Parameterizing orbits in flag varieties

W. Ethan Duckworth

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Abstract

In this document we parameterize the orbits of certain groups acting on partial flag varieties with finitely many orbits. We use the parameters to describe the dimension of these orbits, and the natural partial order on them.

Many crucial results in algebraic geometry and group theory depend upon knowledge about a certain family of objects: projective spaces, their generalization to Grassmannian varieties, and the further generalization to flag varieties. One way to understand these objects is to study various groups which act upon them and to analyze their orbits. This is the purpose of this article. Now we give a technical statement of the context of this problem.

Theorem 1 ([3, 5])**.** *Let $G = \mathrm{GL}(V)$, let X be a maximal rank reductive subgroup of G and P be a proper parabolic subgroup of G. Then $X\backslash G/P$ is finite if and only if one of the following holds.*

1. $X = \mathrm{GL}(V_1)\,\mathrm{GL}(V_2)$,
2. $X = \mathrm{GL}(V_1)\,\mathrm{GL}(V_2)\,\mathrm{GL}(V_3)$ *and P is the stabilizer of a single subspace,*
3. *P is the stabilizer of subspace of dimension 1, or codimension 1.*

In this article we parameterize the X-orbits in each of the cases listed in this theorem, and give dimension formulas and closure relations using these parameters. Thus, the present paper gives more precise information about some of the double cosets studied in [3] and [5]. This paper can also be viewed as an extension of the work by Brion and Helminck [2], who parameterized the X-orbits and orbit closures in varieties G/P, where X was assumed to be the fixed points of an involutive automorphism of G.

In Theorems 6, 8, 12 and 15 we give parameterizations for the orbits arising in Theorem 1. In Corollaries 9, 13, and 16 we give formulas for the dimension of each orbit, and describe the natural partial order in the set of orbits.

Throughout the paper we translate statements about double cosets $X\backslash G/P$ into statements about the collection of X-orbits in G/P, i.e. the X-orbits in a flag variety. We start by giving a little background about these topics.

1 Background and preliminaries

We record some information here that can be found in standard texts, such as [1] and [4]. Let V be a vector space of dimension n over an algebraically closed field of arbitrary characteristic. (Those results that make sense over an arbitrary field are usually true without assuming algebraic closure.) Let $\mathrm{GL}(V)$ be the group of invertible linear transformations on V. Let X be a maximal rank reductive subgroup of $\mathrm{GL}(V)$. Equivalently, there exists a decomposition $V = V_1 \oplus \cdots \oplus V_s$ such that X equals a direct product of groups $X = \mathrm{GL}(V_1) \cdots \mathrm{GL}(V_s)$.

Let $\mathbb{P}(V)$ be the projective space formed on V, i.e. the collection of all 1-spaces in V. For each $r \in \{0, \ldots, \dim V\}$ let $\mathbf{Gr}_r(V)$ be the collection of all r-spaces in V. In particular, $\mathbf{Gr}_0(V)$ is a single point and $\mathbf{Gr}_1(V) = \mathbb{P}(V)$. A flag F in V is a finite sequence $W_1, \ldots, W_k$ of subspaces of V with $W_i < W_{i+1}$ for each i (whenever convenient we will allow $W_0 = 0$, but we assume that $W_i \neq 0$ for $i > 0$). We say the flag is *full* if $\dim W_i = i$ for each i, and *partial* if it is not full. Each flag has a corresponding dimension sequence, $(\dim W_i)_i$. For each sequence $(r_i)_i$ with values in $\{0, \ldots, n\}$ and such that $r_i < r_{i+1}$ for each i, we let $\mathbf{Fl}((r_i)_i, V)$ be the collection of all flags whose dimension sequence equals $(r_i)_i$. In particular $\mathbf{Fl}(r, V) = \mathbf{Gr}_r(V)$. Each collection of flags $\mathbf{Fl}((r_i)_i, V)$ forms a complete, projective variety.

Let P be a parabolic subgroup of $\mathrm{GL}(V)$. By definition, this means that the quotient $\mathrm{GL}(V)/P$ is a complete variety. Equivalently, this means that P is the stabilizer in $\mathrm{GL}(V)$ of a flag. The parabolic subgroup uniquely determines the flag, and thus the dimension sequence. Conversely, two parabolic subgroups have the same dimension sequence if and only if they are conjugate. A parabolic subgroup is a Borel subgroup if it equals the stabilizer of a full flag. Equivalently, it is minimal among the parabolic subgroups. Equivalently, it is maximal among the solvable subgroups. Each parabolic subgroup contains a Levi factor L, which is a maximal among the maximal rank reductive subgroups of P. The subgroup L is not uniquely determined by P, but is unique up to P-conjugacy. We can choose L as the direct product of groups $L = \mathrm{GL}(\widetilde{W}_1) \cdots \mathrm{GL}(\widetilde{W}_s)$ where $\widetilde{W}_{i+1}$ is chosen as a subspace of W_{i+1} such that $W_{i+1} = W_i \oplus \widetilde{W}_{i+1}$. Note that $\dim \mathrm{GL}(V) = n^2$.

Lemma 2. *Let $G = \mathrm{GL}(V)$ act on $\mathbf{Fl}((r_i)_i, V)$ in the natural manner and let P be the stabilizer of an element of $\mathbf{Fl}((r_i)_i, V)$, and let L be a Levi factor of P.*

1. *The action is transitive.*
2. *We have an isomorphism of varieties $G/P \cong \mathbf{Fl}((r_i)_i, V)$.*
3. $\dim \mathbf{Fl}((r_i)_i, V) = \frac{1}{2}(\dim G - \dim L)$.

Part gives a simple way to calculate the dimension of a flag. In particular, one can use this to show that $\mathbf{Gr}_r(V) = r(n - r)$.

Lemma 3 ([3, 3.4])**.** *Let $V = \widetilde{V} \oplus V_s$, let $\pi : V \to \widetilde{V}$ be the natural projection, let $X = \widetilde{X}\,\mathrm{GL}(V_s)$ with $\widetilde{X}$ a subgroup of $\mathrm{GL}(\widetilde{V})$, let W and W' be two r-spaces* *in* V*. If* $(W \cap \widetilde{V}, \pi(W))$ *and* $(W' \cap \widetilde{V}, \pi(W'))$ *are conjugate under* $\widetilde{X}$*, then* W *and* W' *are conjugate under* X*.*

Corollary 4. *If* $X_1 = \mathrm{GL}(V_1)$ *then* W *and* W' *are conjugate under* X *if and only if* $\dim W \cap V_1 = \dim W' \cap V_1$ *and* $\dim \pi_1(W) = \dim \pi_1(W')$*.*

Proof. For $x \in X$ we have that $\dim W \cap V_1 = \dim xW \cap W_1$ and $\dim \pi_1(W) = \dim \pi_1(xW)$ have the same dimension, which proves one direction.

Now we suppose that the equality holds for the indicated dimensions. Note that $(W \cap V_1, \pi_1(W))$ and $(W' \cap V_1, \pi_1(W'))$ are partial flags. By Lemma 2 we have that these flags are conjugate under X_1, whence the result follows by Lemma 3. □

2 Two factor Levi subgroup

Let V be n-dimensional, fix a decomposition $V = V_1 \oplus V_2$ with $X = \mathrm{GL}(V_1)\,\mathrm{GL}(V_2)$, and let $\pi_i : V \to V_i$ be the natural projection with respect to this decomposition. Let P be the stabilizer of an r-space.

For any index i $(0 \le i \le n)$ we let $\mathbf{Gr}_i(V)$ be the collection of all i-spaces in V. Recall that $\mathbf{Gr}_i(V)$ is an irreducible, complete variety of dimension $i(n-i)$. In what follows we often identify G/P with $\mathbf{Gr}_r(V)$ and view $X\backslash G/P$ as the collection of X-orbits in $\mathbf{Gr}_r(V)$.

Lemma 5. *Two* r*-spaces,* W *and* W'*, are in the same* X*-orbit if and only if* $\dim W \cap V_i = \dim W' \cap V_i$ *for* $i = 1, 2$*.*

Proof. "⇒" is clear. For "⇐" we write $W_i = W \cap V_i$ for $i = 1, 2$ and similarly for W'. We decompose W and W' as shown

$$\begin{aligned} W &= W_1 \oplus W_2 \oplus \widetilde{W} \\ W' &= W_1' \oplus W_2' \oplus \widetilde{W'} \end{aligned}$$

where $\widetilde{W}$ and $\widetilde{W'}$ are any direct complements. We pick ordered bases respecting these direct sums

$$\begin{aligned} \mathcal{B} &= B_1 \cup B_2 \cup \widetilde{B} \\ \mathcal{B}' &= B_1' \cup B_2' \cup \widetilde{B'} \end{aligned}$$

These definitions imply that π_i is injective on $W_i \oplus \widetilde{W}$ and on $W_i' \oplus \widetilde{W'}$. Therefore $\pi_i(B_i \cup \widetilde{B})$ and $\pi_i(B_i' \cup \widetilde{B'})$ are linearly independent sets of the same cardinality. For $i = 1, 2$ let $g_i \in \mathrm{GL}(V_i)$ be an order preserving map taking $\pi_i(B_i \cup \widetilde{B})$ to $\pi_i(B_i' \cup \widetilde{B'})$. Then we have that g_i is an order preserving map taking $\pi_i B$ to $\pi_i B'$. Finally, $g = (g_1, g_2) \in X$ takes $\mathcal{B}$ to $\mathcal{B}'$ whence W to W'. □

Theorem 6. *The* X*-orbits on* G/P *are parameterized by a collection of three part compositions* (a_1, a_2, a_3) *of* r *where the correspondence is given by*

$$X\text{-orbit of } W \mapsto (a_2, a_2, a_3)$$

with $a_i = \dim W \cap V_i$ $(i = 1, 2)$ *and* $a_3 = r - a_1 - a_2$.

A composition (a_1, a_2, a_3) *of* r *corresponds to an* X*-orbit if and only if* $a_i + a_3 \leq n_i$ *for* $i = 1, 2$.

Note that the component a_3 is somewhat redundant in this parameterization. However, we include a_3 because using it makes the final statement in the previous theorem convenient.

Proof. The first statement is an immediate consequence of the previous lemma. Now we prove the last statement.

Let W be given, and define a_i, π_i, W_i, $\widetilde{W}$, etc. as above. Since π_i is injective when applied to $\widetilde{W} \oplus W_i$, we have that $\dim \pi_i(\widetilde{W} \oplus W_i) = a_3 + a_i$ and this must be $\leq n_i$.

Conversely, let a_1, a_2, a_3 be given. Pick $W_i \leq V_i$ with $\dim W_i = a_i$, $(i = 1, 2)$. We now construct $\widetilde{W}$. Let $W_3 \leq V_1$ be a subspace of dimension a_3, disjoint from W_1 (which is possible since $a_3 \leq n_1 - a_1$). Then $\pi_1^{-1}(W_3)$ is a subspace of V of dimension $a_1 + a_3$ that contains V_2. Choose a direct complement $\widetilde{W}$ of V_2 in $\pi_1^{-1}(W_3)$. Then $W := W_1 \oplus W_2 \oplus \widetilde{W}$ is the r-space we desire. □

Corollary 7. *Let* $(W_i)_i$ *and* $(U_i)_i$ *be two flags in* V *with* $\dim W_i = \dim U_i$ *for each* i*. These flags are in the same* X*-orbit if and only if for each* $i \geq 1$ *and each* $j = 1, 2$ *we have* $\dim W_i \cap V_j = \dim U_i \cap V_j$.

Proof. The proof is an inductive argument that uses X to zip the two flags together subspace by subspace.

Note that $U_0 = W_0 = 0$. Let $j \geq 0$ such that there exists $x_j \in X$ with $x_j U_i = W_i$ for all $i \leq j$. We will show that there exists $x \in X$ such that x acts as the identity on W_i and $x x_j U_{j+1} = W_{j+1}$, whence the result follows by induction.

To simplify notation we replace $(U_i)_i$ with $(x_j U_i)_i$, and thus assume that $U_i = W_i$ for all $i \leq j$. To simplify further, we write $U = U_i = W_i$. Thus, we have $U < U_{i+1}$ and $U < W_{i+1}$ and wish to construct an element of X than stabilizes U and takes U_{i+1} to W_{i+1}.

For any subspace Z of V, let $\overline{Z} = (Z + U)/U$ be the corresponding subspace in V/U. Then $\overline{V} = \overline{V_1} \oplus \overline{V_2}$ and the stabilizer in X of U induces the group $\overline{X} = \mathrm{GL}(\overline{V_1})\,\mathrm{GL}(\overline{V_2})$ in $\mathrm{GL}(\overline{V})$. Applying Proposition 5 to $\overline{X}$, it suffices to show that

$$\dim \overline{V_k} \cap \overline{U_{i+1}} = \dim \overline{V_k} \cap \overline{W_{i+1}}, \text{ for } k = 1, 2. \tag{1}$$

Equation (1) holds if and only if $\dim(V_k + U) \cap U_{i+1} = \dim(V_k + U) \cap W_{i+1}$. By the modular property of the lattice of subspaces we have $\dim(V_k + U) \cap U_{i+1} = \dim U_{i+1} \cap V_K + U$, which in turn equals $\dim U_{i+1} \cap V_k + \dim U - \dim V_k \cap U$. Since $\dim U_{i+1} \cap V_k = \dim W_{i+1} \cap V_k$ we see that Equation (1) holds, and we are done. □

Let P be the stabilizer of a flag in $\mathrm{GL}(V)$ containing k nontrivial subspaces, and let r_i equal the dimension of the ith subspace of this flag.

Theorem 8. *The X-orbits in G/P are parameterized by a collection of $k \times 3$ arrays $(a_{i,j})$ where the ith row, $(a_{i,1}, a_{i,2}, a_{i,3})$ is a composition of r_i. The correspondence is given by*

$$X.(W_i)_i \mapsto (a_{i,j})$$

with $a_{i,j} = \dim W_i \cap V_j$ for $j = 1, 2$, and $a_{i,3} = r_i - a_{i,1} - a_{i,2}$.

Let $(a_{i,j})$ be a $k \times 3$ array, such that for all i, the ith row is a composition of r_i. Then $(a_{i,j})$ corresponds to an X-orbit in G/P if and only if $a_{i,j} + a_{i,3} \leq n_i$ for all i and for $j = 1, 2$, and $a_{i,j} \leq a_{i+1,j}$ for all i and for $j = 1, 2, 3$.

Proof. The statement about parameterization follows immediately from Proposition 7. If $(a_{i,j})$ is the label of an orbit, then it is clear that it satisfies the conditions.

Now, let $(a_{i,j})$ be a $k \times 3$ array, satisfying the conditions given in the corollary. We construct a flag $(W_i)_i$ such that $\dim W_i = r_i$, and $\dim W_i \cap V_j = a_{i,j}$ for all i and for $j = 1, 2$. The construction is inductive, with the main step similar to the inductive step in the proof of Proposition 7.

By Theorem 6, there exists $W_1 \leq V$ satisfying $\dim W_1 \cap V_i = a_{1,i}$ for $i = 1, 2$ and $\dim W_1 = r_1$. Inductively, we assume that $k \geq 1$ and $W_1 < \cdots < W_k$ are subspaces with $a_{i,j} + a_{i,3} \leq n_i$ for all $i \leq k$ and for $j = 1, 2$, and $a_{i,j} \leq a_{i+1,j}$ for all $i \leq$ and for $j = 1, 2, 3$. For each subspace $Z < V$ let $\overline{Z} = (Z + W_k)/W_k$ be the corresponding subspace of $\overline{V} = V/W_k$. Working in $\overline{V}$, relative to the decomposition $\overline{V} = \overline{V_1} \oplus \overline{V}_2$, there exists a subspace $\widetilde{W}$ with $\dim \widetilde{W} = r_{k+1} - r_k$, $\dim \widetilde{W} \cap \overline{V}_i = a_{k+1,i} - a_{k,i}$ for $i = 1, 2$. Let W_{k+1} be the lift $\widetilde{W}$ of to a subspace of V. ☐

Corollary 9. *Let $(W_i)_i$ be a flag with dimension sequence $(r_i)_i$, with X-orbit $X.(W_i)_i$ and let $(a_{i,j})_{ij}$ be the label of $X.(W_i)_i$. Then $X.(W_i)_i$ has dimension $\dim \mathbf{Fl}((a_{i,1})_i, V_1) + \dim \mathbf{Fl}((a_{i,2})_i, V_2) + \dim \mathbf{Fl}((a_{i,3})_i, V)$. If $(W'_i)_i$ is another flag with dimension sequence $(r_i)_i$ and label $(b_{i,j})$ then $X.(W'_i)_i \leq X.(W_i)_i$ if and only if $b_{i,j} \geq a_{i,j}$ for all $i \geq 1$ and for $j = 1, 2$.*

In the following proof we make use of elementary methods for constructing open and closed subsets of flag varieties. See [6] for a suitable background.

Proof. Let $Z_1 = \{(\widetilde{W_i})_i \in \mathbf{Fl}((r_i)_i, V) \mid \dim \widetilde{W_i} \cap V_j \geq a_{i,j},\ i, j \in \{1, 2\}\}$. Then Z_1 is closed in $\mathbf{Fl}(\underline{(r_i)_i, V})$, contains $(W_i)_i$, and is stable under X. Therefore $X.(W_i)_i \subseteq Z_1$ and $\overline{X.(W_i)_i} \subseteq Z_1$. This shows that if $X.(W'_i)_i \leq X.(W_i)_i$, then $(W'_i)_i \subseteq Z_1$ and so $b_i \geq a_i$ for $i = 1, 2$.

Now we show that if $b_{i,j} \geq a_{i,j}$ for all i and for $j = 1, 2$ then $X.(W'_i)_i \leq X.(W_i)_i$. We start by defining varieties $Y \supset Y_1 \supset Y_2$, and a morphism $\varphi : Y_1 \to Z_1$. Let $Y = \prod_i \mathbf{Gr}_{a_{i,1}}(V_1) \times \mathbf{Gr}_{a_{i,2}}(V_2) \times \mathbf{Gr}_{a_{i,3}}(V)$, let $Y_1 = \{(W_{i,j})_{i,j} \in Y \mid \dim(W_{i,1} + W_{i,2} + W_{i,3}) = r_i,\ \text{for all } i\}$. Note that for $(W_{i,j})_{i,j} \in Y$ we have that $\dim W_{i,1} + W_{i,2} + W_{i,3} = r_i$ if and only if $\dim W_{i,3} \cap (W_{i,1} \oplus W_{i,2}) = 0$. Thus, Y_1 is an open subset of Y. Let $Y_2 = \{(W_{i,j})_{i,j} \in Y_1 \mid W_{i,3} \cap V_j = 0,\ \text{for } j = 1, 2\}$. Then Y_2 is an open subset of Y_1. Since Y is irreducible, we have that Y_1 is as well, and thus Y_2 is dense in Y_1. Define the morphism $\varphi : Y_1 \to \mathbf{Fl}((r_i)_i, V)$ via $\varphi((W_{i,j})_{i,j}) = (W_{i,1} + W_{i,2} + W_{i,3})_i$.

Note that X acts on Y_1 and Y_2, that φ is X-equivariant, that $(W_i)_i \in \varphi(Y_2)$, and that $X.(W_i)_i = \varphi(Y_2)$ (this last by our results showing that the X-orbit of $(W_i)_i$ equals all the flags with the same $3 \times k$ array of dimensions). Next we will justify each containment in the following sequence

$$\overline{X.(W_i)_i} \subseteq Z_1 \subseteq \varphi(Y_1) \subseteq \varphi(\overline{Y_2}) \subseteq \overline{\varphi(Y_2)} \subseteq \overline{X.(W_i)_i},$$

where all closures are taken within Y_1 or Z_1 as appropriate. The first inclusion was proven above. To prove the second inclusion, let $(U_i)_i \in Z_1$, and write $U_i = U_{i,1} \oplus U_{i,2} \oplus U_{i,3}$ with $U_{i,j} = U_i \cap V_{i,j}$ for $j = 1, 2$. Let $U_{i,j} = U'_{i,j} \oplus K_{i,j}$ with $\dim U'_{i,j} = a_{i,j}$. Then $(U'_{i,1}, U'_{i,2}, K_{i,1} \oplus K_{i,2} \oplus U_{i,3})_i \in Y_1$ and $U_i = \varphi(U'_{i,1}, U'_{i,2}, K_{i,1} \oplus K_{i,2} \oplus U_{i,3}) \in \varphi(Y_1)$. The third inclusion follows from the fact that $\overline{Y_2} = Y_1$, as stated above. The fourth inclusion is an elementary statement in point set topology. The fifth inclusion follows from the fact, stated above, that $X.(W_i)_i = \varphi(Y_2)$. Since $(W'_i)_i \in Z_1$, this shows that $(W'_i)_i \in \overline{X.(W_i)_i}$.

Finally, we have $\dim X.(W_i)_i = \dim \overline{X.(W_i)_i} = \dim Y_2 = \dim Y$, which proves the statement in the Corollary about $\dim X.(W_i)_i$. □

3 Three factor Levi subgroup

Throughout this section we have $V = V_1 \oplus V_2 \oplus V_3$ and $X = \mathrm{GL}(V_1)\,\mathrm{GL}(V_2)\,\mathrm{GL}(V_3)$. Let $\pi_i : V \to V_i$ and $\pi_{ij} : V \to V_{ij}$ be the natural projections.

Lemma 10. *Let $X = X_1X_2X_3$ with $X_i = \mathrm{GL}(V_i)$, $V = V_1 \oplus V_2 \oplus V_3$ and let W and W' be two r-spaces. Then W and W' are in the same X-orbit if and only if $(W \cap (V_1 \oplus V_2), \pi_{12}(W))$ and $(W' \cap (V_1 \oplus V_2), \pi_{12}(W'))$ are in the same X_1X_2 orbit.*

Proof. This is an immediate corollary of Lemma 3, where $s = 3$ and $\widetilde{X} = X_1X_2 = \mathrm{GL}(V_1)\,\mathrm{GL}(V_2)$. □

Corollary 11. *With the above notation, W and W' are in the same X-orbit if and only if*

$$\dim W \cap V_i = \dim W' \cap V_i \text{ for } 1 \le i \le 3, \text{ and}$$
$$\dim W \cap (V_i \oplus V_j) = \dim W' \cap (V_i \oplus V_j) \text{ for } 1 \le i < j \le 3.$$

Proof. We will combine Corollary 7 and Lemma 10. We need two equations to show that the two partial flags in $V_1 \oplus V_2$ given in Lemma 10 have the same dimensions, and then we need four equations to show that the spaces in these flags give the same dimensions when intersected with V_1 and V_2. Thus, we see that that W and W' are in the same X-orbit if and only if

$$\begin{aligned}
\dim W \cap (V_1 \oplus V_2) &= \dim W' \cap (V_1 \oplus V_2), \\
\dim \pi_{12}(W) &= \dim \pi_{12}(W'), \\
\dim (W \cap (V_1 \oplus V_2)) \cap V_1 &= \dim (W' \cap (V_1 \oplus V_2)) \cap V_1, \\
\dim (W \cap (V_1 \oplus V_2)) \cap V_2 &= \dim (W' \cap (V_1 \oplus V_2)) \cap V_2, \\
\dim \pi_{12}(W) \cap V_1 &= \dim \pi_{12}(W') \cap V_1, \\
\dim \pi_{12}(W) \cap V_2 &= \dim \pi_{12}(W') \cap V_2.
\end{aligned} \tag{2}$$

It is easy to show that $\dim \pi_{12}(W) = \dim W - \dim W \cap V_3$ and $(W \cap (V_1 \oplus V_2)) \cap V_1 = W \cap V_1$ and $\pi_{12}(W) \cap V_1 = \pi_1(W \cap (V_2 \oplus V_3))$. Note that $\pi_1(W \cap (V_1 \oplus V_3)) = \dim W \cap (V_1 \oplus V_3) - \dim(W \cap (V_1 \oplus V_3) \cap (V_2 \oplus V_3)) = \dim W \cap (V_1 \oplus V_3) - \dim W \cap V_3$. Thus, Equations (2) are equivalent to

$$\begin{aligned}
\dim W \cap (V_1 \oplus V_2) &= \dim W' \cap (V_1 \oplus V_2), \\
\dim W - \dim W \cap V_3 &= \dim W' - \dim W' \cap V_3, \\
\dim W \cap V_1 &= \dim W' \cap V_1, \\
\dim W \cap V_2 &= \dim W' \cap V_2, \\
\dim W \cap (V_1 \oplus V_3) - \dim W \cap V_3 &= \dim W' \cap (V_1 \oplus V_3) - \dim W' \cap V_3, \\
\dim W \cap (V_2 \oplus V_3) - \dim W \cap V_3 &= \dim W' \cap (V_2 \oplus V_3) - \dim W' \cap V_3.
\end{aligned}$$

Note that $\dim W = \dim W'$, so we can eliminate these terms from the second equation. We can also eliminate terms involving $W \cap V_3$ and $W' \cap V_3$ from the last two equations. □

Theorem 12. *With the above notation, the X-orbits on $G/P = \mathbf{Gr}_r(V)$ are parameterized by a collection of 6-tuples $(a_1, a_2, a_3, a_{12}, a_{13}, a_{23})$ where the correspondence is given by*

$$X\text{-orbit of } W \mapsto (a_1, a_2, a_3, a_{12}, a_{13}, a_{23})$$

with $a_i = \dim W \cap V_i$, $(1 \le i \le 3)$, and $a_{ij} = \dim W \cap (V_i \oplus V_j)$, $(1 \le i < j \le 3)$.

A 6-tuple $(a_1, a_2, a_3, a_{12}, a_{13}, a_{23})$ of nonnegative integers corresponds to an X-orbit if and only if $a_{ij} - a_i \le n_j$ and $r - a_{ij} \le n_k$ for $1 \le i < j \le 3$ and $k \ne i, j$.

Proof. The statement about parameterizing follows immediately from Corollary 11.

Given an r-space W, define a_i and a_{ij}, as above. Then Theorem 6, applied three times, to subspaces of $V_1 \oplus V_2$, $V_1 \oplus V_3$ and $V_2 \oplus V_3$, gives $a_{ij} - a_i \le n_j$ for $1 \le i < j \le 3$. To see that $r - a_{ij} \le n_k$, note that $r - a_{ij} = \dim \pi_k(W)$.

Conversely, let a_i and a_{ij} be given. We will construct an r-space W with appropriate dimensions. The proof is similar to the analogous part of Theorem 6, but more complicated because the dimensions of the various subspaces are interrelated.

Choose subspaces $W_i \leq V_i$ with $\dim W_i = a_i$. For $j = 2, 3$, choose $W_{1j} \leq V_1$ of dimension $a_{1j} - a_1 - a_j$, disjoint from W_1 (which is possible since $a_{1j} - a_j \leq n_1$). Then $\pi_1|_{V_1 \oplus V_j}^{-1}(W_{1j})$ is a subspace of $V_1 \oplus V_j$ of dimension $a_{1j} - a_1$ that contains W_j. Choose a direct complement $\widetilde{W}_{1j}$ of W_j in $\pi_1|_{V_1 \oplus V_j}^{-1}(W_{1j})$. Then W_1, W_2, W_3, $\widetilde{W}_{12}$, $\widetilde{W}_{13}$ are independent and $\dim \widetilde{W}_{1j} = a_{1j} - a_1 - a_j$.

Now let K be a subspace of V_1 disjoint from $\pi_1(W_1 \oplus \widetilde{W}_{12} \oplus \widetilde{W}_{13})$ of dimension $r - a_{23}$ (which is possible since $r - a_{23} \leq n_1$). Note that $\pi_1^{-1}(K)$ contains $V_2 \oplus V_3$. Let $\widetilde{\widetilde{W}}_{23}$ be a direct complement of $V_2 \oplus V_3$ in $\pi_1^{-1}(K)$. Then $\dim \widetilde{\widetilde{W}}_{23} = r - a_{23}$ and $\widetilde{\widetilde{W}}_{23}$ is independent from W_1, W_2, W_3, $\widetilde{W}_{12}$, $\widetilde{W}_{23}$. We set $W = W_1 \oplus W_2 \oplus W_3 \oplus \widetilde{W}_{12} \oplus \widetilde{W}_{23} \oplus \widetilde{\widetilde{W}}_{23}$. □

Corollary 13. *Let $X.W$ be the X-orbit of $W \in \mathbf{Gr}_r(V)$ with corresponding label $(a_1, a_2, a_3, a_{12}, a_{13}, a_{23})$. Then the dimension of $X.W$ is given by*

$$\dim X.W = \sum_{i=1}^{3} a_i(n_i - a_i) + \sum_{1 \leq i < j \leq 3} (a_{ij} - a_i - a_j)(n_i + n_j - a_{ij} + a_i + a_j).$$

If $X.W'$ is the X-orbit of another subspace $W' \in \mathbf{Gr}_r(V)$ labelled by the 6-tuple $(b_1, b_2, b_3, b_{12}, b_{13}, b_{23})$. Then $X.W' \leq X.W$ if and only if $b_i \geq a_i$ and $b_{ij} \geq a_{ij}$ for all i and j.

Proof. We define the subvariety

$$F \subset \prod_{i=1}^{3} \mathbf{Gr}_{a_i}(V_i) \times \prod_{1 \leq i < j \leq 3} \mathbf{Gr}_{a_{ij}}(V_i \oplus V_j)$$

consisting of all those 6-tuples $(W_1, W_2, W_3, W_{12}, W_{13}, W_{23})$ such that

$$W_{ij} \cap V_i = W_i \text{ and } W_{ij} \cap V_j \text{ for } 1 \leq i < j \leq 3.$$

As in the proof of Corollary 9, we can define a map from $\varphi : F_1 \to X.W$ where F_1 is an open subset of F consisting of those 6-tuples with $\dim(W_{12} + W_{13} + W_{23}) = \dim W$. In particular, this leads to $\dim X.W = \dim F$, and the statement about containment of orbits follows similarly as in the proof of Corollary 9. Thus, we omit some details concentrate on finding $\dim F$.

The condition $W_{ij} \cap V_i = W_i$ can be split into two conditions: $W_i \leq W_{ij}$ and $\dim W_{ij} \cap V_i \leq a_i$. The second condition is open. Thus, we define $\overline{F}$ consisting of 6-tuples $(W_1, W_2, W_3, W_{12}, W_{13}, W_{23})$ with $W_i \leq W_{ij}$ and $W_j \leq W_{ij}$ for $1 \leq i < j \leq 3$, and we calculate $\dim \overline{F}$.

Let ρ be the projection of $\overline{F}$ to the coordinates (W_1, W_2, W_{12}). The image of ρ is irreducible (we describe it in a moment), and so we can apply the standard result about dimensions of fibres, domain and image: over an open subset the dimension of the domain equals the dimension of each fibre plus the dimension of the image [6]. The image of ρ is isomorphic to a set of triples $(W_1, W_2, \widetilde{W})$

where $\widetilde{W} \leq W_{12}$ satisfies $\dim \widetilde{W} = a_{12} - a_1 - a_2$ and $\widetilde{W} \cap V_i = 0$ for $i = 1, 2$. This is the set of triples studied in Corollary 9 (in Corollary 9, the notation a_3 equaled $\dim W - a_1 - a_2$; here we would replace a_3 with $a_{12} - a_1 - a_2$). Therefore $\dim \rho(F) = a_1(n_1 - a_1) + a_2(n_2 - a_2) + (a_{12} - a_1 - a_2)(n_1 + n_2 - a_{12} + a_1 + a_2)$.

The fibres of ρ consist of triples (W_3, W_{13}, W_{23}) subject to the remaining conditions. We can break these triples down further using projections and fibres. The W_3 component leads to $\mathbf{Gr}_{a_3}(V_3)$. The W_{13} component leads to the set of all subspace $\overline{W_{13}}$ of dimension $a_{13} - a_1 - a_3$ in $V_1 \oplus V_3 / W_1 \oplus W_3$. This has dimension $\dim \mathbf{Gr}_{a_{13}-a_1-a_3}(V_1 \oplus V_3/W_1 \oplus W_3)$ and W_{23} leads to $\dim \mathbf{Gr}_{a_{23}-a_2-a_3}(V_2 \oplus V_3/W_2 \oplus W_3)$. Adding the dimensions of all these varieties, we get the desired result. □

4 Many factor Levi subgroups

We use similar notation as in the previous sections. Note, that the two cases under consideration in this section are essentially equivalent: the action of X on V has a dual action on the dual space V^*, and 1-spaces in V correspond to $(n-1)$-spaces in V^*. Thus, we restrict our attention to the case where P is the stabilizer of a 1-space. Let $V = V_1 \oplus \cdots \oplus V_m$, $X = \mathrm{GL}(V_1) \cdots \mathrm{GL}(V_m)$ and let $\pi_i : V \to V_i$ be the natural projection.

Lemma 14. *Two* 1*-spaces* W *and* W' *are in the same* X*-orbit if and only if for all* i *we have* $\dim W \cap \bigoplus_{j \neq i} V_j = \dim W' \cap \bigoplus_{j \neq i} V_j$*. This condition is also equivalent to* $\dim \pi_i W = \dim \pi_i W'$ *for all* i*.*

Proof. This follows from Lemma 3 and induction on m. □

Theorem 15. *The* X*-orbits on* G/P *are parameterized by a collection of* m*-tuples* $(a_1, \ldots, a_m)$ *where* $a_i = \dim W \cap (\bigoplus_{j \neq i} V_j)$*.*

An m*-tuple* $(a_1, \ldots, a_m)$ *corresponds to an* X*-orbit if and only if* $a_i \in \{0, 1\}$ *for all* i*, with at least one* a_i *not equal to* 1*.*

Proof. The first statement follows from Lemma 14. For the second statement, if W is a 1-space, then it is clear that a_i equals 0 or 1. Conversely, suppose we are given an m-tuple of $(a_1, \ldots, a_m)$ with each a_i equal to 0 or 1. For each i, we pick $v_i \in V_i$ with the condition $v_i = 0 \iff a_i \neq 0$. Then let W be the 1-space spanned by $v_1 + \cdots + v_m$. □

Corollary 16. *Let* $W \in G/P$ *have label* $(a_1, \ldots, a_m)$ *and orbit* $X.W$*. Let* $i_1, \ldots, i_s$ *be those indices such that* $a_i = 0$*. Then*

$$\dim X.W = \dim \mathbb{P}(V_{i_1} \oplus \cdots \oplus V_{i_s}).$$

If $W' \in G/P$ *has label* $(b_1, \ldots, b_m)$ *and orbit* $X.W'$ *then* $X.W' \leq X.W$ *if and only if* $b_i \geq a_i$

Proof. This result follows in the same manner as Corollaries 9 and 13, but it is perhaps easier to understand if we translate the statements to projections instead of intersections. The $X.W$ orbit is dense in $V_{i_1} \oplus \cdots \oplus V_{i_s}$ since the projection of W to each V_i is nontrivial. This implies that the closure of $X.W$ will contain $X.W'$ if W' has nontrivial projections to a subset of $V_{i_1}, \ldots, V_{i_s}$. $\square$

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