \int_0^e \frac{3e^x \, dx}{1 + e^{2x}} \\
= \int_1^{e^e} \frac{3 \, dw}{1 + w^2} \\
= \left[ 3 \tan^{-1} w \right]_1^{e^e} \\
= 3 \tan^{-1} (e^e) - 3 \tan^{-1}(1) \\
= 3 \tan^{-1} (e^e) - \frac{3\pi}{4}.
\]
6. A carpet which is 8 meters long is rolled up. When \( x \) meters have been unrolled, a force of \( e^x(64 - x^2) \) Newtons is required to unroll it further. How much work does it take to unroll the entire carpet?

Using the formula

\[
\text{Work} = \int_a^b F(x) \, dx,
\]

We get

\[
\text{Work} = \int_0^8 e^x (64 - x^2) \, dx
= \int_0^8 64 e^x \, dx - \int_0^8 \frac{d}{dx} x^2 e^x \, dx
= [64e^x]_0^8 - \int_0^8 x^2 e^x \, dx.
\]

This integral needs to be evaluated using integration by parts. Let

\[
\begin{align*}
u &= x^2, & \quad dv &= e^x \, dx, & \quad v &= e^x.
\end{align*}
\]

Then we get

\[
\text{Work} = [64e^x]_0^8 - \int_0^8 x^2 e^x \, dx
= [64e^x]_0^8 - \left( \left[ x^2 e^x \right]_0^8 - \int_0^8 2xe^x \, dx \right)
= [64e^x]_0^8 - \left[ x^2 e^x \right]_0^8 + \int_0^8 2xe^x \, dx
\]

This next integral also needs to be done by parts. Let

\[
\begin{align*}
u &= 2x, & \quad du &= 2 \, dx, & \quad dv &= e^x \, dx, & \quad v &= e^x.
\end{align*}
\]

Then we get

\[
\text{Work} = [64e^x]_0^8 - \left[ x^2 e^x \right]_0^8 + \int_0^8 2xe^x \, dx
= [64e^x]_0^8 - \left[ x^2 e^x \right]_0^8 + [2xe^x]_0^8 - \int_0^8 2e^x \, dx
= [64e^x]_0^8 - \left[ x^2 e^x \right]_0^8 + [2xe^x]_0^8 - [2e^x]_0^8
= (64e^8 - 64) - (64e^0 - 0) + (16e^8 - 0) - (2e^8 - 2)
= 14e^8 - 62.
\]

So the work done is \( 14e^8 - 62 \) Joules.
7. Consider the region obtained by rotating the area enclosed by the graphs of \( y = x^2 - 2 \) and \( y = 6 - x^2 \) and \( x = 0 \) about the line \( y = 8 \).

a) Draw a picture of a typical cross-section of the solid at the point \( x = x_i \).

The area we're interested in is the shaded area in the top left diagram below. A cross-section is generated by rotating the line at \( x = x_i \) in the middle diagram around the line \( y = 8 \), as shown in the top right diagram. These cross-sections are rings, as in the bottom figure.

![Diagram of cross-sections](image)

b) What is the area of your cross-section? (Your answer should be in terms of \( x_i \).)

For a ring, we have

\[
\text{Area} = \pi r_o^2 - \pi r_i^2,
\]

where \( r_o \) is the outside radius and \( r_i \) is the inside radius. In this case, we have

\[
r_o = 8 - (x_i^2 - 2) = 10 - x_i^2 \quad \text{and} \quad r_i = 8 - (6 - x_i^2) = 2 + x_i^2.
\]

Hence, the area of the ring is

\[
\text{Area} = \pi (10 - x_i^2)^2 - \pi (2 + x_i^2)^2.
\]
c) Write down a Riemann sum with \( n \) terms which approximates the volume of the solid. Your answer should be in terms of \( \Delta x \) and the \( x_i \).

We know that we have

\[
V \approx \sum_{i=1}^{n} A(x_i) \Delta x,
\]

where \( A(x_i) \) is the cross-sectional area at the point \( x_i \). Thus we get

\[
V \approx \sum_{i=1}^{n} \left( \pi \left( 10 - x_i^2 \right)^2 - \pi \left( 2 + x_i^2 \right)^2 \right) \Delta x.
\]

d) Now write down a Riemann sum only in terms of \( i \) and \( n \) which approximates the volume of the solid. (This time, the expressions \( x_i \) and \( \Delta x \) should not appear in your sum.)

For this part, we need to write the answer to part c) in terms of only \( i \) and \( n \). First, note that we have

\[
\Delta x = \frac{b - a}{n} = \frac{2 - 0}{n} = \frac{2}{n}
\]

(where you can find that the graphs intersect at \( x = 2 \) by setting the functions equal to each other). Also, note that we then have

\[
x_0 = 0, \quad x_1 = 2/n, \quad x_2 = 4/n, \quad x_3 = 6/n, \quad \ldots
\]

and that following the pattern we have \( x_i = 2i/n \) in general. Thus our sum is

\[
V \approx \sum_{i=1}^{n} \left( \pi \left( 10 - \frac{4i^2}{n^2} \right)^2 - \pi \left( 2 + \frac{4i^2}{n^2} \right)^2 \right) \frac{2}{n}.
\]
e) Write down an integral representing the exact volume of the solid. You \textbf{do not} need to evaluate the volume.

To do this, we simply have to convert the answer from part c) into an integral. This gives us

\[ V = \int_0^2 \pi(10 - x^2)^2 - \pi(2 + x^2)^2 \, dx. \]
8. Let $f(t)$ be the function graphed below. Between $t = 0$ and $t = 4$, the graph of $f(t)$ is the same as the graph of $y = t - 4$. The function has a global maximum at the point $(5, 2)$, and the area of the shaded region is 2. If $t < 0$ or $t > 6$, then $f(t)$ is undefined.

\[
\begin{align*}
(5, 2) \\
\end{align*}
\]

\[
\begin{align*}
y = f(t) \\
\end{align*}
\]

\[
\begin{align*}
a) \text{ Evaluate } \int_{1}^{6} f(t) \, dt. \\
\text{The integral can be broken up into two areas - the shaded area and a triangle.} \\
\text{We know that the shaded area is equal to 2. For the triangle, we note that} \\
\text{the height is 3 and the base is also 3. Hence the area is } \frac{9}{2}. \text{ Since this area is} \\
\text{below the } x\text{-axis, we count it negatively when evaluating the integral. Thus} \\
\text{we have} \\
\int_{1}^{6} f(t) \, dt = - \frac{9}{2} + 2 = - \frac{5}{2}.
\end{align*}
\]

\[
\begin{align*}
b) \text{ Consider the volume formed by rotating the graph of } f(t) \text{ around the} \\
t\text{-axis. Let } h(x) \text{ be the volume of the part of the solid between } t = 0 \text{ and} \\
t = 2x. \text{ Write down a formula expressing } h(x) \text{ as an integral.} \\
\text{The cross-sections of the solid are circles with a radius of } |f(t_i)|. \text{ Thus the} \\
\text{volume we want is} \\
\int_{0}^{2x} \pi |f(t)|^2 \, dt = \int_{0}^{2x} \pi f(t)^2 \, dt.
\end{align*}
\]
c) Write down an expression for \( h'(x) \).

We can write \( h(x) = m(n(x)) \), where we have

\[
m(x) = \int_0^x \pi f(t)^2 \, dt \quad \text{and} \quad n(x) = 2x.
\]

Moreover, we know that

\[
m'(x) = \pi f(x)^2 \quad \text{and} \quad n'(x) = 2.
\]

Therefore, we find that

\[
h'(x) = m'(n(x)) \cdot n'(x) = \pi f(2x)^2 \cdot 2 = 2\pi f(2x)^2.
\]

d) Evaluate \( h(4) \) (not \( h'(4) \)).

This problem is hard to interpret because we need to evaluate the volume of the solid formed by rotating the graph of \( f(t) \) between \( t = 0 \) and \( t = 8 \) around the \( t \)-axis, but \( f(t) \) is undefined for part of this domain. Everybody received full credit for this part of the problem.
e) Find a number which you know \textbf{for a fact} is \textbf{larger} than $h(6)$. Show how you know that your answer is larger than $h(6)$.

For the same reason as in part d), it is hard to make sense out of this question. Everybody received full credit for this part of the problem.
9. Suppose that \( f(x) \) is a function with the property that \( f(-x) = -f(x) \) for every value of \( x \). Show that

\[
\int_{-1}^{1} f(x) \, dx = 0.
\]

**Hint:** Make a substitution.

To do this, we make the substitution \( u = -x \), \( du = -dx \). Then we get

\[
\int_{-1}^{1} f(x) \, dx = - \int_{1}^{-1} f(-u) \, du
\]

\[
= \int_{-1}^{1} f(-u) \, du
\]

\[
= \int_{-1}^{1} -f(u) \, du
\]

\[
= - \int_{-1}^{1} f(u) \, du
\]

\[
= - \int_{-1}^{1} f(x) \, dx.
\]

Then we have

\[
\int_{-1}^{1} f(x) \, dx = - \int_{-1}^{1} f(x) \, dx
\]

\[
2 \int_{-1}^{1} f(x) \, dx = 0
\]

\[
\int_{-1}^{1} f(x) \, dx = 0.
\]