

UNIPOTENT CLASSES IN THE CLASSICAL GROUPS PARAMETERIZED BY SUBGROUPS

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ABSTRACT. This paper describes how to use subgroups to parameterize unipotent classes in the classical algebraic group in characteristic 2. These results can be viewed as an extension of the Bala-Carter Theorem, and give a convenient way to compare unipotent classes in a group G with unipotent classes of a subgroup X where G is exceptional and X is a Levi subgroup of classical type.

AMS subject: 14L35, 20G15

Keywords: Algebraic groups, unipotent classes, Bala-Carter
January 25, 2008

1. INTRODUCTION AND STATEMENT OF RESULTS

The unipotent classes in a classical group are often described using Jordan blocks, but there are problems with this approach. For instance, the group SO_{2n} , in all characteristics, and the groups O_n and Sp_n in characteristic two, have distinct unipotent classes with the same Jordan blocks sizes. Furthermore, Jordan blocks are defined using the natural module and are, therefore, not intrinsic to the group. For example, suppose that G is an exceptional algebraic group, and $X \leq G$ is a Levi subgroup of classical type. If we describe the unipotent classes in X using Jordan blocks, then this description does not make it easy to translate the list of X -classes into unipotent classes in G .

In good characteristic the Bala-Carter Theorem avoids the problems just described by describing the unipotent classes in all simple algebraic groups using (pairs of) subgroups. However, Bala-Carter does not hold in bad characteristic.

The main goal of this paper is to extend the results of Bala-Carter for groups of type B_n , C_n and D_n in characteristic 2 and to use this to translate the list of X -classes into unipotent classes in G .

For the rest of the paper let k be an algebraically closed field of characteristic $p \geq 0$. All the groups discussed here will be linear algebraic groups over k . Each of the groups O_n , SO_n , and Sp_n has a natural module of dimension n which possesses a bilinear form and a quadratic form. A subspace is **nonsingular** if it has trivial radical with respect to the bilinear form, and it is **nongenerate** if 0 is the only element of the radical which maps to 0 under the quadratic form. A subspace is **totally singular** if the restrictions of the quadratic and bilinear forms to the subspace are both identically zero. We use the following standard conventions to distinguish between certain isomorphic but non-conjugate subgroups of SO_{2n} . A subgroup of SO_{2n} denoted by GL_m induces the full general linear group on a pair of disjoint totally singular m -spaces which are in duality via the bilinear form.

A subgroup of SO_{2n} denoted by SO_m induces the special orthogonal group on a non-degenerate m -space.

A subgroup of GL_n , O_n , or Sp_n , denoted by Cl_m , equals GL_m , O_m , or Sp_m , respectively. We denote by Cl_m° the identity component of Cl_m . Recall that GL_n and Sp_n are connected. For the orthogonal groups we have $\mathrm{O}_m^\circ = \mathrm{SO}_m$. For $m \geq 2$, the group O_m equals O_m° if and only if m is odd and $p = 2$. The notation O_1 means the trivial group; we view it as acting on a 1-dimensional vector space.

Let G be one of GL_n , SO_n , or Sp_n . Let $\mathcal{R}(G)$ (“ \mathcal{R} ” is chosen to stand for “regular”) be the set of closed subgroups $X \leq \mathrm{Cl}_n$ where X is a direct product of the following form:

- (i) If $G = \mathrm{GL}_n$ then $X = \mathrm{GL}_{n_1} \cdots \mathrm{GL}_{n_s}$ with $n_1 + \cdots + n_s = n$,
- (ii) If $G = \mathrm{SO}_n$ and $p \neq 2$ or $G = \mathrm{Sp}_n$ then $X = \mathrm{GL}_{n_1} \cdots \mathrm{GL}_{n_s} \mathrm{Cl}_{m_1}^\circ \cdots \mathrm{Cl}_{m_r}^\circ$ with $2n_1 + \cdots + 2n_s + m_1 + \cdots + m_r = n$,
- (iii) If $G = \mathrm{SO}_n$ and $p = 2$ then $X = \mathrm{GL}_{n_1} \cdots \mathrm{GL}_{n_s} \mathrm{Cl}_{m_1} \cdots \mathrm{Cl}_{m_r}$ with $2n_1 + \cdots + 2n_s + m_1 + \cdots + m_r = n$; if n is even then r even.

If $X \in \mathcal{R}(G)$ we write $X = X_1 X_2 \cdots$ where the X_i are the factors of X given in the definition of $\mathcal{R}(G)$. The factors X_i and the sequences (n_1, \dots, n_s) and (m_1, \dots, m_r) are uniquely determined (up to permutation) by X and the definitions.

A unipotent element in a connected reductive group G is **regular** if the dimension of its centralizer equals the rank of G . A connected reductive group has a single class of regular unipotent elements [11]. When $p = 2$ and n is even we also consider the group O_n to have a unique regular unipotent class in the non-identity component: an element in this class also has dimension of its centralizer equal to the rank. (See Table 1 for the Jordan blocks of these classes; see [10, I.4.8] for more information about regular classes in non-connected reductive groups; see [3] for dimension of centralizer formulas).

Theorem 1.1. *Let G be one of GL_n , SO_n , or Sp_n . Let G act on $\mathcal{R}(G)$ via conjugation. The following map is surjective*

$$\begin{aligned} \Psi_1 : \{G\text{-classes in } \mathcal{R}(G)\} &\longrightarrow \{\text{unipotent } G\text{-classes}\} \\ X = X_1 X_2 \cdots &\longmapsto \text{conjugacy class of } u_1 u_2 \cdots \end{aligned}$$

where each u_i is a regular unipotent element in X_i , in the non-identity component when possible.

Define a right inverse Φ_1 of Ψ_1 as follows. Given a unipotent G -class C , let $\Phi_1(C)$ equal the unique G -class in $\Psi_1^{-1}(C)$ which has a maximal number of factors of the form GL_{n_i} . The image of Φ_1 equals all the G -classes in $\mathcal{R}(G)$ which satisfy the following conditions:

- (i) if $G = \mathrm{GL}_n$, then all of $\mathcal{R}(G)$ is in the image,
- (ii) if $G = \mathrm{SO}_n$ and $p \neq 2$, then the sequence (m_1, \dots, m_r) has distinct, odd parts,
- (iii) if $G = \mathrm{Sp}_n$ and $p \neq 2$, then the sequence (m_1, \dots, m_r) has distinct even parts,
- (iv) if $p = 2$ and G equals SO_n or Sp_n , then at most one part of the sequence (m_1, \dots, m_r) equals 1, and the rest of the parts are even with multiplicity at most 2.

Remarks 1.2. (a) If n is even, $G = \mathrm{SO}_n$, and $p = 2$, then the groups in $\mathcal{R}(G)$ are not always subgroups of G . For the statement of Theorem 1.1, this is unavoidable.

For example, when $p = 2$, there is a distinguished unipotent class in SO_{16} with Jordan block sizes given by 6, 4, 4, 2 and this class cannot be represented by regular elements in a subgroup with factors of the form GL_{n_i} and SO_{m_i} (c.f. Remark 1.4 and Example 5.2).

(b) Remark 2.1 shows that when n is even, $G = \mathrm{SO}_n$ and $p = 2$, we have that $\Psi_1(X)$ is in SO_n and not just in O_n .

The next result is similar to Theorem 1.1, except that we use Richardson classes of all distinguished parabolic subgroups, instead of using only regular classes. Carter [1] provides a list of distinguished parabolic subgroups for good characteristics, and we use his list even in bad characteristics (note that his second formula for D_n has a slight mistake).

Let $D(G)$ (“ D ” standing for “distinguished”) be the set of closed connected subgroups P of G such that $P \leq X$ for some $X \in \mathcal{R}(G)$ (using notation as in the definition of $\mathcal{R}(G)$) with the following changes: (1) if $G = \mathrm{SO}_n$ with n even and $p = 2$ we do not require that r be even; (2) we require $r \leq 3$ and that if we factor $X = X_1 X_2 \cdots$ then P equals a direct product $P = P_1 P_2 \cdots$ where for each i we have that P_i is a distinguished parabolic subgroup of X_i° . The factors P_i are uniquely determined (up to permutation) by P .

In a connected reductive group, each parabolic subgroup has a unique dense orbit in its unipotent radical [7] which we call the **Richardson class**.

Theorem 1.3. *Let G be one of GL_n , SO_n , or Sp_n . Let G act on $D(G)$ via conjugation. The following map is surjective,*

$$\begin{aligned} \Psi_2 : \{G\text{-classes in } D(G)\} &\longrightarrow \{\text{unipotent } G\text{-classes}\} \\ P = P_1 P_2 \cdots &\longmapsto \text{conjugacy class of } u = u_1 u_2 \cdots \end{aligned}$$

where P is described above and each u_i represents the Richardson class of P_i .

Remarks 1.4. (a) The restriction of Ψ_2 to the subset of $D(G)$ consisting of those P with $r \leq 1$ is injective and may be identified with the map in the Bala–Carter Theorem from (pairs of) subgroups to unipotent classes (see [1] and [2]).

(b) The distinguished unipotent class in SO_{16} mentioned in Remark 1.2 does not equal the Richardson class of any parabolic subgroup. This indicates the need in Theorem 1.3 for more than one parabolic factor in P .

(c) When we define below the right inverse Φ_2 of the map Ψ_2 we refer to partitions consisting of Jordan blocks sizes. In principle, one can avoid mentioning Jordan blocks and still describe a subset of the domain of Ψ_2 , upon which Ψ_2 is injective, thus implicitly describing an inverse of Ψ_2 . But the description so obtained seems less natural than the presentation of Φ_2 given below.

A partition of n is a sequence of natural numbers which add to n . We write a partition β as $(\beta_1, \beta_2, \dots)$ and assume that $\beta_i \geq \beta_{i+1}$ for all i unless otherwise indicated. We call each β_i a part of β (however we sometimes have to keep track of the index i in addition to the value of β_i , see below). Let α and β be partitions of m and n respectively. We define $\alpha \oplus \beta$ to be a partition of $m + n$ obtained by taking the union of the parts, counting multiplicity, of α and β .

Definition 1.5. Let G equal SO_n or Sp_n and let β be a partition. We will define a decomposition $\beta = \beta^{(1)} \oplus \beta^{(2)} \oplus \beta^{(3)}$. If $p \neq 2$, we set $\beta^{(2)} = \beta^{(3)} = 0$. If $p = 2$ and $G = \mathrm{Sp}_n$ then we set $\beta^{(3)} = 0$ and we define $\beta^{(2)}$ by the requirement that all

parts of $\beta^{(2)}$ be distinct and that a part of β equal a part of $\beta^{(2)}$ if and only if the part has multiplicity greater than 1 in β .

If $p = 2$ and $G = \mathrm{SO}_n$ we first define a map f from parts of β to 0 or 1. We allow the abusive notation that $\beta_i = \beta_{i+1}$ but that $f(\beta_i) \neq (\beta_{i+1})$; in these cases the subscript in β_i is implicitly part of the definition of f .

Define $f(\beta_1) = 1$. Let j be given such that $f(\beta_1), \dots, f(\beta_{j-1})$ have been defined, let β_k be the last of these parts which maps to 1, let ℓ and i be the number of parts in $\beta_1, \dots, \beta_{j-1}$ which map to 1 and to 0 respectively. Define $f(\beta_j)$ as follows (where we allow β_{j+1}, β_{j+2} , etc. to equal 0):

$$f(\beta_j) = \begin{cases} 0 & \text{if } \ell \text{ is even and } \beta_k - \beta_j \leq 2 & (1) \\ 0 & \text{if } \ell \text{ is even, } i \text{ is odd, } \beta_{j+1} \in \{0, 1\} & (2) \\ 0 & \text{if } \ell \text{ is even, } i \text{ is odd, } \beta_{j+1} - \beta_{j+3} \leq 2, \beta_{j+3} \neq 0, \beta_j - \beta_{j+3} \geq 3 & (3) \\ 1 & \text{in all other cases.} \end{cases}$$

(The result $f(\beta_j) = 1$ is meant to be the generic case, with conditions (1), (2) and (3) viewed as exceptions; see Example 5.1.)

Finally, when $p = 2$ and $G = \mathrm{SO}_n$ we apply f to β , let $\beta^{(1)}$ and δ equal the pre-image of 1 and 0 respectively. Apply f to δ , let $\beta^{(2)}$ and $\beta^{(3)}$ be the pre-image of 1 and 0 respectively.

In the following theorem the notation $|\beta^{(i)}|$ denotes the sum of the parts of $\beta^{(i)}$.

Theorem 1.6. *If $G = \mathrm{GL}_n$ then Ψ_2 is bijective. Otherwise, we define a right inverse Φ_2 of Ψ_2 as follows. Let $u \in G$ be unipotent, let $L = \mathrm{GL}_{n_1} \cdots \mathrm{GL}_{n_s} \mathrm{Cl}_m^\circ$ be a minimal Levi subgroup containing u and factor $u = u_1 \cdots u_s u_0$ with $u_i \in \mathrm{GL}_{n_i}$ for $1 \leq i \leq s$ and $u_0 \in \mathrm{Cl}_m^\circ$. Let β be the Jordan blocks of u_0 in the natural module for Cl_m° and write $\beta = \beta^{(1)} \oplus \beta^{(2)} \oplus \beta^{(3)}$ as in Definition 1.5. Then u is contained in a subgroup of L of the form $X = X_1 \cdots X_{s+3}$ with $X_i = \mathrm{GL}_{n_i}$ for $1 \leq i \leq s$ and $X_{s+i} = \mathrm{Cl}_{|\beta^{(i)}|}^\circ$ for $1 \leq i \leq 3$. We factor u_0 further such that $u = u_1 \cdots u_{s+3}$ where $u_i \in X_i$ for each i . For $1 \leq i \leq s+3$ there exists a distinguished parabolic subgroup P_i of X_i , unique up to conjugacy, whose Richardson class is represented by u_i . We define $\Phi_2(u)$ to equal the G -orbit of $P = P_1 \cdots P_{s+3}$.*

When $p \neq 2$, Theorems 1.3 and 1.6 are equivalent to the Bala–Carter Theorem for the classical groups. By an **extra class** we mean one which is not parameterized by Bala–Carter. In the notation of Theorem 1.6 a class is extra if and only if $\beta \neq \beta^{(1)}$.

Corollary 1.7. *Two unipotent classes in SO_n are conjugate under O_n , but not under SO_n , if and only if these classes correspond under the map of Theorem 1.1 to a pair of Levi subgroups in $\mathcal{R}(G)$ (or, under the map in Theorem 1.6, to a pair of Borel subgroups in these Levi factors) which are also conjugate under O_n but not SO_n .*

We note that Corollary 1.7 is well known for $p \neq 2$ as it follows from the usual Bala–Carter Theorem.

In Section 3 we give an explicit, combinatorial formula for the Jordan blocks of $\Psi_1(X)$ or $\Psi_2(P)$. This formula is used to determine the parabolics P_i in Theorem 1.6. Proposition 4.5 establishes certain canonical properties possessed by the decomposition $\beta^{(1)} \oplus \beta^{(2)} \oplus \beta^{(3)}$.

2. JORDAN BLOCK PARAMETERIZATION OF UNIPOTENT CLASSES

In this section we recall one method of parameterizing unipotent classes in G , following [10, I.2.5ff] (though we extend the method there to include the odd-dimensional orthogonal case in characteristic 2).

For the moment we fix G equal to one of O_n or Sp_n defined over an algebraically closed field k of characteristic $p \geq 0$ and we denote by B the associated bilinear form. Calculations involving $u - 1$ are made by viewing the natural module for G as a $k[u]$ -module.

Let δ equal 1 if $G = Sp_n$ or if $G = O_n$ and $p = 2$. Let δ equal -1 otherwise.

Let $u \in G$ be unipotent and let λ equal the Jordan block sizes of u . For each part x of λ let $\varepsilon_u(x)$ be defined as follows: if x is odd let $\varepsilon_u(x) = -\delta$, if x is even then $\varepsilon_u(x) \in \{0, \delta\}$ with $\varepsilon_u(x) = 0$ if and only if $p = 2$ and $B((u - 1)^{x-1}v, v) = 0$ for all $v \in \ker(u - 1)^x$. Usually, if the element u has been fixed or is irrelevant, we will write ε instead of ε_u . This gives a map Υ from unipotent classes in G to pairs (λ, ε) . The map Υ is injective.

Let λ be a partition of n and ε a map from the set $\{\lambda_i \mid i \geq 1\}$, to the set $\{-1, 0, 1\}$. Then (λ, ε) is in the image of Υ if and only if two of the following are satisfied:

- (i) $G = O_n$, $p \neq 2$, every even part of λ has even multiplicity,
- (ii) $G = O_n$, $p = 2$ or $G = Sp_n$, every odd part of λ strictly greater than 1 has even multiplicity,
- (iii) $p \neq 2$, $\varepsilon(x) = -\delta$ if x is odd and $\varepsilon(x) = \delta$ if x is even,
- (iv) $p = 2$, $\varepsilon(x) = -1$ if x is odd, $\varepsilon(x) = 1$ if x is even with odd multiplicity and $\varepsilon(x) \in \{0, 1\}$ if x is even with even multiplicity.

This completes the parameterization of unipotent classes in O_n and Sp_n . Now we relate the unipotent classes of O_n to the classes in SO_n .

If $p \neq 2$ or if n is odd, then every unipotent class in O_n is contained in SO_n .

If $p = 2$ and n is even, then a unipotent element in O_n is contained in SO_n if and only if it has an even number of Jordan blocks.

We apply the definition of ε to SO_n without change. Let $u \in SO_n$ be unipotent, let C be the O_n -class of u , and let λ be the Jordan block sizes of u . Then C equals a single SO_n -class, unless each part x of λ is even and $\varepsilon(x) \neq 1$, in which case C forms two SO_n -classes.

Remarks 2.1. We pause to clarify one aspect of the map Ψ_1 from Theorem 1.1. If $G = SO_n$ with n even and $p = 2$ then some elements of $\mathcal{R}(G)$ are subgroups of O_n but not subgroups of G . However the regular unipotent class which we have chosen in such subgroups is contained in SO_n , so the map is properly defined.

3. JORDAN BLOCKS OF RICHARDSON CLASSES OF DISTINGUISHED PARABOLICS.

Let u be a regular unipotent element in G . In Table 1 we describe the possible Jordan blocks of u .

Let G equal SO_n or Sp_n , let m be the rank of G , and let P be a parabolic subgroup of G . Let L be a Levi factor for P and write $L = GL_{n_1} \cdots GL_{n_s} H_{m_0}$ where H_{m_0} is one of SO_{2m_0} , SO_{2m_0+1} , or Sp_{2m_0} (we allow $m_0 = 0$ and $H_{m_0} = 1$). In this manner P determines a partition of m given by $m = n_1 + \cdots + n_s + m_0$. We write this partition as $(1^{c(1)}, 2^{c(2)}, \dots, N^{c(N)}) \oplus m_0$ where $N = \max\{n_i \mid 1 \leq i \leq s\}$ and $c(x)$ is the multiplicity of the part x in the sequence n_1, \dots, n_s . We assume

TABLE 1. Jordan blocks of regular unipotent elements

Jordan blocks	Groups and conditions
n	$\mathrm{GL}_n, \mathrm{Sp}_n,$ SO_n with n odd and $p \neq 2,$ O_n with u in the non-identity component, n even and $p = 2$
$n - 1, 1$	SO_n with n odd and $p = 2$ or n even and $p \neq 2$
$n - 2, 2$	SO_n with n even, $n \geq 4, p = 2$ (and $\varepsilon(2) = 1$ if $n = 4$)
$1, 1$	SO_2 with $p = 2$

now that P is distinguished. Then we have $c(i) \geq 1$ if and only if $1 \leq i \leq N$.
 ??? ref ??? Let λ be the partition of n whose parts equal the Jordan block sizes of the Richardson class of P . Recall that the dual of λ is the partition λ^* of n where λ_i^* equals the number of j such that $\lambda_j \geq i$. Recall also that $\lambda = (\lambda^*)^*$. In Table 2 we describe λ in terms of its dual. In this table we have written parts of λ^* which may have multiplicity 0; for example, when $G = \mathrm{SO}_{2m+1}$ and $p \neq 2$ we can have $c(2m_0 + 1) = 0$. Thus Table 2 implicitly defines a map from G -classes of distinguished parabolics to partitions of n . This map is injective and its image is described in Table 3 (see [2]).

Table 2 allows one to calculate the Jordan blocks of $\Psi_2(P)$ where P and Ψ_2 are as in Theorem 1.6. Table 3 will allow us verify, in Section 4, the assertion in Theorem 1.6 that there exists a distinguished parabolic subgroup P_i of X_i , unique up to conjugacy, whose Richardson class is represented by u_i .

4. MAIN PROOFS

Let G be connected and reductive and let L be a Levi subgroup of G (we allow $L = G$). A unipotent element $u \in L$ is **distinguished** (in L) if u is not contained in any proper Levi subgroup of L . If we omit mention of L then we assume $L = G$. Then u is distinguished if and only if each maximal torus of $C_G(u)$ is contained in $Z(G)$ (see Lemma 4.1). If $Z(G) = 1$ this is equivalent to requiring that $C_G(u)$ have no nontrivial torus (this is the usual definition) which is also equivalent to requiring that $C_G(u)^0$ be a unipotent group.

For many questions, the following lemma reduces the study of unipotent classes in G to the study of distinguished classes.

Lemma 4.1 ([1, 5.9.2, 5.9.3]). (i) *Let S be a torus. Then $L = C_G(S)$ is a Levi subgroup.*

(ii) *If u is a unipotent element and S a maximal torus of $C_G(u)$ then u is distinguished in $L = C_G(S)$. Furthermore, any Levi subgroup in which u is distinguished is conjugate to L via an element of $C_G(u)^\circ$.*

Corollary 4.2. *Define a map from G -classes of pairs (L, C) consisting of a Levi subgroup, L , of G and a distinguished unipotent L -class, C , to unipotent G -classes by extending C . This map gives a bijection.*

TABLE 2. Jordan blocks of the Richardson class of a distinguished parabolic.

G	λ
GL_m	$m_0 = 0 \quad (1^{c(1)}, 2^{c(2)}, \dots)^* = (m) = (n)$
SO_{2m}	$m_0 = 0, p \neq 2: \left(1^{2c(1)-2}, 2^{2c(2)+1}\right)^*$ $m_0 = 0, p = 2: \left(1^{2c(1)-4}, 2^{2c(2)+2}\right)^*$ $m_0 > 0, p \neq 2: \left(1^{2c(1)}, \dots, (2m_0 - 1)^{2c(2m_0-1)}, (2m_0)^{c(2m_0)+1}\right)^*$ $m_0 > 0, p = 2: \left(1^{2c(1)-2}, 2^{2c(2)+2}, 3^{2c(3)-2}, 4^{2c(4)+2}, \dots, \right.$ $\left. (2m_0 - 1)^{2c(2m_0-1)-2}, (2m_0)^{2c(2m_0)+2}\right)^*$
SO_{2m+1}	$p \neq 2: \left(1^{2c(1)}, 2^{2c(2)}, \dots, (2m_0)^{2c(2m_0)}, (2m_0 + 1)^{2c(2m_0+1)+1}\right)^*$ $p = 2: \left(1^{2c(1)-2}, 2^{2c(2)+2}, 3^{2c(3)-2}, 4^{2c(4)+2}, \dots, \right.$ $\left. (2m_0)^{2c(2m_0)+2}, (2m_0+1)^{2c(2m_0+1)}\right)^* \oplus (1)$
Sp_{2m}	$m_0 = 0: \left(1^{2c(1)}, 2^{2c(2)}, \dots, N^{2c(N)}\right)^*$

TABLE 3. Partitions which equal Jordan blocks of the Richardson class of a distinguished parabolic

G	Set of partitions λ ($\varepsilon(x) = 1$ for all parts x unless otherwise noted)
$G = \mathrm{GL}_m$	All partitions of m
$G = \mathrm{SO}_{2m+1}, p \neq 2$	Partitions of $2m + 1$ consisting of distinct odd parts
$G = \mathrm{SO}_{2m+1}, p = 2$	Partitions of $2m + 1$ which have exactly one part equal to 1, $\varepsilon(1) = -1$, the rest of the parts are even and of multiplicity at most 2, and, if i is even and $\lambda_{i+1} \geq 1$ then $\lambda_i - \lambda_{i+1} \geq 3$
$G = \mathrm{Sp}_{2m}$	Partitions of $2m$ consisting of distinct even parts
$G = \mathrm{SO}_{2m}, p \neq 2$	Partitions of $2m$ consisting of distinct odd parts
$G = \mathrm{SO}_{2m}, p = 2$	Partitions of $2m$ which have an even number of parts, each part is even and of multiplicity at most 2, and, if i is even and $\lambda_{i+1} \geq 1$ then $\lambda_i - \lambda_{i+1} \geq 3$

In the following lemma the notation $\lambda(u_i)$ for $i \geq 1$ denotes the Jordan block sizes of u_i in the natural module for GL_{n_i} . The notations $\lambda(u)$ and $\lambda(u_0)$ denote the Jordan block size of u and u_0 in the natural module for G and Cl_m° respectively.

Lemma 4.3. *Let G equal SO_n or Sp_n and let $u \in G$ be unipotent. The following hold:*

- (i) *Let $\mathrm{GL}_{n_1} \cdots \mathrm{GL}_{n_s} \mathrm{Cl}_m^\circ$ be a Levi subgroup of G containing u and write $u = u_1 u_2 \cdots u_s u_0$ with $u_i \in \mathrm{GL}_{n_i}$ for $i \geq 1$ and $u_0 \in \mathrm{Cl}_m^\circ$. Then $\lambda(u) = \bigoplus_{i \geq 1} \lambda(u_i)^2 \oplus \lambda(u_0)$ where $\lambda(u_i)^2$ means that each part of $\lambda(u_i)$ has been doubled in multiplicity.*

- (ii) If $p \neq 2$ then u is distinguished if and only if each part of $\lambda(u)$ has multiplicity 1 (whence each part is odd if $G = \mathrm{SO}_n$ and each part is even if $G = \mathrm{Sp}_n$).
- (iii) If $p = 2$ then u is distinguished if and only if at most one part of $\lambda(u)$ equals 1, and each remaining part x has multiplicity at most 2 and $\varepsilon(x) = 1$ (whence x is even).
- (iv) Let V be the natural module for G , let x be a Jordan block of u and suppose that u stabilizes a decomposition $V = V_1 \perp V_2$. Let $u_1 = u|_{V_1}$ and $u_2 = u|_{V_2}$. Then $\varepsilon_u(x) = 1$ if and only if $\varepsilon_{u_1}(x) = 1$ or $\varepsilon_{u_2}(x) = 1$.
- (v) Let n_1 be the rank of G , let GL_{n_1} be a Levi factor of G , and let u be distinguished in GL_{n_1} . Suppose that n_1 is even and that $p = 2$. Then $\varepsilon(n_1) = 0$.

Proof. Parts (i), (ii) and (iii) are in [10, II.7.10].

Part (iv) (sketch). The crucial case is where x is even and $p = 2$, which we now assume. Writing any $v \in V$ as $v = v_1 + v_2$ with $v_i \in V_i$, it is easy to show that

$$\begin{aligned} B((u-1)^{x-1}v, v) &= 0 \quad \forall v \in \ker(u-1)^x \\ \iff B((u_i-1)^{x-1}v_i, v_i) &= 0 \quad \forall v_i \in \ker(u_i-1)^x \text{ for } i = 1, 2. \end{aligned}$$

The result now follows from the definition of ε (c.f. Section 2).

Part (iv). From part (i) we know that the multiplicity of n_1 is 2. Since u is *not* distinguished in G we have by part (iii) that $\varepsilon(n_1) \neq 1$. \square

The following result is essentially equivalent to Corollary 1.7.

Corollary 4.4. *Let n be even, let u_1 and u_2 be unipotent elements in $G = \mathrm{SO}_n$ and for $i = 1, 2$ let L_i be a minimal Levi subgroup of G containing u_i . Then u_1 and u_2 are conjugate under O_n but not SO_n if and only if L_1 and L_2 are conjugate under O_n but not SO_n .*

Proof. Recall that an O_n -class of Levi subgroups splits into two SO_n -classes if and only if the class is represented by $\mathrm{GL}_{n_1} \cdots \mathrm{GL}_{n_s}$, with each n_i even. (This can be shown by viewing each Levi subgroup as a stabilizer of subspaces and then using standard arguments about the geometry of classical groups and Witt's Theorem, or by more abstract arguments about conjugacy of subgroups and root systems of algebraic groups.) Recall from Section 2, that an unipotent O_n -class splits into two SO_n -classes if and only if each Jordan block size x is even and satisfies $\varepsilon(x) \neq 1$.

Suppose that L_1 and L_2 are conjugate under O_n but not under SO_n . Then $L_1 \cong L_2 \cong \mathrm{GL}_{n_1} \cdots \mathrm{GL}_{n_s}$ with each n_i even. Then $\lambda(u_1) = \lambda(u_2) = (n_1^2, n_2^2, \dots, n_s^2)$ and, by Lemma 4.3 (iv) and (v), we have $\varepsilon(n_i) \neq 1$ for each i . Therefore u_1 and u_2 are conjugate under O_n but not SO_n .

Suppose that u_1 and u_2 are conjugate under O_n but not under SO_n . By Corollary 4.1(ii) we have that L_1 and L_2 are conjugate under O_n .

Now we claim that L_1 and L_2 cannot have any factor of the form SO_{2m} . Otherwise u_1 and u_2 could each be written as a product with one factor distinguished in SO_{2m} . By Lemma 4.3 and Table 3 this would give rise to at least one part x of $\lambda(u_1)$ and $\lambda(u_2)$ with $\varepsilon(x) = 1$. But then u_1 and u_2 would be conjugate under SO_{2n} .

Now $\lambda(u_1)$ and $\lambda(u_2)$ are equal (since u_1 and u_2 are conjugate under O_n) with parts (n_1^2, \dots, n_s^2) where $L_1 \cong L_2 \cong \mathrm{GL}_{n_1} \cdots \mathrm{GL}_{n_s}$, where all n_i are even (since u_1

and u_2 are not conjugate under SO_n). This implies that L_1 is not conjugate to L_2 under SO_n . \square

Proof of Theorems 1.1, 1.3, and 1.6. There is essentially nothing to show for the case $G = \mathrm{GL}_n$, so we assume now that G equals one of SO_n or Sp_n .

Let C be a unipotent G -class, $u \in C$ a unipotent element, let $\mathrm{GL}_{n_1} \cdots \mathrm{GL}_{n_s} H_m$ be a minimal Levi subgroup containing u with H equal to SO_{2m} , SO_{2m+1} or Sp_{2m} as appropriate. We factor u as $u = u_1 \cdots u_s u_0$ with $u_i \in \mathrm{GL}_{n_i}$ for $1 \leq i \leq s$ and $u_0 \in \mathrm{Cl}_m^\circ$. Note that L has the maximal number of factors of the form GL_{n_i} among elements of $\mathcal{R}(G)$ which contain u , as required for Φ_1 ; that u_i is regular in GL_{n_i} , for $i \geq 1$; and that u_0 is distinguished in Cl_m° . Let α be the Jordan blocks of $u_1 \cdots u_s$ in the natural module for $\mathrm{GL}_{n_1} \cdots \mathrm{GL}_{n_s}$ and let β be the Jordan blocks of u_0 in the natural module for Cl_m° . Then the parts α equals (n_1, \dots, n_s) , β satisfies the properties described in Lemma 4.3 parts (ii) and (iii), and, if $G = \mathrm{SO}_n$, $p = 2$ and n is even, then β has an even number of parts. These observations about β , combined with Lemma 4.3, (and with the remaining part of this proof) also establish the assertions in Theorem 1.1 about the image of Φ_1 .

Note that $u = u_1 \cdots u_s$ is equivalent to $\beta = 0$ and $u = u_0$ is equivalent to $\alpha = 0$. Applying Lemma 4.3 it suffices to construct Φ_1 and Φ_2 under the assumption that $\alpha = 0$ or $\beta = 0$. In other words, if Φ_1 and Φ_2 have been so constructed, then the class represented by $\Phi_i(u)$ equals the class represented by $\Phi_i(u_1 \cdots u_s)\Phi_i(u_0)$.

If $\beta = 0$ we define $\Phi_i(u) = \mathrm{GL}_{n_1} \cdots \mathrm{GL}_{n_s}$.

Suppose $\alpha = 0$. For $p \neq 2$ or for $G = \mathrm{Sp}_{2n}$ we define $\Phi_1(u) = \mathrm{Cl}_{m_1}^\circ \cdots \mathrm{Cl}_{m_r}^\circ$. For $p = 2$ and $G = \mathrm{SO}_n$ we define $\Phi_1(u) = \mathrm{Cl}_{m_1} \cdots \mathrm{Cl}_{m_r}$.

For $\Phi_2(u)$ we define $\beta = \beta^{(1)} \oplus \beta^{(2)} \oplus \beta^{(3)}$ as in Theorem 1.6. Then each $\beta^{(i)}$ satisfies the properties in Table 3 (we prove this for the case $G = \mathrm{SO}_n$ and $p = 2$ in Proposition 4.5 below) and each part x satisfies $\varepsilon(x) = 1$. Therefore one may apply the formulas in Table 2 to find a unique parabolic subgroup P_i of $\mathrm{Cl}_{|\beta^{(i)}|}^\circ$ such that the Jordan blocks of the Richardson class of P_i equals $\beta^{(i)}$ (recall that Table 2 defines an injective map from G -classes of distinguished parabolics to the partitions of n described in Table 3). \square

We say a partition β satisfies the **difference condition** if for all even i such that $\beta_{i+1} \geq 1$ we have $\beta_i - \beta_{i+1} \geq 3$. If $p = 2$ and β equals the Jordan blocks of the Richardson class of a distinguished parabolic subgroup of an orthogonal group, then β satisfies the difference condition (see Table 3).

Proposition 4.5. *Let β be a partition with at most one part equal to 1 and all other parts even with multiplicity at most 2. If β does not have a part equal to 1 then we assume that β has an even number of parts. Apply Definition 1.5 for the case $p = 2$ and $G = \mathrm{SO}_n$ to decompose β as $\beta = \beta^{(1)} \oplus \delta = \beta^{(1)} \oplus \beta^{(2)} \oplus \beta^{(3)}$. The following hold:*

- (i) *If β satisfies the difference condition then $\beta = \beta^{(1)}$.*
- (ii) *If $\beta^{(i)}$ does not contain 1 then it has an even number of parts.*
- (iii) *Each $\beta^{(i)}$ satisfies the difference condition.*
- (iv) *$\beta^{(3)} = 0$ if and only if it is possible for β to be decomposed into two partitions each of which satisfies the difference condition*

Proof. Recall that we apply f to parts of β , and that when we do so we refer only to β_j , but we keep track (implicitly) of j . Thus, $f(\beta_j)$ depends not only upon the value β_j but also on j .

Part (i). Suppose that β satisfies the difference condition. An inductive argument on j shows that $f(\beta_j) = 1$ for all j .

In the remaining proof, we use the phrase “ β_j is added to $\beta^{(1)}$ ” to mean $f(\beta_j) = 1$. We also use obvious variations on this.

Part (ii). Since this decomposition is defined by applying f recursively, it suffices to prove this claim for $\beta^{(1)}$ and δ . Condition (2) in the definition of f guarantees that the last part of β will be added to $\beta^{(1)}$ or δ in such a way that both have an even number of parts, or that both have an even number of parts greater than 1. This proves part (ii).

In the remaining proof, we will use the notation “ $\beta_j \mapsto \beta_\ell^{(1)}$ ” to mean that $f(\beta_j) = 1$ and that β_j becomes the ℓ^{th} part of $\beta^{(1)}$. (Another way to say this is that ℓ parts from β_1, \dots, β_j map to 1, or, equivalently, map to $\beta^{(1)}$.)

Claim (a). Let ℓ be even. Then $\beta_j \mapsto \beta_\ell^{(1)}$ if and only if $\beta_{j-1} \mapsto \beta_{\ell-1}^{(1)}$. Proof: This follows immediately from the definition of f .

We say that β has a **bad sequence** if there exists an even number i such that $\beta_i = \beta_{i+1} > \beta_{i+2} = \beta_{i+3}$ with $\beta_{i+1} - \beta_{i+2} = 2$ and $\beta_{i+2} \neq 0$.

Claim (b). If δ violates the difference condition then β has a bad sequence. Proof: Let i be even with $\delta_{i+1} \geq 1$ and $\delta_i - \delta_{i+1} \leq 2$. Let β_j map to δ_i , let ℓ and k be as in the definition of f (i.e. ℓ is the number of parts from $\beta_1, \dots, \beta_{j-1}$ which map to 1; k is the last of these parts which map to 1; note that i in the definition of f has been replaced by $i - 1$ in the present context). Then we have $\beta_k \mapsto \beta_\ell^{(1)}$. Since $f(\beta_j) = 0$ we have that ℓ is even, whence $j = \ell + i$ is even

Suppose $\delta_i = \delta_{i+1}$. Then $\delta_i \geq 2$, $\beta_j = \beta_{j+1}$ and β_{j+1} maps to δ_{i+1} . Since i is even and $f(\beta_{j+1}) = 0$, we have that $\beta_\ell^{(1)} - \beta_{j+1} = 2$, whence k equals $j - 1$ or $j - 2$ (i.e. β_{j-1} or β_{j-2} maps to $\beta_\ell^{(1)}$).

We summarize this information below as follows. Each part β_a is sent to a part $\beta_b^{(1)}$ in $\beta^{(1)}$ or a part δ_b in δ . We indicate this by writing β_a or β_a respectively.

We write $a \overset{2}{>} b$ to indicate that $a - b = 2$. With this notation we have:

$$\beta_{j-1} \overset{2}{>} \beta_j = \beta_{j+1}, \text{ or } \beta_{j-2} = \beta_{j-1} \overset{2}{>} \beta_j = \beta_{j+1}.$$

$$\beta_\ell^{(1)} \quad \delta_i \quad \delta_{i+1} \quad \beta_\ell^{(1)} \quad \delta_{i-1} \quad \delta_i \quad \delta_{i+1}$$

The second subsequence in β is a bad sequence; whence in this case we are done proving claim (b). In the first subsequence we have, by claim (a), that $\beta_{j-2} \mapsto \beta_{\ell-1}^{(1)}$, thus, an even number of parts from $\beta_1, \dots, \beta_{j-3}$ map to 1 and an odd number of these parts map to 0 (recall that ℓ is even and j is even). Since $f(\beta_{j-2}) = 1$ we have that $\beta_{j-2} = \beta_{j-1}$ (if $\beta_{j-2} > \beta_{j-1}$ then condition (3) in the definition of f would have caused $f(\beta_{j-2}) = 0$). This means that the first subsequence is $\beta_{j-2} = \beta_{j-1} \overset{2}{>} \beta_j = \beta_{j+1}$ which is a bad sequence.

We suppose now that $\delta_i - \delta_{i+1}$ equals 1 or 2 and show that this leads to a contradiction. Case 1: $\beta_j = \beta_{j+1}$. It is easy to show that conditions (2) or (3)

could not have caused $f(\beta_j) = 0$, whence we have that $\beta_\ell^{(1)} - \beta_j = 2$. Thus $f(\beta_{j+1}) = 0$ and $\delta_i = \delta_{i+1}$, a contradiction.

Case 2: $\beta_j > \beta_{j+1}$ and $\beta_\ell^{(1)} > \beta_j$. Then $\beta_\ell^{(1)} - \beta_{j+1} \geq 3$ whence, $f(\beta_{j+1}) = 1$, whence, by claim (a), we have that $f(\beta_{j+2}) = 1$. This implies that $\delta_i - \delta_{i+1} > 2$ contrary to assumption.

Case 3: $\beta_j > \beta_{j+1}$ and $\beta_\ell^{(1)} = \beta_j$. We have $\beta_{j-1} \mapsto \beta_\ell^{(1)}$, whence, by claim (a), we have that $\beta_{j-2} \mapsto \beta_{\ell-1}^{(1)}$. This shows that f maps an even number of the parts $\beta_1, \dots, \beta_{j-3}$ to 1 and an odd number of these parts to 0. Note that $\beta_{j-1} - \beta_{j+1} \leq 2$, that $\beta_{j+1} \neq 0$, and that $\beta_{j-2} - \beta_{j+1} \geq 3$. But this would imply that $f(\beta_{j-2}) = 0$, a contradiction.

This finishes the proof of claim (b).

Claim (c). The partition δ has no bad sequence. Proof: Suppose, for contradiction, that i is even, $\delta_i = \delta_{i+1} > \delta_{i+2} = \delta_{i+3}$ with $\delta_{i+3} \neq 0$. Let $\beta_j \mapsto \delta_i$. Then we also have $\beta_{j+1} \mapsto \delta_{i+1}$, $\beta_{j+2} \mapsto \delta_{i+2}$ and $\beta_{j+3} \mapsto \delta_{i+3}$. Then condition (1) in the definition of f cannot apply to either of β_{j+2} and β_{j+3} . But conditions (2) and (3) both require that an odd number of parts have already been mapped to δ . Thus, it is not possible for both β_{j+2} and β_{j+3} to be affected by conditions (2) and (3), a contradiction.

Part (iii). The definition of f makes it clear that $\beta^{(1)}$ and $\beta^{(2)}$ satisfy the difference condition. By claim (c), δ has no bad sequence, whence, by claim (b), $\beta^{(3)}$ satisfies the difference condition.

Part (iv). “ \Rightarrow ” follows from part (iii). Conversely, by claim (b), it suffices to show that if β has a bad sequence then it cannot be written as the sum of two partitions each of which satisfies the difference condition. Suppose that β has a bad sequence $\beta_j = \beta_{j+1} > \beta_{j+2} = \beta_{j+3}$. Fix a decomposition $(\beta_1, \dots, \beta_{j-1}) = \mu \oplus \nu$ where μ has an odd number of parts. At most one part of $\beta_j, \beta_{j+1}, \beta_{j+2}$ and β_{j+3} can be added to μ without violating the difference condition and at most two of these parts can be added to ν without violating the difference condition. \square

5. UNIPOTENT CLASSES IN CLASSICAL SUBGROUPS OF EXCEPTIONAL GROUPS

In this section we give two examples of Theorem 1.6, and then return to our main application, translating the labels for unipotent classes in a classical Levi subgroup of an exceptional group G into the labels for unipotent classes in G .

First we recall some Bala-Carter notation. If R is a type of root system then we use “ R ” to label the regular class in the simple group of type R . If R is of type A_n, B_n, C_n or D_n then we use $R(a_j)$ to denote the Richardson class of the distinguished parabolic whose Levi subgroup has only simple factors of rank 1 and a total semisimple rank of j .

Example 5.1. Here we illustrate the map f , and the decomposition $\beta = \beta^{(1)} \oplus \beta^{(2)} \oplus \beta^{(3)}$ it gives rise to as described in Definition 1.5. Throughout we assume that $G = \mathrm{SO}_n$ and $p = 2$.

Suppose that $\beta = (8, 4, 1)$ equals the Jordan block sizes of an unipotent element in SO_{13} . We start with $f(8) = 1$. Now $\beta_k = 8$, $\ell = 1$ and $i = 0$, therefore $f(4) = 1$. Now $\beta_k = 4$, $\ell = 2$ and $i = 0$. Since $4 - 1 > 2$ we have $f(1) = 1$. Therefore $\beta = \beta^{(1)}$ and $\beta^{(2)} = \beta^{(3)} = 0$. This agrees with the fact that the unipotent class with Jordan blocks given by $(8, 4, 1)$ is parameterized by the Bala-Carter Theorem; it is the class $B_6(a_2)$.

TABLE 4. Extra unipotent classes in D_8 , $p = 2$

Jordan blocks ($\varepsilon(4) = \varepsilon(2) = 1$ in all cases)	$\beta = \beta^{(1)} \oplus \beta^{(2)} \oplus \beta^{(3)}$	Bala-Carter type label de- termined by Theorems 1.3 and 1.6
$(8, 4, 2^2)$	$(8, 4, 2^2) = (8, 4) \oplus (2, 2)$	$D_6(a_1)D_2$
$(6, 4^2, 2)$	$(6, 4^2, 2) = (6, 4) \oplus (4, 2)$	$D_5(a_1)D_3$
$(6, 4, 2^2, 1^2)$	$(6, 4, 2^2) = (6, 4) \oplus (2, 2)$	$D_5(a_1)D_2$
$(4^2, 2^4)$	$(4^2, 2^2) = (4, 4) \oplus (2, 2)$	$A_1D_4(a_1)D_2$
$(4^2, 2^2, 1^4)$	$(4^2, 2^2) = (4, 4) \oplus (2, 2)$	$D_4(a_1)D_2$

Suppose that $\beta = (12, 12, 10, 8, 6, 6, 4, 2)$ and we have $G = \text{SO}_{60}$. For convenience we keep track of the results by writing an array: the middle row has the original partition β , the first row contains the parts that f maps 1, and the third row contains the parts that f maps to 0. The calculation of f proceeds sequentially from left to right.

$$\begin{array}{rcccccccc}
 \text{map to 1 :} & 12 & 12 & & 6 & 6 & & & \\
 \beta : & 12 & 12 & 10 & 8 & 6 & 6 & 4 & 2 \\
 \text{map to 0 :} & & & 10 & 8 & & & 4 & 2
 \end{array}$$

For instance, to calculate $f(10)$ one has that $\ell = 2$ and condition (1) causes $f(10) = 0$. Similarly, condition (3) causes $f(8) = 0$, and condition (2) causes $f(2) = 0$. The decomposition is $\beta = (12, 12, 6, 6) \oplus (10, 8, 4, 2)$. Each of these partitions corresponds to the Richardson class of a distinguished parabolic subgroup of a group of type D . The usual Bala-Carter notation does not apply to these parabolics, but they can be described by the Dynkin diagrams below (where each x represent a crossed off node):

$$\begin{array}{cccccccc}
 & & & \circ & & & & \circ \\
 \text{x} \circ & \text{x} \circ & \text{x} \circ \circ & \text{x} \circ \circ \circ & \text{x} \circ \circ \circ & \text{x} & & & \text{x} & \text{x} \circ & \text{x} \circ \circ \circ & \text{x} \circ \circ \circ & \text{x} \\
 & & & \circ & & & & \circ & & & & &
 \end{array}$$

Finally, consider the partition $(6, 4, 4, 2, 2, 1)$ in SO_{19} . Applying f once gives $\beta^{(1)} = (6, 4)$ and $\delta = (4, 2, 2, 1)$. Applying f to δ gives $\beta^{(2)} = (4, 2)$ and $\beta^{(3)} = (2, 1)$.

In the following example and lemmas, we describe Levi subgroups using notation which specifies only their Lie type. Thus, a Levi subgroup denoted by, for example, B_3T_1 , has Lie type of B_3 and a central, one dimensional torus T_1 .

Example 5.2. If $G = D_8$ and $p = 2$, then there are five extra classes. In Table 4 we give the Jordan blocks, the decomposition of β and a Bala-Carter type label.

We sketch how one can see that these classes are extra. By Lemma 4.3, the class $(4^2, 2^2, 1^4)$ is distinguished in the Levi subgroup D_6T_2 . Since it does not satisfy the properties in Table 3 applied to D_6 , it cannot be in the image of the Bala-Carter map.

Similarly the class $D_5(a_1)D_2$ is distinguished in the Levi subgroup D_7T_1 , the class $A_1D_4(a_1)D_2$ is distinguished in the Levi subgroup $A_1D_6T_1$, and the classes $D_6(a_1)D_2$ and $D_5(a_1)D_3$ are distinguished in D_8 .

We turn now to the exceptional groups E_7 , E_8 and F_4 and use Lawther [4] (who draws on the work of [5], [6], [8], [9]) for the number of unipotent classes, the

number of extra unipotent classes, and their representatives. We note that E_6 has no extra classes in any characteristic (see [2]).

Lemma 5.3. *Let $p = 2$, $X = B_3T_1$, and $G = F_4$. Then X has two extra unipotent classes. One of these is distinguished in a B_2T_2 Levi subgroup, and we denote this class by D_2 . The other is distinguished in X and we denote this class by D_3 . The classes D_2 and D_3 equal the F_4 -classes Lawther denotes by $\tilde{A}_1^{(2)}$ and $B_2^{(2)}$, respectively.*

Proof. We proceed as in Examples 5.1 and 5.2 to find the extra classes and decompose their partitions. They are $(2^2, 1^3) = (2^2) \oplus (1^3)$ and $(4, 2, 1) = (4, 2) \oplus (1)$. The class $(2, 2)$ is regular in SO_4 , whence we label it as D_2 . The class $(4, 2)$ is regular in SO_6 , whence we label it as D_3 .

The classes D_2 and D_3 are distinguished in the B_2 and B_3 Levi subgroups of F_4 and are not in the image of the Bala-Carter map. The same comment applies to the classes $\tilde{A}_1^{(2)}$ and $B_2^{(2)}$. Applying Corollary 4.2 we conclude that the D_2 and $\tilde{A}_1^{(2)}$ classes are equal, as are the D_3 and $B_2^{(2)}$ classes. \square

Lemma 5.4. *Let $p = 2$, $X = D_6T_1$, and $G = E_7$. Then X has one extra unipotent class. It is distinguished in X and denoted by $D_4(a_1)D_2$. Using Lawther's notation this is the class $A_3 + A_2^{(2)}$ in E_7 .*

Proof. If $G = E_7$ and $p = 2$ then there is one extra class. The Levi subgroup D_6T_1 also has one extra class (see Example 5.2), which is distinguished in D_6T_1 . By Corollary 4.2 these extra classes are the same class, whence we label it as $D_4(a_1)D_2$. Lawther [4] denotes this class by $A_3 + A_2^{(2)}$. \square

Lemma 5.5. *Let $p = 2$, $X = D_7T_1$ and $G = E_8$. Then X has 2 extra classes. One of these is the class $D_4(a_1)D_2$ described in the previous lemma. The other is distinguished in X and denoted by $D_5(a_1)D_2$. Using Lawther's notation these are the classes $A_3 + A_2^{(2)}$ and $D_4 + A_2^{(2)}$ respectively.*

Proof. Example 5.2 shows that X has two extra classes and the previous lemma shows that $D_4(a_1)D_2$ and $A_3 + A_2^{(2)}$ are the same class. Thus, it remains to show that $D_5(a_1)D_2$ and $D_4 + A_2^{(2)}$ are the same class.

Here is one way to verify this. In the natural module for D_7 the class $D_5(a_1)D_2$ has Jordan blocks given by $(6, 4, 2, 2)$. Decomposing these blocks as $(6, 2) \oplus (4, 2)$ we see that this class can be represented by a regular unipotent element in a $SO_8 SO_6$ subgroup. By [11] such a regular element can be represented as a product of elements in root groups with the roots forming a basis for a $D_4 + D_3$ subsystem. The roots given in [6] for the $D_4 + A_2^{(2)}$ class form the basis for a $D_4 + D_3$ root system. In E_8 all $D_4 + D_3$ root systems are conjugate, so the root groups used to represent the class we have called $D_5(a_1)D_2$ can be conjugated to the roots given by Mizuno. \square

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