Statement of Research Interests

Michael P. Knapp

My research lies in the broad field of number theory, in the area of Diophantine equations. Specifically, I study conditions under which systems of homogeneous polynomials have nontrivial p-adic integral zeros. In August 2002, I was awarded a three-year research grant by the National Science Foundation to support this research. Additionally, I was awarded a summer research grant by Loyola College to support my research in 2005. A much more detailed account of this research than is presented here is contained in my NSF grant proposal, which can be found on my home page on the internet at the URL http://www.evergreen.loyola.edu/˜mpknapp/professional/.

The majority of my work focuses on a conjecture usually attributed to Emil Artin. This conjecture stated that for every prime number p, any system of R homogeneous polynomials (also called forms) of degrees $d_1, \ldots, d_R$ with p-adic integral coefficients has a nontrivial $\mathbb{Q}_p$-integral zero provided only that the number of variables at least $d_1^2 + \cdots + d_R^2 + 1$. This can be thought of as a generalization of linear algebra in local fields, since setting $d_1 = \cdots = d_R = 1$ yields the familiar theorem that a system of R homogeneous linear equations in at least $R + 1$ variables has a nontrivial solution.

Although this conjecture has been proven false even in the case of only one form, there are a few ways in which the conjecture is "almost" correct. For example, although the Artin bound of $d_1^2 + \cdots + d_R^2 + 1$ is incorrect, it can be shown that there does exist some function $g(d_1, \ldots, d_R)$ of the degrees such that a nontrivial $\mathbb{Q}_p$-integral zero exists for each p provided only that there are at least $g(d_1, \ldots, d_R)$ variables. This leads naturally to the problem of evaluating (or at least bounding) this function. Further, this problem can be extended to the situation where the underlying field is an algebraic extension of $\mathbb{Q}_p$ rather than $\mathbb{Q}_p$ itself, or to other fields entirely.

While a list of my publications may be found in my curriculum vitae, I would now like to describe some of my more recent results. The first of these are in my paper "Pairs of homogeneous additive equations." The first result in this paper is not explicitly about Artin's conjecture but has great independent interest. We give conditions under which a system of two homogeneous additive polynomials (i.e. having no cross-terms) is guaranteed to have a nonsingular zero over a finite field. In 1966, H. Davenport & D. Lewis proved a theorem of this sort for finite fields with a prime number of elements. In their paper, they asked whether their result would hold for any finite field, and gave an example showing that one of their key lemmas was false in the general case. My paper shows that their result indeed extends directly to any finite field.

The other theorem in this paper applies this result about finite fields to obtain a result about Artin's conjecture in extensions of $\mathbb{Q}_p$. Specifically, if L is any (possibly infinite) algebraic extension of $\mathbb{Q}_p$, then Artin's conjecture is true for systems of two additive forms of equal degree defined over L, provided only that p does not divide the degree. While this condition is somewhat disappointing, it should be noted that even over $\mathbb{Q}_p$, Artin's conjecture has not been proven to be true in this situation when p divides the degree. One must also realize that extensions of $\mathbb{Q}_p$ are even more difficult to handle than $\mathbb{Q}_p$ itself due to the possibility of ramification.

My most recent result concerns zeros of systems of additive forms over $\mathbb{Q}_p$, where the forms may have different degrees. As usual, one desires to find a number of variables (in terms of the degrees) which guarantees that the system possesses a nontrivial zero. A result from 1983 due to D. Lewis & H. Montgomery implies that any such bound that holds for any number of forms of any degrees and over any field $\mathbb{Q}_p$ must exhibit exponential growth. Recently, I have been able to show that if the prime p is restricted to be mildly large, in particular if

$$p > (\text{largest degree}) - (\text{smallest degree}) + 1,$$

then only a polynomial bound is required. Hence this result shows that exponential growth is only required for relatively small primes and also bounds the size of these primes.

There are many problems in this field which I would like to work on in the future. For example, I am just beginning a project to study a variant of Artin's conjecture in which the underlying field, rather than $\mathbb{Q}_p$, is a function field over a finite field. Working on this problem would involve studying a version of the Hardy-Littlewood circle method which works over such fields. As another project, I would like to extend my work with two additive forms described above to systems of three forms, both when the underlying field is $\mathbb{Q}_p$ and when it is an extension of $\mathbb{Q}_p$. In addition to being interesting in its own right, I hope that this work will give me insight about how to deal with the general situation. I have done some preliminary work on this topic, and have found that there are some very interesting issues with three forms which do not arise when there are only two forms. These are only two of the many open problems in this field. I hope to make significant progress on these and many of the others during the coming years.