Jordan Blocks of Richardson Classes in the Classical Groups and the Bala-Carter Theorem

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Abstract

This paper provides new, relatively simple, proofs of some important results about unipotent classes in simple linear algebraic groups. We derive the formula for the Jordan blocks of the Richardson class of a parabolic subgroup of a classical group. This result was originally due to Spaltenstein. Secondly, we derive the description of the natural partial order of unipotent classes in the general linear group in terms of their Jordan blocks. This result was originally due to Gerstenhaber. Finally, we obtain a proof of the Bala-Carter Theorem which holds even in certain bad characteristics. This proof requires the prior knowledge of the number of unipotent classes, unlike the original proofs due to Bala, Carter and Pommerening.

Keywords: classical groups, unipotent classes, Jordan blocks, Richardson classes, partial order of unipotent classes, Bala-Carter Theorem.

1 Introduction

In 1974, Richardson [16] made a vital contribution to the study of unipotent classes in a reductive algebraic group by associating to each parabolic subgroup a unipotent class. We give now a precise statement of Richardson's result and discuss below some of its implications.

All algebraic groups in this paper are linear algebraic groups defined over a fixed algebraically closed field k of arbitrary characteristic p.

Richardson's Theorem. *Let G be a connected reductive group, P a parabolic subgroup with unipotent radical Q and Levi factor L. The following hold:*

- (i) *There exists a unique unipotent G-class C such that $C \cap Q$ is open and dense in Q.*
- (ii) *$C \cap Q$ forms a single P-class.*
- (iii) *If $u \in C \cap Q$ then $C_G(u)^\circ = C_P(u)^\circ$; these centralizers have dimension $\dim L$.*
- (iv) *Let Z be the center of G, let Q' be the derived subgroup of Q. Then $\dim L/Z \geq \dim Q/Q'$.*

We call C and $C \cap Q$ the **Richardson G-class** and **Richardson P-class** respectively.

For many questions it is of fundamental importance to be able to find the Jordan block sizes of a unipotent class. The next result indicates how to do this for Richardson classes, but first we recall some standard terminology.

A **partition** of n is a sequence of natural numbers which add to n; for convenience we often assume that the sequence is weakly decreasing. We write a partition λ as $(\lambda_1, \lambda_2, \dots)$ (here we assume that $\lambda_1 \geq \lambda_2 \geq \dots$), or as $(1^{c(1)}, 2^{c(2)}, 3^{c(3)}, \dots)$ where $c(x)$ is the multiplicity of x in λ. We call each λ_i a part of λ, however any statement involving the number of parts with some property refers to the number of indices i such that the part λ_i has the property. The **dual** of λ is a partition of n, which we write as λ^*, where λ_i^* equals the number of parts of λ which are $\geq i$ (i.e. λ_i^* equals the size of the set $\{j \mid \lambda_j \geq i\}$). One may show that $(\lambda^*)^* = \lambda$.

Given a parabolic subgroup of GL_n, SO_{2n}, SO_{2n+1}, or Sp_{2n} we write the Levi factor as $\mathrm{GL}_{n_1} \dots \mathrm{GL}_{n_s} \mathrm{Cl}_m$ where Cl_m is one of 1, SO_{2m}, SO_{2m+1}, or Sp_{2m} and we use the following conventions. In all cases if $m = 0$ we set $\mathrm{Cl}_m = 1$. If $G = \mathrm{GL}_n$ then $m = 0$. Otherwise we fix a Dynkin diagram Δ for G and label the nodes $\alpha_1, \dots, \alpha_n$ as in [3]. Let J be a subset of Δ such that P is conjugate to the standard parabolic associated with J. If $G \in \{\mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}\}$ and $\alpha_n \notin J$ then let $m = 0$. If $G = \mathrm{SO}_{2n}$ and J contains neither α_{n-1} or α_n then let $m = 1$ and let $\mathrm{Cl}_m = \mathrm{SO}_{2m} \cong \mathrm{GL}_1$. (This choice for m might strike some as unusual. In fact, Theorem 1 and the formulas given in Table 1 are still valid if, instead of $m = 1$, one sets $m = 0$, but some of the proofs we give below would require additional cases). If $G = \mathrm{SO}_{2n}$ and J contains exactly one of α_{n-1} or α_n then let $m = 0$. In the remaining cases let m be the largest integer such that such that $\alpha_{n-m+1}, \dots, \alpha_n$ are contained in J and let Cl_m denote SO_{2m}, SO_{2m} or Sp_{2m} according as G is SO_{2n}, SO_{2n+1} or Sp_{2n} respectively. Now $L = \mathrm{GL}_{n_1} \dots \mathrm{GL}_{n_s} \mathrm{Cl}_m$ with the n_i and m uniquely determined. In this way, each parabolic subgroup of G determines a partition $n = n_1 + \dots + n_s + m$. We call this the **Levi partition** of P and write it either as $\Lambda = (n_1, \dots, n_s) \oplus m$ or $\Lambda = (1^{c(1)}, 2^{c(2)}, \dots) \oplus m$ where $c(x)$ is the multiplicity of x in the n_i and the notation "$\oplus$" indicates that the partition of n is given by an ordered pair consisting of $(n_1, \dots, n_s)$ and the number m.

Each of the groups GL_n, SO_{2n}, SO_{2n+1}, Sp_{2n} has a natural module, of dimension n, $2n$, $2n+1$, $2n$ respectively. In this paper all Jordan blocks will be with respect to the natural module. If u is an element of one of these groups we abbreviate the phrase "let λ be the partition consisting of the Jordan block sizes of u" with the shorter phrase "let λ be the Jordan blocks of u."

The first goal of this paper is to provide a new proof of the following result.

Theorem 1. *Let G be one of GL_n, SO_{2n}, SO_{2n+1}, Sp_{2n} defined over k. We exclude the case SO_{2n+1} if $\operatorname{char} k = 2$. Let P be a parabolic subgroup of G, let Λ be the Levi partition of P, define $\psi(\Lambda)$ as in Table 1, and let λ be the Jordan blocks of the Richardson class of P. Then λ equals $\psi(\Lambda)^*$, the dual of $\psi(\Lambda)$.*

Remarks 1.1. (a) This result is originally due to Spaltenstein [19, II.7.4]. Spaltenstein's formulas appear rather different from those presented here and have some minor typographical mistakes. Spaltenstein also determines the extra information needed to parameterize the unipotent classes in bad characteristic. (b) To extend this result to the case of $G = \mathrm{SO}_{2n+1}$ and $p = 2$, one calculates $\psi(\Lambda)$ using the formula for Sp_{2n} and lets λ equal $\psi(\Lambda)^* \oplus 1$. (c) In the statement of Theorem 1 we allow P to equal a Borel subgroup or G itself, thus recovering the Jordan blocks of regular elements or of the identity respectively.

Table 1: Jordan blocks of a Richardson class

$$\Lambda = (1^{c(1)}, 2^{c(2)}, \dots) \oplus m, \text{ a partition of } n$$

		$G = \mathrm{GL}_n,\ \psi$ is the identity	
		$G = \mathrm{SO}_{2n},\ p \neq 2$	
$\psi(m)$	$=$	$2m$	
$\psi(j^{c(j)})$	$=$	$j^{2c(j)}$	if j is even or $j \leq 2m$
$\psi(j^{c(j)})$	$=$	$j+1,\ j^{2c(j)-2},\ j-1$	if j is odd and $j > 2m$
		$G = \mathrm{SO}_{2n},\ p = 2$	
$\psi(m)$	$=$	$2m$	
$\psi(j^{c(j)})$	$=$	$j^{2c(j)}$	if j is even
$\psi(j^{c(j)})$	$=$	$j+1,\ j^{2c(j)-2},\ j-1$	if j is odd and $j \leq 2m$
$\psi(j^{c(j)})$	$=$	$j+1,\ j-1$	if j is odd, $j > 2m$ and $c(j) = 1$
$\psi(j^{c(j)})$	$=$	$(j+1)^2,\ j^{2c(j)-4},\ (j-1)^2$	if j is odd, $j > 2m$ and $c(j) \geq 2$
		$G = \mathrm{SO}_{2n+1},\ p \neq 2$	
$\psi(m)$	$=$	$2m+1$	
$\psi(j^{c(j)})$	$=$	$j^{2c(j)}$	if j is odd and $j > 2m+1$
$\psi(j^{c(j)})$	$=$	$j^{2c(j)}$	if $j \leq 2m+1$
$\psi(j^{c(j)})$	$=$	$j+1,\ j^{2c(j)-2},\ j-1$	if j is even and $j > 2m+1$
		$G = \mathrm{Sp}_{2n}$	
$\psi(m)$	$=$	$2m$	
$\psi(j^{c(j)})$	$=$	$j^{2c(j)}$	if j is even or $j \geq 2m$
$\psi(j^{c(j)})$	$=$	$j+1,\ j^{2c(j)-2},\ j-1$	if j is odd and $j < 2m$

The next result describes the partial order of unipotent classes using the partial order on partitions. Given two unipotent classes C_1 and C_2 we define $C_1 \leq C_2$ if C_1 is a subset of the closure of C_2. Given two partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ we define $\lambda \leq \mu$ if for all $j \geq 1$ we have $\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i$. This is called the **dominance** order. We note that $\lambda \leq \mu$ if and only if $\lambda^* \geq \mu^*$.

The second goal of this paper is to provide a new proof of the following result.

Theorem 2. *Let C_λ and C_μ be two unipotent classes in GL_n with λ and μ the Jordan blocks of C_λ and C_μ respectively. Then $\lambda \leq \mu$ if and only if $C_\lambda \leq C_\mu$.*

This result is originally due to Gerstenhaber [5]. There are similar results for the other classical groups, see [6, 7, 19].

We now introduce the necessary terminology to state the Bala-Carter Theorem.

Let G be a reductive group with root system Φ and root base Δ. Fix $J \subseteq \Delta$ and let P be the standard parabolic subgroup corresponding to J. Let $\beta \in \Phi$ and write $\beta = \sum_{\alpha \in \Delta} n_\alpha\, \alpha$. The P-height of β is defined to be $\sum_{\alpha \in \Delta - J} n_\alpha$.

Let Z be the center of G, let L be a Levi factor for P, Q the unipotent radical of P, and $\Phi(Q)$ the roots of Q. We say P is **distinguished** if $\dim L/Z$ equals the number of roots in $\Phi(Q)$ with P-height equal to 1.

By Richardson's Theorem (iv), we have, for all P, that $\dim L/Z \geq \dim Q/Q'$. If P is distinguished then

$$\dim L/Z(G) = \dim Q/Q'. \tag{1}$$

The converse holds provided $p \neq 2$ if the Dynkin diagram of G contains double bonds, and provided $p \neq 3$ if the Dynkin diagram of G contains triple bonds, see [1, 2].

The work in [2, 4] takes Equation 1 as the definition of distinguished, but then applies this definition only with the restrictions on p just described. Thus, the definition we have given here takes the usual list of distinguished parabolics and uses this same list in all characteristics. We refer the reader to [2, 4] for a list of the distinguished parabolics (note however that [4] has a mistake in the second formula for groups of type D_n).

A **Levi subgroup** of G is a Levi factor of a parabolic subgroup of G. A unipotent element $u \in G$, contained in a Levi subgroup L, is **distinguished** in L if u is not contained in any proper Levi subgroup of L. If L equals G and G has trivial center this is equivalent to having $C_G(u)$ contain no non-trivial torus (see Lemma 6.1 below).

For a reductive group G let BC-pairs(G) denote the pairs (L, P) where L is a Levi subgroup of G and P is a distinguished parabolic subgroup of L. Let ψ (or ψ_G) denote the map from such pairs to unipotent classes in G obtained by extending the Richardson L-class of P to a G-class.

The third goal of this paper is to prove the following result.

Theorem 3. *Let G be a simple algebraic group and let $\psi = \psi_G$ be as just defined. The following hold:*

(i) *If X is a Levi subgroup of G then* BC-pairs$(X) \subseteq$ BC-pairs(G) *and the following diagram commutes:*

$$\begin{tikzcd}
\text{X-classes in BC-pairs(X)} \arrow[r, "\psi_X"] \arrow[d] \arrow[dr, phantom, "\circ"] & \text{unipotent classes in X} \arrow[d] \\
\text{G-classes in BC-pairs(G)} \arrow[r, "\psi"] & \text{uniptotent classes in G}
\end{tikzcd}$$

where the vertical maps extend an X-class to the corresponding G-class.

(ii) *Let $\psi(L, P) = C$ and $u \in C \cap L$. Then u is distinguished in L.*

(iii) *The map ψ is injective. It is a bijection except in the following cases: $G \in \{B_n, C_n, D_n\}$ and $p = 2$; (G, p) is one of $(E_7, 2)$, $(E_8, 2)$, $(E_8, 3)$, $(F_4, 2)$ or $(G_2, 3)$ in which cases there are 1, 4, 1, 4 and 1 extra classes respectively.*

Remarks 1.2. (a) This result, with the requirement that $p = 0$ or $p > 3(h - 1)$, where h is the Coxeter number, is due to Bala and Carter [2]. It was extended to all good primes by Pommerening [15]. The statement in part (iii) extends this work by showing that ψ a bijection when $(G, p) \in \{(E_6, 2), (E_6, 3), (E_7, 3), (E_7, 5), (E_8, 5), (F_4, 3), (G_2, 2)\}$ and by showing that ψ is injective in the remaining cases. The version we have stated here is known to specialists, c.f. [11], but its proof does not seem to appear in print. Finally, the proof given in the present paper (see Section 6) requires *a priori* knowledge of the number of unipotent classes for each simple algebraic group. The standard proof (as in [4, 15]) of the slightly weaker result constructs the inverse of ψ directly and is independent of other classifications. (b) Although part (i) is obvious, we state it here to bring attention to some of the following point. Since Jordan blocks depend upon a fixed module they are not always useful for comparing unipotent classes in X with unipotent classes in G. However, by part (i), such comparisons are easy using Bala-Carter. (c) If X is a maximal rank reductive subgroup of G which is not a Levi subgroup, one may often restrict to a subset of BC-pairs(X) and still obtain a commutative diagram like that in part (i). (d) Parts (i) and (iii) show that in most cases the intersection of a unipotent G-class with a Levi subgroup forms a single unipotent class for the Levi subgroup. If this is not the case then $G = E_r$, L is of type D_n and the unipotent class is of type A_{n-1}.

2 Recollections and Conventions

Throughout this section G is one of GL_n, SO_{2n}, SO_{2n+1}, Sp_{2n} and V is the natural module for G. Throughout this section (and the next) we exclude the case $G = \mathrm{SO}_{2n+1}$ with $p = 2$.

Remarks 2.1. Totally singular and nonsingular subspaces. Let G be one of the groups SO_{2n}, SO_{2n+1}, Sp_{2n} and let β and φ be the associated bilinear form and quadratic form. A subspace $W \leq V$ is **totally singular** if $\varphi\big|_W$ is identically zero (which implies that $\beta\big|_{W\times W}$ also equals zero); it is **nonsingular** if $\beta\big|_{W\times W}$ has trivial radical. If W is nonsingular then $\mathrm{Cl}(W)$ denotes the classical group of the same type as G defined on W using $\beta\big|_{W\times W}$ or $\varphi\big|_W$, as appropriate.

If G equals GL_n and W is any subspace of V then we consider W to be both totally singular and nonsingular and we define $\mathrm{Cl}(W)$ to equal $\mathrm{GL}(W)$.

Remarks 2.2. Parabolic subgroups and natural flags. A **flag** is a sequence of nested subspaces of V. Let f be the flag $0 = W_0 < W_1 < \cdots < W_\ell = V$. Then f has length ℓ and is **totally singular** if for each i either W_i or $W_i^\perp$ is totally singular. A subgroup of G is parabolic if and only if it is the stabilizer of a totally singular flag. Let $L = \mathrm{GL}_{n_1} \cdots \mathrm{GL}_{n_s} \mathrm{Cl}_m$ be the Levi factor of a parabolic subgroup P with Cl_m defined as in Section 1. A natural flag for P is defined inductively as follows. If $0 = W_0 < \cdots < W_{i-1}$ have been defined, with $i \leq s$, then we let W_i be a totally singular subspace, stabilized by P, with $\dim W_i/W_{i-1} = n_i$. For $s < i < 2s + 1$ we let $W_i = W_{2s+1-i}^\perp$. Note that in certain cases we have $W_s^\perp = W_s$ and in the case $G = \mathrm{GL}_n$ we have $W_i = V$ for all $i \geq s$. In these cases the indexing of the W_i subspaces just used is *not* the indexing we will use for discussing the length ℓ. The length of the natural flag equals s if $G = \mathrm{GL}_n$; the length equals $2s$ if $m = 0$ and $G \in \{\mathrm{SO}_{2n}, \mathrm{Sp}_{2n}\}$; and the length equals $2s + 1$ if $m \geq 1$ or $G = \mathrm{SO}_{2n+1}$. The unipotent radical of a parabolic equals the set of elements which act trivially upon each factor in a natural flag.

Remarks 2.3. Parity conditions of Jordan blocks. In the classical groups the unipotent classes are described using partitions. We will mention only a few facts here and refer the reader to [4, 8, 19, 24] for more complete information. Let G equal SO_{2n}, SO_{2n+1} or Sp_{2n}, let C be a unipotent class of G and let λ be equal the Jordan blocks of C. The **parity conditions** on λ refer to the following requirements: if G is orthogonal and $p \neq 2$ then each even part of λ has even multiplicity; if $G = \mathrm{Sp}_{2n}$ and p is arbitrary or $G \in \{\mathrm{SO}_{2n}, \mathrm{SO}_{2n+1}\}$ and $p = 2$ then each odd part of λ, greater than 1, has even multiplicity; if $G = \mathrm{SO}_{2n}$ then λ has an even number of parts.

Remarks 2.4. Dimension of centralizer formulas. Let u be a unipotent element with Jordan blocks λ and let $C_G(u)$ be the centralizer of u. We recapitulate some formulas for $\dim C_G(u)$ here.

If $G = \mathrm{GL}_n$ then $\dim C_G(u) = \sum (\lambda_i^*)^2$ (see [4]).

We assume now that $G \neq \mathrm{GL}_n$. Let $\mathrm{op}(\lambda)$ be the number of odd parts of λ and define $\delta_m(\lambda)$ as follows:

$$\begin{array}{ll} \text{Case I. (orthogonal, } p \neq 2): & \delta_m(\lambda) = -\operatorname{op}(\lambda) \\ \text{Case II. (orthogonal, } p = 2): & \delta_m(\lambda) = \operatorname{op}(\lambda) - 2\lambda_1^* \\ \text{Case III. (symplectic)}: & \delta_m(\lambda) = \operatorname{op}(\lambda) \end{array}$$

Let $\delta(u)$ be defined via $\dim C_G(u) = \frac{1}{2}(\sum_{i\geq 1}(\lambda_i^*)^2 + \delta(u))$. If $p \neq 2$ then $\delta(u) = \delta_m(\lambda)$ (see [4]).

Let $p = 2$, let G equal one of SO_{2n} or Sp_{2n}. Following Spaltenstein [19, I.2.8], we define an index ε as follows. Fix a part x of λ. Let $\varepsilon(x) = -1$ if x is odd and let $\varepsilon(x) = 1$ if x is even with odd multiplicity. If x is even and has even multiplicity let $\varepsilon(x) = 0$ if $\beta\big((u-1)^{x-1}v, v\big) = 0$ for all $v \in \ker(u-1)^x$ and let $\varepsilon(x) = 1$ otherwise. Let $\mathrm{sp}(u)$ ("singular parts") be the number of parts x of λ with $\varepsilon(x) = 0$.

For $p = 2$ and we have $\delta(u) = \delta_m(\lambda) + 2\,\mathrm{sp}(u)$ (see [12, 19]), thus in all cases we have $\delta(u) \geq \delta_m(\lambda)$.

Remarks 2.5. Jordan chains. Let $u \in G$ be unipotent. A **Jordan chain** of u is a finite sequence of vectors $\{v_i\}_{i=1}^{\ell}$ in V such that $v_1 \neq 0$ and $(u-1)v_i = v_{i-1}$ for all $i \geq 1$ (where $v_0 = 0$). The existence and uniqueness of the Jordan canonical form is equivalent to the existence of a basis consisting of Jordan chains of unique lengths.

Lemma 2.6. *Let V be a vector space, $u \in \mathrm{GL}(V)$ be a unipotent element, d be the size of the largest Jordan block of u. Let W be a u-stable subspace of V such that u has exactly one Jordan block on W and such that this block has size equal to d. Then W has a u-stable direct complement.*

Proof. Let u act on V/W and fix a basis of V/W consisting of Jordan chains $\{v_i^j + W\}_{i=1}^{\ell_j}$ where for each $i, j \geq 1$ we have $(u-1)v_i^j + W = v_{i-1}^j + W$ and we set $v_0^j + W = 0 + W$. Let j be fixed, let $\ell = \ell_j$ and let $v_i = v_i^j$. Let $w = (u-1)^{\ell} v_{\ell}$. Since $(u-1)^{\ell} v_{\ell} + W = v_0 + W = W$ we see that $w \in W$. Since d is the largest Jordan block of u, we have that $\ker(u-1)^d = V$, whence $w \in \ker\big((u-1)\big|_W\big)^{d-\ell}$. Since u has one Jordan block on W we have for each $i \geq 0$ that $(u-1)^i(W) = \ker\big((u-1)\big|_W\big)^{d-i}$, whence $w \in (u-1)^{\ell}(W)$. Let $w_{\ell} \in W$ such that $(u-1)^{\ell} w_{\ell} = w$. For $i \geq 0$ let $\widetilde{v}_{\ell-i} = (u-1)^i(v_{\ell} - w_{\ell})$. Note that $(u-1)\widetilde{v}_1 = \widetilde{v}_0 = (u-1)^{\ell}(v_{\ell} - w_{\ell}) = w - w = 0$.

We apply this same procedure to each Jordan chain $\{v_i^j + W\}_{i=1}^{\ell_j}$. Note that $v_i^j + W = \widetilde{v}_i^j + W$ for each i, j, whence the $\widetilde{v}_i^j + W$ form a basis for V/W. Let K be the span of all $\widetilde{v}_i^j$. Then $\dim K = \dim V/W$. Furthermore $K \cap W = 0$ as the projection $\pi : V \to V/W$ sends K onto V/W. Finally, K is u-stable since $(u-1)\widetilde{v}_i^j = \widetilde{v}_{i-1}^j$ for $i \geq 2$ and $(u-1)\widetilde{v}_1^j = 0$. $\square$

3 Proof of Theorem 1

Throughout this section we use the following notation. The group G is one of GL_n, SO_{2n+1}, SO_{2n}, Sp_{2n}; we exclude the case SO_{2n+1} when $p = 2$. Let V be the natural module for G and β the bilinear form on V if $G \neq \mathrm{GL}_n$. Let P be a proper parabolic subgroup (we allow $P = G$ in the statement of Theorem 1, but if this holds there is nothing to prove). Let Q be the unipotent radical of P, let $L = \mathrm{GL}_{n_1} \ldots \mathrm{GL}_{n_s} \mathrm{Cl}_m$ be a Levi factor with Cl_m determined as in Section 1, fix a natural flag $0 = W_0 < \cdots < W_{\ell} = V$ for P, denote this flag by f and let ℓ be the length of f. Let $\Lambda = (n_1, \ldots, n_s) \oplus m = (1^{c(1)}, 2^{c(2)}, \ldots) \oplus m$ be the Levi partition Λ.

For any $g \in Q$ we let $\lambda(g) = (\lambda_1(g), \lambda_2(g), \ldots)$ be the Jordan blocks of g. We fix $u \in Q$ which represents the Richardson P-class. We fix $\lambda = (\lambda_1, \lambda_2, \ldots) = \lambda(u)$ and $\mu = (\mu_1, \mu_2, \ldots) = \psi(\Lambda)^*$. To prove Theorem 1 we show that $\lambda = \mu$.

If $G \in \{\mathrm{GL}_n, \mathrm{SO}_{2n+1}\}$ let $r = 1$ and if $G \in \{\mathrm{SO}_{2n}, \mathrm{Sp}_{2n}\}$ let $r = 2$. We show first that $(\lambda_1, \ldots, \lambda_r) = (\mu_1, \ldots, \mu_r)$, and then induct on the dimension of V.

Lemma 3.1. *Let $g \in Q$, let X be a g-stable subspace of V and let $g\big|_X$ have r Jordan blocks. Then* $\dim X \leq \sum_{i=1}^{\ell} \min\{r, \dim W_i/W_{i-1}\}$ *with equality holding* *if and only if* $\dim X \cap W_j = \sum_{i=1}^{j} \min\{r, \dim W_i/W_{i-1}\}$ *for all* $j \geq 1$. *In particular we have* $\sum_{i=1}^{r} \lambda_i \leq \sum_{i=1}^{\ell} \min\{r, \dim W_i/W_{i-1}\}$.

Proof. (Here r can be any positive integer.) This is elementary linear algebra and induction together with the fact that g acts trivially upon each factor W_i/W_{i-1}. □

Corollary 3.2. *We have* $\sum_{i=1}^{\ell} \min\{r, \dim W_i/W_{i-1}\} = \mu_1 + \cdots + \mu_r$ *and* $(\lambda_1, \ldots, \lambda_r) \leq (\mu_1, \ldots, \mu_r)$ *(dominance order).*

Proof. We begin by giving a formula for $(\mu_1, \ldots, \mu_r)$. Recall that μ_1 equals the number of parts in $\psi(\Lambda)$ and μ_2 equals the number of parts in $\psi(\Lambda)$ which are greater than or equal to 2. It is easy to verify the following formulas for μ_1 and μ_2.

(i) $G = \mathrm{GL}_n$ or $G = \mathrm{SO}_{2n+1}$ and $p \neq 2$. We have $\mu_1 = \ell$.
(ii) $G = \mathrm{SO}_{2n}$ and $p \neq 2$. If $m = 0$ and $c(1) \geq 1$ then $(\mu_1, \mu_2) = (\ell - 1, \ell - 2c(1) + 1)$. Otherwise we have $(\mu_1, \mu_2) = (\ell, \ell - 2c(1))$.
(iii) $G = \mathrm{SO}_{2n}$ and $p = 2$. If $m = 0$ and $c(1) \geq 2$ then $(\mu_1, \mu_2) = (\ell - 2, \ell - 2c(1) + 2)$. If $m = 0$, $c(1) = 1$, or $m \geq 1$, $c(1) \geq 1$ then $(\mu_1, \mu_2) = (\ell - 1, \ell - 2c(1) + 1)$. If $c(1) = 0$ then $(\mu_1, \mu_2) = (\ell, \ell)$.
(iv) $G = \mathrm{Sp}_{2n}$. If $m \geq 1$ and $c(1) \geq 1$ then $(\mu_1, \mu_2) = (\ell - 1, \ell - 2c(1) + 1)$. Otherwise $(\mu_1, \mu_2) = (\ell, \ell - 2c(1))$.

For example, suppose $G = \mathrm{SO}_{2n}$, $p = 2$, $m = 0$ and $c(1) \geq 2$. Note that $\ell = 2s$. Then the number of parts of Λ is $s = c(1) + c(2) + \ldots$. We see that $\psi(1^{c(1)}) = 2^2, 1^{2c(1)-4}$, whence ψ has taken $c(1)$ parts of Λ and produced $2c(1)-2$ parts of $\psi(\Lambda)$. For $j \geq 2$ we have that $\psi(j^{c(j)})$ equals $2c(j)$ parts of $\psi(\Lambda)$. Thus, the number of parts of $\psi(\Lambda)$ is $2c(1) - 2 + 2c(2) + 2c(3) + \cdots = 2s - 2$, whence $\mu_1 = \ell - 2$. The number of parts of $\psi(\Lambda)$ strictly greater than 1 equals μ_1 minus the number of 1's. The number of 1's is $2c(1) - 4$ whence $\mu_2 = \ell - 2c(1) + 2$.

We return now to the general argument. Note that $\sum_{i=1}^{\ell} \min\{1, \dim W_i/W_{i-1}\} = \ell$. The previous lemma shows that $\lambda_1 \leq \ell$. If G equals GL_n or SO_{2n+1} we are done. In the remaining cases we have $\sum_{i=1}^{\ell} \min\{2, \dim W_i/W_{i-1}\} = 2\ell - 2c(1) = \mu_1 + \mu_2$. The previous lemma shows that $\lambda_1 \leq \ell$ and that $\lambda_1 + \lambda_2 \leq 2\ell - 2c(1)$. It remains to show that $\lambda_1 \leq \mu_1$ in those case where $\mu_1 < \ell$.

We assume for the remainder of the proof that $\mu_1 < \ell$. Thus we are in one of the following cases: (1) $G = \mathrm{SO}_{2n}$, $p \neq 2$, $m = 0$ and $c(1) \geq 1$, (2) $G = \mathrm{SO}_{2n}$, $p = 2$, $c(1) \geq 1$, (3) $G = \mathrm{Sp}_{2n}$, $m \geq 1$, $c(1) \geq 1$. Note that in each case $c(1) \geq 1$. We assume now that $\lambda_1 = \ell$ and obtain a contradiction in each case. Since $c(1) \geq 1$ and $\lambda_1 + \lambda_2 \leq 2\ell - 2c(1)$ we see that λ_1 has multiplicity 1. Thus, the parity conditions on Jordan blocks (see Remarks 2.3) show that λ_1 cannot be odd if $G = \mathrm{Sp}_{2n}$ or if $p = 2$, and λ_1 cannot be even if $G = \mathrm{SO}_{2n}$ and $p \neq 2$. In case (1) we have $\ell = 2s$, whence $\lambda_1 \neq \ell$. In case (3) we have $\ell = 2s + 1$, whence $\lambda_1 \neq \ell$. In case (2), if $m \geq 1$, then $\ell = 2s + 1$, whence $\lambda_1 \neq \ell$. We now finish obtaining a contradiction in case (2) with the additional assumption that $m = 0$. Since $m = 0$ the natural flag f contains the sequence of subspaces $W_{s-1} < W_s < W_{s+1}$ with $W_s^{\perp} = W_s$ and $W_{s+1}^{\perp} = W_{s-1}$. Since $\lambda_1 = \ell = 2s$ the element u has a Jordan chain $v_1, \ldots, v_{2s}$. Since u acts trivially upon each factor in the flag, and since the Jordan chain has as many elements as there are terms in the flag, we see that for each i we have $v_i \in W_i - W_{i-1}$. Let $\widetilde{W} = \langle v_{s+1} \rangle^{\perp} \cap W_s$. Then $\widetilde{W}$ is an $(n-1)$-dimensional totally singular subspace which is stabilized by u. Let $\widetilde{u}$ be the element of $\mathrm{SO}(\widetilde{W}^{\perp}/\widetilde{W})$ induced by u. Then $\widetilde{u}$ is a unipotent element. We will show that $\widetilde{u}$ is nontrivial, contradicting the fact that $\mathrm{SO}_2 = \mathrm{SO}(\widetilde{W}^{\perp}/\widetilde{W})$ is a torus. Note that $v_s, v_{s+1} \in \widetilde{W}^{\perp}$. Let $\widetilde{v}_s$ and $\widetilde{v}_{s+1}$ be the projections of v_s and v_{s+1} to $\widetilde{W}^{\perp}/\widetilde{W}$. Since $v_s - v_{s+1} \notin W_s$

we have $\widetilde{v}_s \neq \widetilde{v}_{s+1}$. But $\widetilde{u}$ takes $\widetilde{v}_s$ to $\widetilde{v}_{s+1}$, whence $\widetilde{u} \neq 1$, establishing our contradiction.

We have shown that if $\mu_1 < \ell$ then $\lambda_1 < \ell$ which finishes the proof for those cases where $\mu_1 = \ell - 1$.

In the final case we have $\mu_1 = \ell - 2$, $G = \mathrm{SO}_{2n}$, $p = 2$ and $m = 0$ and $\ell = 2s$. If $\lambda_1 = \ell - 1$, this would imply that λ_1 has multiplicity 1 and is odd, contradicting the parity conditions on Jordan blocks (see Remarks 2.3). □

Lemma 3.3. *Let ν and τ be two partitions of an even number with $\nu < \tau$ and define $\delta_m(\nu)$, $\delta_m(\tau)$ as in Remarks 2.4. We assume that τ has two parts and that the following parity conditions hold: in case I each even part of ν or τ has even multiplicity; in cases II and III each odd part of ν or τ has even multiplicity; in cases I and II, ν and τ each have an even number of parts. Then*

$$\sum_{i\geq 1}(\nu_i^*)^2 + \delta_m(\nu) > \sum_{i\geq 1}(\tau_i^*)^2 + \delta_m(\tau).$$

Proof. Let $d(\nu^*, \tau^*) = \sum \big((\nu_i^*)^2 - (\tau_i^*)^2\big)$. Then it suffices to prove

$$d(\nu^*, \tau^*) > \delta_m(\tau) - \delta_m(\nu). \tag{2}$$

Since τ has only two parts it is easy to establish the following bounds:

$$\delta_m(\tau) - \delta_m(\nu) \quad \leq \quad \begin{cases} \nu_1^* & \text{Case I.} \\ 2\nu_1^* - 2 & \text{Case II.} \\ 2 & \text{Case III.} \end{cases} \tag{3}$$

Let $\nu^* = (1^{a_1}, 2^{a_2}, \dots)$, $\tau^* = (1^{b_1}, 2^{b_2})$. Since ν and τ are partitions of the same number we have $\sum i a_i = b_1 + 2b_2$. Using this equation to eliminate b_1 we have:

$$d(\nu^*, \tau^*) = \sum (i^2 - i)a_i - 2b_2. \tag{4}$$

Since $\nu < \tau$ we have $\sum a_i \leq b_1 + b_2$, whence:

$$d(\nu^*, \tau^*) \geq \sum (i-2)(i-1)a_i. \tag{5}$$

Note that $\nu_1^* = \max\{\, i \,|\, a_i > 0\}$ and that the terms in Equation 5 are ≥ 0, whence $d(\nu^*, \tau^*) \geq (i-2)(i-1)a_i \geq (i-2)(i-1)$ for each i such that $\nu_1^* \geq i \geq 1$.

Since $\nu < \tau$ we have that $\nu_1^* \geq 2$. We consider cases according to ν_1^*.

Let $\nu_1^* = i \geq 5$. Equation 5 shows that $d(\nu^*, \tau^*) \geq (i-2)(i-1) > 2i - 2$. We have $2i - 2 \geq \max\{\nu_1^*, 2\nu_1^* - 2, 2\}$, whence $2i - 2 \geq \delta_m(\tau) - \delta(\nu)$ by Equation 3, whence Equation 2 holds.

Let $\nu_1^* = i = 4$. Then $d(\nu^*, \tau^*) \geq (i-2)(i-1)a_4 = 6a_4 \geq 6$. Comparing this with Equation 3 we are done except for case II, which we now assume holds. If $a_4 \geq 2$ we are done. If $a_4 = 1$ the smallest part of ν equals 1, and the parity conditions imply that $\mathrm{op}(\nu) \geq 2$. Thus $\delta_m(\tau) - \delta_m(\nu) \leq 4$ and we are done.

Let $\nu_1^* = i = 3$. We may assume that cases I and II do not hold. Now $d(\nu^*, \tau^*) \geq (i-2)(i-1)a_3 = 2a_3$. If $a_3 \geq 2$ we are done. If $a_3 = 1$ then the smallest part of ν equals 1 and the parity conditions imply that $\mathrm{op}(\nu) \geq 2$. Thus $\delta_m(\tau) - \delta_m(\nu) \leq 0 < 2 \leq d(\nu^*, \tau^*)$.

Let $\nu_1^* = i = 2$. Then equation 4 becomes $d(\nu^*, \tau^*) = 2(a_2 - b_2)$. We have $\nu = (a_1 + a_2, a_2)$, $\tau = (b_1 + b_2, b_2)$. We have that $a_1 + 2a_2 = b_1 + 2b_2$ and, since $\nu < \tau$, that $a_1 + a_2 < b_1 + b_2$. This implies that $a_2 > b_2$, whence $d(\nu^*, \tau^*) = 2(a_2 - b_2) \geq 2$. Since $\nu < \tau$ we have that $b_1 \neq 0$. In case I, $b_1 \neq 0$ implies that b_2 is odd, whence $\mathrm{op}(\tau) = 2$, whence $\delta_m(\tau) - \delta_m(\nu) \leq 0$ and we are done. Suppose now that we are in case II or III. Then $\delta_m(\tau) - \delta_m(\nu) = \mathrm{op}(\tau) - \mathrm{op}(\nu) \leq 2$.

If $a_2 - b_2 > 1$ then $d(\nu^*, \tau^*) > 2$ and we are done. Otherwise, let $a_2 - b_2 = 1$. Since $b_1 \neq 0$ we have that b_2 is even, whence a_2 is odd, whence $\operatorname{op}(\nu) = 2$ and $\delta_m(\tau) - \delta(\nu) \leq 0$. $\square$

Corollary 3.4. *Theorem 1 holds if the number of parts of μ is at most r; in particular the theorem holds if P is a Borel subgroup.*

Proof. Let $\mu_1^* = 1$ and note that P is a Borel subgroup. Then $\dim C_G(u) = n$ by Richardson's Theorem (iii). All unipotent elements with centralizer dimension equal to n form a single conjugacy class (the regular class, see [4, 20]). Since unipotent elements with Jordan blocks given by μ have centralizer dimension equal to n, we conclude that $\lambda = \mu$.

We assume now that $r = 2$ and $\mu_1^* = 2$. Whence we have G equal to SO_{2n} or Sp_{2n}. Define $\delta(u)$, $\delta_m(\lambda)$ and $\delta_m(\mu)$ as in Remarks 2.4. Using the formulas for μ in the proof of Corollary 3.2 it is easy to verify that μ satisfies the parity conditions in the previous lemma. For example, suppose $G = \mathrm{SO}_{2n}$, $p \neq 2$. Then $\mu_1^* = 2$ implies that $m = 0$, whence $\ell = 2s$. We consider two cases: $c(1) \geq 1$ and $c(1) = 0$. If $c(1) \geq 1$ then we have $\mu = (2s - 1, 2s - 2c(1) + 1)$; neither part is even and there are two parts since $2s - 2c(1) + 1 > 0$. If $c(1) = 0$ then we have $\mu = (2s, 2s)$; the even part has even multiplicity and there are two parts.

If $\lambda < \mu$ then we claim that the following equalities and inequalities hold, yielding a contradiction:

$$\dim L = \dim C_G(u) = \frac{1}{2}\left(\sum(\lambda_i^*)^2 + \delta(u)\right) \geq \frac{1}{2}\left(\sum(\lambda_i^*)^2 + \delta_m(\lambda)\right) > \frac{1}{2}\left(\sum(\mu_i^*)^2 + \delta_m(\mu)\right) = \dim L. \tag{6}$$

The first equality follows from Richardson's Theorem (iii). The second equality and the first inequality follow from Remarks 2.4. The strict inequality follows from the previous lemma. The final equality follows from a simple calculation involving the Levi partition. For example, suppose $G = \mathrm{SO}_{2n}$, $p \neq 2$. Since $\mu_1^* = 2$ we have $m \leq 1$ and $0 = c(3) = c(4) = \dots$. We consider the case $m = 1$. Then $\Lambda = (1^{c(1)}, 2^{c(2)}) \oplus 1$. We have $\mu^* = \psi(\Lambda) = (1^{2c(1)}, 2^{2c(2)}) \oplus 2$ and $\mu = (2c(2) + 2c(1) + 1, 2c(2) + 1)$ whence $\delta_m(\mu) = -\operatorname{op}(\mu) = -2$. Thus the expression involving μ^* and μ in Equation 6 becomes $4c(2) + c(1) + 1$. Finally, $\dim L = 4c(2) + c(1) + 2 \cdot m^2 - m = 4c(2) + c(1) + 1$. $\square$

Lemma 3.5. (i) *Let $H \leq \mathrm{GL}(V)$ be an algebraic group, U a closed subset of unipotent elements. Let X be the finite partition of U defined by Jordan blocks (an element of X consists of all those $g \in U$ with the same Jordan blocks). Let X inherit a partial order from Jordan blocks and a topology from U (given $x, y \in X$ with Jordan blocks ν and τ respectively, we have $x \leq y$ if and only if $\nu \leq \tau$; a subset $\{x_1, \dots, x_t\} \subseteq X$ is open if and only if $x_1 \cup \dots \cup x_t$ is open in U). Let $O \subseteq X$. If O contains the set $\{y \in X \mid \exists x \in O,\ x \leq y\}$ or the set $\{y \in X \mid \exists x \in O,\ x \geq y\}$ then O is open or closed respectively.*

(ii) *Let $H \leq \mathrm{GL}(V)$ be an algebraic group, let C_ν and C_τ be any two unipotent classes of H with Jordan blocks given by the partitions ν and τ respectively. If $C_\nu \leq C_\tau$ then $\nu \leq \tau$.*

(iii) *Let $Q_{\lambda,r}$ equal the subset of Q consisting of those elements g such that there exists a nonsingular g-stable subspace X with $g\big|_X$ having Jordan blocks λ_1, …, λ_r. If $Q_{\lambda,r}$ is nonempty then it contains the Richardson P-class in Q.*

Sketch of proof. Part (i). It is easy to show that for each $j, b \geq 0$ the subset $\{g \in U \mid \dim\ker(g-1)^j \geq b\}$ is closed in U. (The reader may prove this by characterizing rank in terms of determinants of minors of a matrix or, as in [19, III.8.1], by using the upper semi-continuity of dimension applied to the endomorphism of $U \times V$ given by $(g, v) \mapsto (g, (g-1)^j v)$.) Let $h \in U$. Then for each j, the set of $\{g \in U \mid \dim\ker(g-1)^j \geq \dim\ker(h-1)^j\}$ is closed in U. Let g be in this set, let ν and τ be the Jordan blocks of g and h respectively. Note that $\dim\ker(g-1)^j = \sum_{i=1}^{j} \nu_i^*$ and $\dim\ker(h-1)^j = \sum_{i=1}^{j} \tau_i^*$, whence $\nu^* \geq \tau^*$, whence $\nu \leq \tau$. Thus, the set of $g \in U$ with Jordan blocks $\leq \tau$ is closed in U.

We restate this last conclusion thus: given $x \in X$ the set of elements y with $x \geq y$ is closed in X. Suppose O contains $\{y \in X \mid \exists x \in O,\ x \geq y\}$. Let $O = \{x_1, \ldots, x_t\}$ and let $[x_i]$ denote the set of all elements in X which are $\leq x_i$. It's easy to show that $O = \bigcup [x_i]$ whence O is closed.

Part (ii). Let $u_\nu \in C_\nu$ and $u_\tau \in C_\tau$. Since $C_\nu \subseteq \overline{C_\tau}$ we have that u_ν is contained in any H-stable, closed subset of U that contains u_τ. By part (i), the set of all $g \in U$ with Jordan blocks $\leq \tau$ is closed.

Part (iii). (Here r can be any positive integer.) Every P-stable, nonempty, open subset of Q contains the Richardson P-class. Clearly $Q_{\lambda,r}$ is P-stable; we show that it is open.

Fix a subset $X_0 \subset V$ consisting of r vectors and for each $g \in Q$ define $g^\infty X_0$ to be the subspace of V generated by all $g^i x$ for $i \geq 0$ and $x \in X_0$. The set $O_1^{X_0}$ of g in Q such that $g^\infty X_0$ is nonsingular, is open in Q. (One way to prove this is to characterize nonsingularity of $g^\infty X_0$ in terms of the matrix of $\beta\big|_{g^\infty X_0}$ having a nonzero determinant.) By parts (i) and (ii) the set $O_2^{X_0}$ of g in Q such that $g\big|_{g^\infty X}$ has Jordan blocks equal to $\lambda_1, \ldots, \lambda_r$ is also open in Q. (By part (ii), and the definition of a Richardson class, we have $\lambda(g) \leq \lambda$. Thus $O_2^{X_0}$ contains all $g \in Q$ such that $\lambda(g) \geq \lambda(h)$ for some $h \in O_2^{X_0}$.) The set $Q_{\lambda,r}$ is obtained by intersecting $O_1^{X_0}$ and $O_2^{X_0}$, and taking the union of all such intersections as X_0 ranges over all subsets of r vectors in V. $\square$

Lemma 3.6. *There exists an element $g \in Q$ and a nonsingular subspace X such that $\lambda(g\big|_X) = (\mu_1, \ldots, \mu_r) = (\lambda_1, \ldots, \lambda_r)$.*

Proof. Claim: it suffices to produce a nonsingular subspace $X \leq V$ and a totally singular flag $f_X = \{0 < X_1 < X_2 < \cdots < X_\ell = X\}$ which is a natural flag in X such that $X_i \leq W_i$ for each i and such that if Λ_X is the Levi partition associated with this flag then $\psi(\Lambda_X)^* = (\mu_1, \ldots, \mu_r)$.

Proof of claim. Let P_X be the stabilizer of f_X in $\mathrm{Cl}(X)$ and let g represent the Richardson P_X-class. We may apply Corollary 3.4 to see that $\lambda(g) = (\mu_1, \ldots, \mu_r)$. If $G = \mathrm{GL}_n$ let Y be any direct complement of X and for all other G let $Y = X^\perp$. We identify $\mathrm{Cl}(X)$ as a subgroup of G in the obvious way: given $h \in \mathrm{Cl}(X)$ and $v \in V$ we write $v = x + y$ with $x \in X$ and $y \in Y$ and define hv to equal $hx + y$. Since g acts trivially upon each factor in f_X we see that g acts trivially upon each factor in f, whence $g \in Q$. By Corollary 3.2 we have that $(\lambda_1, \ldots, \lambda_r) \leq (\mu_1, \ldots, \mu_r)$ and by Lemma 3.5(ii) we have that $\lambda(g) \leq (\lambda_1, \ldots, \lambda_r)$, whence equality holds.

The construction of f_X is essentially a matter of picking subspaces of the correct dimension. If $G \neq \mathrm{GL}_n$ we also need this construction to produce totally singular spaces, spaces with appropriate radicals etc. In each case one can choose X_j such that $\dim X_j = \sum_{i=1}^{j} \min\{r, \dim W_i/W_{i-1}\}$ for $1 \leq i \leq \ell$. For example, suppose $G = \mathrm{Sp}_{2n}$. Set $X_0 = 0$. Let $j \geq 1$ such that X_{j-1} has been constructed. If $1 \leq j \leq s$ let X_j be a subspace of W_j which contains X_{j-1} and which is of the proper dimension. If $s+1 \leq j \leq \ell$ we consider first the nonsingular space $X_{\ell-j}^\perp / X_j$. The subspace $X_{j-1}/X_{\ell-j}$ has radical $X_{\ell-(j-1)}/X_{\ell-j}$ which has dimension $d := \min\{r, \dim W_{\ell-(j-1)}/W_{\ell-j}\} = \min\{r, \dim W_j/W_{j-1}\}$. Thus, there exists $\overline{X_j}$, a nonsingular subspace of $X_{\ell-j}^{\perp}/W_{\ell-j}$ which contains $X_{j-1}/X_{\ell-j}$ and which has dimension $\dim X_{j-1}/X_{\ell-j} + d = \sum_{i=1}^{j} \min\{r, \dim W_i/W_{i-1}\} - \dim X_{\ell-j}$. Let X_j be the lift of $\overline{X_j}$ to W_j.

We now sketch how to verify that $\psi(\Lambda_X)^* = (\mu_1, \dots, \mu_r)$.

When G equals GL_n or SO_{2n+1} then $r = 1$ and $\dim X_j/X_{j-1} = 1$ for all j. Thus $L_X = \mathrm{GL}_1 \cdots \mathrm{GL}_1$. If $G = \mathrm{GL}_n$ then $\ell = s$, $\Lambda_X = 1^s$, $\lambda_1(g) = (1^{\ell})^* = \ell = \mu_1$. If $G = \mathrm{SO}_{2n+1}$ then $\ell = 2s + 1$, $\Lambda_X = (1^s) \oplus 0$, $\psi(\Lambda_X) = (1^{2s}) \oplus 1$, $\lambda(g) = \ell = \mu_1$.

When G equals SO_{2n} or Sp_{2n} then $r = 2$. For $1 \leq j \leq s$ we have that $\dim X_j/X_{j-1}$ equals 1 or 2, with 2 occurring $s-c(1)$ times. If $m = 0$ then $\Lambda_X = (2^{s-c(1)}, 1^{c(1)})$, if $m \geq 1$ then $\Lambda_X = (2^{s-c(1)}, 1^{c(1)}) \oplus 1$. It is now relatively simple to verify the desired formula for $\lambda(g)$, although there are a number of cases, as given at the beginning of the proof of Corollary 3.2. For example, if $G = \mathrm{Sp}_{2n}$, $m \geq 1$ and $c(1) \geq 1$ then $\ell = 2s + 1$, $\psi(\Lambda_X) = (2^{2(s-c(1))}, 2, 1^{2c(1)-2}, 0) \oplus 2$ and $\lambda(g) = (2s, 2s + 2 - 2c(1)) = (\ell - 1, \ell - 2c(1) + 1)$. □

Corollary 3.7. *$\lambda = \mu$.*

Proof. If G has rank 1 then P is a Borel subgroup and we are done by Corollary 3.4. We assume now that Theorem 1 is true for each connected classical group whose natural module has dimension strictly less than $\dim V$.

By Lemmas 3.5 and 3.6 there exists a nonsingular u-stable subspace X with $\lambda(u\big|_X) = (\lambda_1, \dots, \lambda_r) = (\mu_1, \dots, \mu_r)$.

If G is symplectic or orthogonal let $Y = X^{\perp}$, otherwise let Y be a u-stable direct complement of X as given by Lemma 2.6.

Recall that we have a fixed natural flag f consisting of subspaces W_i and recall that $r = 1$ if G equals GL_n or SO_{2n+1}. For each $i \geq 0$ let $X_i = W_i \cap X$ and $Y_i = W_i \cap Y$. Let f_X and f_Y be the flags consisting of the X_i and Y_i subspaces respectively. We show first that the flags f_X and f_Y are totally singular and that $W_i = X_i \oplus Y_i$.

If G equals GL_n then the flags f_X and f_Y are automatically totally singular. It remains to show that $W_i = X_i + Y_i$. Let $w_i \in W_i$ and write $w_i = x + y$ for some $x \in X$, $y \in Y$. We have $(u-1)^i w_i = 0$ whence $(u-1)^i x = 0$. Since u has a single Jordan block on X this implies $x \in X_i$, whence $y = w_i - x_i \in W_i$ so $y \in Y_i$.

Suppose that G is an orthogonal or symplectic group. For $j \leq s$ we have that X_j is contained in W_j, whence X_j is totally singular. For $j > s$ we wish to show that $X_j = X_{\ell-j}^{\perp}$ where the perpendicular space is taken within X. Since X_j is contained in $W_j = W_{\ell-j}^{\perp}$ we have $X_j \leq X_{\ell-j}^{\perp}$. We prove equality by dimension. We have $\dim X = \lambda_1 + \cdots + \lambda_r = \mu_1 + \cdots + \mu_r$. By Corollary 3.2 this equals $\sum_{i=1}^{\ell} \min\{r, \dim W_i/W_{i-1}\}$, whence by Lemma 3.1, we have $\dim X_j = \sum_{i=1}^{j} \min\{r, \dim W_i/W_{i-1}\}$. Since $\dim X_{\ell-j}^{\perp} = \dim X - \dim X_{\ell-j}$ we have $\dim X_j = \dim X_{\ell-j}^{\perp}$ if and only if $\sum_{i=1}^{j} \min\{r, \dim W_i/W_{i-1}\} = \sum_{i=\ell-j+1}^{\ell} \min\{r, \dim W_i/W_{i-1}\}$. This last equation holds by definition of the natural flag f. Similar calculations, using the formula $\dim Y_i = \dim W_i + \dim(W_i^{\perp} \cap X) - \dim X$, show that f_Y is totally singular and that $W_i = X_i + Y_i$.

Let P_Y be the stabilizer of f_Y in $\mathrm{Cl}(Y)$. The Levi partition Λ_Y of P_Y is easily calculated using the dimensions of the Y_i. One may verify that Λ_Y equals $(\max\{n_i - r, 0\} \mid 1 \leq i \leq s) \oplus m{-}1$ if $m \geq 1$ and $G \neq \mathrm{SO}_{2n+1}$, and equals $(\max\{n_i - r, 0\} \mid 1 \leq i \leq s) \oplus m$ otherwise. Let C_Y be the Richardson P_Y-class. By induction on $\dim V$ we may apply Theorem 1 to see that $\lambda(C_Y) = \psi(\Lambda_Y)^*$. A case by case calculation shows that $\psi(\Lambda_Y) = (\max\{\mu_i^* - r, 0\} \mid 1 \geq i)$ whence $\psi(\Lambda_Y)^* = (\mu_{r+1}, \mu_{r+2}, \dots)$ (for the reader familiar with Young diagrams, this asserts that removing the first r rows of the diagram for μ is equivalent to removing the first r columns of μ^*).

Let P_X be stabilizer of f_X in $\mathrm{Cl}(X)$. Let Q_X and Q_Y be the unipotent radicals of P_X and P_Y respectively. Let $C \subset Q$ and $C_X \subset Q_X$ be the Richardson P-class and P_X-class respectively. We identify the product $\mathrm{Cl}(X)\,\mathrm{Cl}(Y)$ as a subgroup of G, and Q_XQ_Y as a subgroup of Q. Note that $C \cap (Q_XQ_Y)$ is an open subset of Q_XQ_Y and dense as well since it contains u. It is stable under P_X and P_Y whence it intersects C_XC_Y. Let v be in this intersection. Then $\lambda(C) = \lambda(v) = \lambda(C_XC_Y) = \lambda(C_X) \oplus \lambda(C_Y)$. □

4 Dominance order

Theorem 4. *Every unipotent class in* GL_n *is a Richardson class.*

This is also proven in [19, II.5.14] and in [9, 5.5].

Proof. If a unipotent class has Jordan blocks given by the partition ν then it is the Richardson class of any parabolic with Levi partition equal to ν^*. □

Lemma 4.1. *Let J be an algebraic group acting morphically on a variety X, with the action denoted by $g.x$ for $g \in J$ and $x \in X$. For $O \subseteq X$ and $K \subseteq J$ we denote by $\overline{O}$ the closure of O and by $K.O$ the set $\{g.x \,|\, g \in K,\ x \in O\}$. Let O_1 and O_2 be two subsets of X with $O_1 \subseteq \overline{O_2}$. For any subset $K \subset J$ we have that $K.O_1 \subseteq \overline{K.O_2}$.*

Sketch of proof. Elementary calculations show that

$$K.O_1 \subseteq K.\overline{O_2} = \bigcup_{g\in K} g.\overline{O_2} = \bigcup_{g\in K} \overline{g.O_2} = \overline{\bigcup_{g\in K} g.O_2} = \overline{K.O_2}\,.$$

□

Proof of Theorem 2. By Lemma 3.5(ii) we have that $C_\nu \le C_\tau$ implies $\nu \le \tau$. By transitivity, it suffices to prove the converse under the assumption that τ covers ν, i.e. that there is no partition strictly between τ and ν. We assume this and apply [10, 1.4.10] to conclude that there exist i, j with $\tau_j = \nu_j - 1$ and $\tau_i = \nu_i + 1$ and $\tau_t = \nu_t$ for all $t \neq i, j$. Let $n_1 = \nu_i + \nu_j = \tau_i + \tau_j$. Let C_1 and C_2 be the unipotent GL_{n_1}-classes with Jordan blocks given by (ν_i, ν_j) and (τ_i, τ_j) respectively. Let $n_2 = n - n_1$, let C_0 be the unipotent GL_{n_2} class with Jordan blocks given by all the other parts of τ (which equal all the other parts of ν). View $\mathrm{GL}_{n_1}\mathrm{GL}_{n_2}$ as a subgroup of GL_n and note that C_ν and C_τ are the extensions to GL_n of C_0C_1 and C_0C_2 respectively. It suffices to show that $C_1 \le C_2$, for then the previous Lemma, with $K = C_0$ shows that $C_0C_1 \le C_0C_2$, and then the previous Lemma again, with $K = \mathrm{GL}_n$ and the action by conjugation, shows that $C_\nu \le C_\tau$.

We now assume that $\nu = (\nu_1, \nu_2)$ and $\tau = (\tau_1, \tau_2)$ (we allow $\tau_2 = 0$) with $\nu_1 = \tau_1 - 1$ and $\nu_2 = \tau_2 + 1$. Then the difference between τ^* and ν^* is that ν^* has one extra part equal to 2 and two fewer parts equal to 1. Let $g \in C_\nu$. By Theorem 4, we may find flags $f_\nu : 0 < W_2 < W_3 < \dots$ and $f_\tau : 0 < W_1 < W_2 < W_3 < \dots$ such that f_ν and f_τ have corresponding Levi partitions of ν^* and τ^*, f_ν and f_τ are identical to the right of W_3, and g represents the Richardson class corresponding to f_ν (in particular g acts trivially upon each factor in f_ν). Then g is in the unipotent radical associated with f_τ, which in turn is contained in $\overline{C_\tau}$. Whence $C_\nu \subseteq \overline{C_\tau}$. □

Table 2: Jordan blocks of distinguished Richardson classes

Image of Ψ
GL_n The partition of n consisting of a single block
$G = \mathrm{SO}_{2n+1},\ p \neq 2$ Partitions of $2n+1$ consisting of distinct odd parts
$G = \mathrm{SO}_{2n+1},\ p = 2$ Partitions of $2n+1$ of the form $1 \oplus \lambda$ such that: each part of λ is even; the multiplicity of each part of λ is at most 2; if i is even then $\lambda_i - \lambda_{i+1} \geq 4$.
$G = \mathrm{Sp}_{2n}$ Partitions of $2n$ consisting of distinct even parts
$G = \mathrm{SO}_{2n},\ p \neq 2$ Partitions of $2n$ consisting of distinct odd parts
$G = \mathrm{SO}_{2n},\ p = 2$ Partitions λ of $2n$ such that: λ has an even number of parts; each part of λ is even; the multiplicity of each part is at most 2; if i is even with $\lambda_{i+1} \neq 0$ then $\lambda_i - \lambda_{i+1} \geq 4$.

5 Richardson Classes of Distinguished Parabolics

Lemma 5.1. *Let G be one of GL_n, SO_{2n+1}, SO_{2n}, and Sp_{2n}. Let Ψ denote the map which takes each distinguished parabolic class to the Jordan blocks of its Richardson class. Then Ψ is injective and has image given in Table 2. Furthermore, if $p = 2$, $G \in \{\mathrm{SO}_{2n}, \mathrm{Sp}_{2n}\}$, x is the size of one of the Jordan blocks, and ε is the Spaltenstein index then $\varepsilon(x) \neq 0$.*

For $p \neq 2$, the descriptions in Table 2 of the image of Ψ are stated in [2], but it is not stated there that each of these partitions equals the Jordan blocks of the Richardson class of a distinguished parabolic.

Proof. If $G = \mathrm{GL}_n$, then the only distinguished parabolic is the Borel subgroup, which corresponds to the regular class, which has Jordan blocks as stated.

We give the proof for SO_{2n} and leave the other cases to the reader.

Let λ be a partition of $2n$. Let M be the maximal part of λ^*. Then λ has distinct odd parts if and only if the following hold:

$$\text{M is even, λ^* contains the set $\{1,\dots,M\}$, the part M has odd multiplicity and all other parts have even multiplicity.} \tag{7}$$

Let P be a distinguished parabolic with Levi partition $\Lambda = (n_1, \dots, n_s) \oplus m = (1^{c(1)}, 2^{c(2)}, \dots) \oplus m$. Let N be the largest n_i in Λ. Using the description of distinguished parabolics given in [4], and recalling our conventions about $m = 1$ given in Section 1, we have that

$$m \geq 1,\ N \in \{2m-1, 2m\},\ c(i) \geq 1 \text{ for } 1 \leq i \leq N. \tag{8}$$

Suppose $p \neq 2$ and apply ψ defined in Table 1. We obtain

$$m \geq 1,\ \psi(\Lambda) = (1^{2c(1)}, 2^{2c(2)}, \dots, (2m-1)^{2c(2m-1)}, (2m)^{c(2m)+1}). \tag{9}$$

Now it is easy to verify that $\lambda^* = \psi(\Lambda)$ satisfies Equation 7.

Conversely, let $p \neq 2$ and let λ be a partition of $2n$ which satisfies Equation 7. Now we solve $\lambda^* = (1^{2c(1)}, 2^{2c(2)}, \dots, (2m-1)^{2c(2m-1)}, (2m)^{c(2m)+1})$ for m and $c(1), \dots, c(2m)$ and define $\Lambda = (1^{c(1)}, 2^{c(2)}, \dots) \oplus m$. It is easy to check that Λ satisfies Equation 8. Among all the parabolics which have Levi partition equal to Λ, there is a unique (up to conjugacy) parabolic P which is distinguished. We define $\Psi^{-1}(\lambda)$ to be the conjugacy class of P. It is easy to check that Ψ^{-1} is the inverse map to Ψ.

Suppose now that $p = 2$. The argument is similar to the case $p \neq 2$, so we briefly indicate the relevant formulas. Given λ, a partition of $2n$, with maximal part M, λ satisfies the conditions in Table 2 for $G = \mathrm{SO}_{2n}$, $p = 2$, if and only if

$$\text{M is even, each part in λ^* has even multiplicity, $\lambda_i^* - \lambda_{i+1}^* \leq 2$ for all i, every even part less than M has multiplicity at least 4.} \tag{10}$$

If P is distinguished we have

$$m \geq 1,\ \psi(\Lambda) = (1^{2c(1)-2}, 2^{2c(2)+2}, \dots, (2m-1)^{2c(2m-1)-2}, (2m)^{2c(2m)+2}). \tag{11}$$

Then $\lambda = \Psi(P) = \psi(\Lambda)^*$ satisfies Equation 10. Conversely, given λ satisfying Equation 10 one may solve for Λ such that $\psi(\Lambda)^* = \lambda$ and verify that Λ satisfies Equation 8.

We now address the statements in the lemma regarding ε. By the definition of ε (see Remarks 2.4), and the description given in Table 2, there is nothing to prove unless $p = 2$ and $G = \mathrm{SO}_{2n}$, which we now assume. Let L be the Levi factor of a distinguished parabolic, let $\lambda^* = \psi(\Lambda)$ with notation as in Equations 8 and 9. By Richardson's Theorem (iii) we know $\dim L = \dim C_G(u)$ where u is an element of the Richardson class in G. As stated in Remark 2.4 we have that $\dim C_G(u) = \frac{1}{2}\sum_{i\geq 1}(\lambda_i^*)^2 + \frac{1}{2}\operatorname{op}(\lambda) + \operatorname{sp}(u) - \lambda_1^*$. We have that $\operatorname{op}(\lambda) = 0$ and $\lambda_1^* = 2m$ so it suffices to show that $\frac{1}{2}\sum_{i\geq 1}(\lambda_i^*)^2 = \dim L + 2m$. We have $\dim L = \sum_{i=1}^N i^2 c(i) + 2m^2 - m$. Recall that $N \in \{2m-1, 2m\}$ whence $\sum_{i=1}^N i^2 c(i) = \sum_{i=1}^{2m} i^2 c(i)$. We take each sum over $i \in \{1, \dots, 2m\}$ with additional restrictions as noted:

$$\begin{aligned}
\frac{1}{2}\sum(\lambda_i^*)^2 &= \frac{1}{2}\left(\sum_{i\text{ odd}} i^2(2c(i)-2) + \sum_{i\text{ even}} i^2(2c(2)+2)\right) \\
&= \sum_{i\text{ odd}} i^2(c(i)-1) + \sum_{i\text{ even}} i^2(c(i)+1) = \sum i^2 c(i) - \sum_{i\text{ odd}} i^2 + \sum_{i\text{ even}} i^2.
\end{aligned}$$

It is easy to show that $\sum_{i=1}^m -(2i-1)^2 + (2i)^2 = 2m^2 + m$, whence $\frac{1}{2}\sum(\lambda_i^*)^2 = \dim L + 2m$. □

Corollary 5.2. *Let G be a simple algebraic group and consider the map which takes each distinguished parabolic class to its Richardson class. This map is injective.*

Proof. For the classical groups this follows from the previous lemma. For the exceptional groups, we observe that no two distinct distinguished parabolics have the same dimension of Levi factor. By Richardson's Theorem (iii) the dimension of the Levi factor equals the dimension of the centralizer of an element in the unipotent class, whence the result follows by dimension. □

6 Proof of the Bala-Carter-Pommerening Theorem

Throughout this section, G denotes a connected reductive group, unless indicated otherwise.

Lemma 6.1. (i) *Let S be a torus in G. Then $L = C_G(S)$ is a Levi subgroup.*

(ii) *If u is a unipotent element and S a maximal torus of $C_G(u)$ then u is distinguished in $L = C_G(S)$. Furthermore, any Levi subgroup in which u is distinguished is conjugate to L via an element of $C_G(u)^\circ$.*

Proof. For part (i) one may adapt [4, 5.9.2]. For part (ii) one may adapt [4, 5.9.3]. □

Corollary 6.2. *Define a map from G-classes of pairs (L, C) consisting of a Levi subgroup L of G and a distinguished unipotent L-class C to unipotent G-classes by extending C. This map gives a bijection.*

Lemma 6.3. *Let P be a distinguished parabolic of G. Let $\overline{G} = G/Z(G)$, $\overline{P} = P/Z(G)$, let $\overline{Q}$ be the unipotent radical of $\overline{P}$ and let u represent the dense class of $\overline{P}$ upon its unipotent radical $\overline{Q}$. Then $C_{\overline{G}}(u)^\circ = C_{\overline{P}}(u)^\circ = C_{\overline{Q}}(u)^\circ$. In particular the Richardson class of P is distinguished in G.*

Proof. It is easy to reduce to the case $Z(G) = 1$ and adapt the proof given in [4, 5.8.7]. □

Proof 6.4 (Proof of Theorem 3). Part (i). This is by definition of the map ψ.

Part (ii). We have $\psi(L, P) = C$ and $u \in C \cap L$. Let $M \leq L$ be a minimal Levi subgroup containing u. We wish to show that $L = M$. By definition, C is obtained by extending to G the Richardson class in L of P. If $v \in L$ represents this Richardson class in L then v is distinguished in L by Lemma 6.3. Since u is conjugate to v (in G) we have $\operatorname{rank} C_G(u) = \operatorname{rank} C_G(v)$. By Lemma 6.1 we have $\dim Z(M) = \operatorname{rank} C_G(u) = \operatorname{rank} C_G(v) = \dim Z(L)$ whence $L = M$.

Part (iii). Corollary 5.2 shows that ψ, restricted to those pairs where $L = G$, is injective and part (ii) shows that the image of this restriction is a subset of the distinguished classes of G. Then Corollary 6.2 shows that ψ defined on all of BC-pairs(G) is injective.

For surjectivity, we have two cases. If G is a classical group, we use the description of distinguished unipotent classes in [19, II.7.10] and apply Lemma 5.1 to see that ψ, applied to those pairs (L, P) where $L = G$, has image equal to all the distinguished classes of G. Then Corollary 6.2 shows that ψ is surjective. If G is exceptional it is simpler to count all pairs (L, P) and compare this to the number of unipotent classes in G as found in [11], which draws on [13], [14], [17], [18], [23]. □

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