A CLASSIFICATION OF CERTAIN FINITE DOUBLE COSET COLLECTIONS IN THE EXCEPTIONAL GROUPS.

W. ETHAN DUCKWORTH

Abstract. Let G be an exceptional algebraic group, X a maximal rank reductive subgroup and P a parabolic subgroup. This paper classifies when $X\backslash G/P$ is finite. Finiteness is proven using a reduction to finite groups and character theory. Infiniteness is proven using a dimension criterion which involves root systems.

1. Classification Theorem

The results in this paper are part of a program to classify certain families of finite double coset collections.

All groups in this paper are affine algebraic groups defined over an algebraically closed field. Let G be simple of exceptional type, X a maximal rank reductive subgroup and P a parabolic subgroup. Our main result is a classification of when $X\backslash G/P$ is finite (notation explained after the theorem).

Theorem 1. *Let G be an exceptional algebraic group, X a maximal rank reductive subgroup and $P \neq G$ a parabolic subgroup. Then $X\backslash G/P$ is finite if and only if X is spherical or one of the following holds:*

(i) $G = E_6$, $X \in \{A_5T_1,\ A_1A_4T_1,\ A_2A_2A_2,\ D_4T_2\}$ *and* $P \in \{P_1,\ P_6\}$.
(ii) $G = E_7$, $X \in \{D_6T_1,\ A_1D_5T_1,\ A_6T_1,\ A_2A_5\}$ *and* $P = P_7$.
(iii) $G = F_4$, $(X, P) \in \{(L_1, P_4),\ (L_4, P_1)\}$.

Throughout this paper we use the notation L_i to denote the Levi subgroup (unique up to G-conjugacy) obtained by crossing off the node i from the Dynkin diagram of G (we label the nodes as in [2]). Similarly P_i is corresponding parabolic, and L_{i_1,i_2} means we have crossed off i_1 and i_2. Notation of the form A_5T_1 means a subgroup of type A_5, with a central one dimensional torus $T_1$1.

The cases where X is spherical have been classified [3, 5, 11, 13] and are shown in Table 1. Theorem 1 complements the results in [6], which covered the case where G was of classical type.

A number of the cases in Theorem 1, as well as those in [6], can be concisely described using information from root systems. Let G be any simple algebraic group. Fix a maximal torus T, a root system Φ, a Dynkin diagram Δ and root groups U_α, $\alpha \in \Phi$. Let $\widetilde{\Delta}$ be the extended Dynkin diagram of G; i.e. Δ together with $-\delta$ where δ is the highest root. Let L be a subgroup of G containing T. We say that L is a **pseudo-Levi subgroup** if $L = \langle T, U_\alpha \mid \alpha \in \Phi_0\rangle$ where Φ_0 is a

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TABLE 1. Maximal rank reductive spherical subgroups defined in all characteristics

$X \leq G$ with X spherical in G					
D_5T_1	$\leq$	E_6	D_8	$\leq$	E_8
A_1A_5	$\leq$	E_6	A_1C_3	$\leq$	F_4
E_6T_1	$\leq$	E_7	B_4	$\leq$	F_4
A_7	$\leq$	E_7	A_2	$\leq$	G_2
A_1D_6	$\leq$	E_7	$A_1\widetilde{A}_1$	$\leq$	G_2
A_1E_7	$\leq$	E_8			

closed root subsystem generated by Δ_0 for some $\Delta_0 \subseteq \tilde{\Delta}$. Write the high root δ of Φ as $\sum_{\alpha\in\Delta} n_\alpha \alpha$, set $n_\delta = 1$. Given a pseudo-Levi subgroup L and corresponding subset $\Delta_0 \subseteq \tilde{\Delta}$, we define the **pseudo-height** of L to be

$$\mathrm{pht}(L) = -1 + \sum_{\alpha\in\widetilde{\Delta}-\Delta_0} n_\alpha.$$

In other words, the pseudo-height is -1 plus the sum of the coefficients of the crossed off nodes. Note that pseudo-height of a Levi subgroup equals its height as usually defined [1]. Furthermore, if $L = \langle T, U_\alpha \mid \alpha \in \pm\Delta_0\rangle$ is a Levi factor, then $Q = \langle U_\alpha \mid \alpha \in \Phi^+ - \Delta_0\rangle$ is the unipotent radical of a parabolic subgroup that has L as a Levi factor. We define $\mathrm{nilp}(L)$ to be the nilpotency class of Q. Then we have

$$\mathrm{pht}(L) = \mathrm{ht}(L) \leq \mathrm{nilp}(L) \tag{1}$$

where equality holds if $(G,p) \notin \{(B_n,2),(C_n,2),(F_4,2),(G_2,3)\}$ [1]. We extend pht, ht and nilp to parabolic subgroups in the obvious manner. Finally, if G is reductive we define pht, ht, nilp to be the maximal value obtained by restricting to a simple factor of G.

Theorem 2. *Let G be a simple algebraic group and X a pseudo-Levi subgroup.*

1. $\mathrm{pht}(X) \leq 1$ *if and only if $X\backslash G/P$ is finite for all parabolics P.*
2. *If* $\mathrm{pht}(X) \leq 2$ *and* $\mathrm{ht}(P) \leq 1$ *then $X\backslash G/P$ is finite.*
3. *If $G \neq A_n$ and* $\mathrm{pht}(X) \geq 3$ *then $X\backslash G/P$ is infinite.*

2. The finiteness theorem

The hardest part of Theorem 1 is establishing the finiteness assertions. These follow from the next theorem. In this section we use the notation $\mathbb{F}_{p^n}$ for the field with p^n elements (with p is a prime), and $\overline{\mathbb{F}_p}$ for the algebraic closure.

Theorem 3. *Let G, X and P be affine algebraic group schemes, all reduced and of finite type over $\mathbb{Z}$, with G simple, X and P subgroup schemes, X maximal rank and reductive and P parabolic. Let k be any algebraically closed field. Let W be the Weyl group of G.*

Suppose that the following two conditions hold.

(1) *there exists C, such that for all $p > 0$, $w \in W$, $s \in X(\overline{\mathbb{F}_p})$ a semisimple element with $G(\overline{\mathbb{F}_p})_s$ (the centralizer in $G(\overline{\mathbb{F}_p})$), a maximal centralizer subgroup of $G(\overline{\mathbb{F}_p})$, we have $\left|X(\overline{\mathbb{F}_p})_s\backslash G(\overline{\mathbb{F}_p})_s/{}^wP(\overline{\mathbb{F}_p})_s\right| < C$*

(2) *for all $p > 0$, $u \in X(\overline{\mathbb{F}_p})$, $1 \neq u$, we have $\dim X(\overline{\mathbb{F}_p})_u \geq \dim(G(\overline{\mathbb{F}_p})/P(\overline{\mathbb{F}_p}))_u$.*

Then $|X(k)\backslash G(k)/P(k)| < \infty$.

When this theorem is applied to various parts of Theorem 1, condition (1) allows for an inductive argument in the following sense. The groups that appear as $G_s(\overline{\mathbb{F}_p})$ are smaller that $G(\overline{\mathbb{F}_p})$, whence results that we have proven previously for $G_s(\overline{\mathbb{F}_p})$ can be used. In this paper we will start with G of type E_6, in which case the subgroups $G_s(\overline{\mathbb{F}_p})$ are classical groups in [6]. Verifying condition (2) is a little more tedious, but the dimension calculations that are involved are fairly standard (see Proposition 13).

The rest of this section comprises the proof of Theorem 3.

Lemma 4. *To prove Theorem 3 it suffices to show that $|X(F_{p^n})\backslash G(F_{p^n})/P(F_{p^n})|$ is bounded independently of p or n.*

Proof. By [5, Prop. 13], $|X(k)\backslash G(k)/P(k)|$ is finite if and only if $\left|X(\overline{\mathbb{F}_p})\backslash G(\overline{\mathbb{F}_p})/P(\overline{\mathbb{F}_p})\right|$ is finite and bounded independently of p. By [5, Lemma 14] we have

$$\frac{1}{C_1}\limsup_{n\to\infty}|X(\mathbb{F}_{p^n})\backslash G(\mathbb{F}_{p^n})/P(\mathbb{F}_{p^n})| \leq \left|X(\overline{\mathbb{F}_p})\backslash G(\overline{\mathbb{F}_p})/P(\overline{\mathbb{F}_p})\right| \leq \limsup_{n\to\infty}|X(\mathbb{F}_{p^n})\backslash G(\mathbb{F}_{p^n})/P(\mathbb{F}_{p^n})|$$

where C_1 is an upper bound on the number of connected components of stabilizers of the group $X(\overline{\mathbb{F}_p}) \times P(\overline{\mathbb{F}_p})$ acting on $G(\overline{\mathbb{F}_p})$.

To finish the proof we show now that C_1 is bounded independently of p.

Let $g \in G(\overline{\mathbb{F}_p})$. Then the stabilizer in $X(\overline{\mathbb{F}_p}) \times P(\overline{\mathbb{F}_p})$ of g is isomorphic to $X(\overline{\mathbb{F}_p}) \cap {}^gP(\overline{\mathbb{F}_p})$. Since X and P are both schemes of finite type over $\mathbb{Z}$, we have that the intersection of $X(\overline{\mathbb{F}_p})$ and ${}^gP(\overline{\mathbb{F}_p})$ has a bounded number of connected components [10, Prop. 13.1.7, Cor. 9.7.9] or [12, Prop. 1.4], independent of p. This shows that we can fix C_1 is independently of p. □

Through the remainder of this section we will be working over $\overline{\mathbb{F}_p}$ for a fixed p, therefore we abbreviate our notation and write X, G and P instead of $X(\overline{\mathbb{F}_p})$, $G(\overline{\mathbb{F}_p})$, $P(\overline{\mathbb{F}_p})$. Furthermore, let $\sigma : G \to G$ be a standard Frobenius morphism. We write G_σ for the set of σ-fixed points in G, so that $G_\sigma = G(\mathbb{F}_{p^n})$ for some n. We assume that X and P are σ-stable.

For any $g \in G$ we write G_g for g-fixed points in G, i.e. the centralizer of g. We extend this notation to X, G/P, etc. as well.

Definition 5. Define an equivalence relation on semisimple elements in X_σ as follows: s and t are equivalent if $G_{\sigma,\,s}$ and $G_{\sigma,\,t}$ are conjugate under X_σ. Choose a set $S_\sigma \subset T$ of representatives of these equivalence classes; we assume that the identity 1 is one of the elements of S_σ. Finally let $Z(s,\sigma)$ equal the set $\{t \in G \mid G_{\sigma,\,t} = G_{\sigma,\,s}\}$.

Proposition 6 ([5, Cor. 17]). *With the preceding definitions we have that*

$$\frac{1}{|W|}\sum \frac{|Z(s,\sigma)|}{|X_{\sigma,s,u}|} 1_{P_\sigma}^{G_\sigma}(su) \quad \leq \quad |X_\sigma \backslash G_\sigma / P_\sigma| \quad \leq \quad \sum \frac{|Z(s,\sigma)|}{|X_{\sigma,s,u}|} 1_{P_\sigma}^{G_\sigma}(su),$$

where each sum is taken over the elements $s \in S_\sigma$, and the representatives u of the unipotent classes $[u] \subseteq X_{\sigma,s}$.

Lemma 7. *Let s and P be σ-fixed. Let G_s act on $(G/P)_s$ in the natural fashion. Note that σ permutes the G_s-orbits and that the orbit of P is fixed by σ. Each G_s-orbit in $(G/P)_s$ may be represented by ${}^wP$ for some $w \in W$.*

Proof. Let $g \in G$ with $s \in {}^gP$. Then $s^g \in P$, whence $s^{gq} \in T$ for some $q \in P$. Since the Weyl group controls fusion in T ([4, 3.7.1]) we have $s^{gq} = s^w$ for some $w \in W$. Therefore $gqw^{-1} \in G_s$ and $g = xwq^{-1}$ for some $x \in G_s$. So the G_s-orbit of ${}^gP$ equals the G_s-orbit of ${}^wP$. □

Definition 8. For each σ-stable G_s-orbit in $(G/P)_s$ we may apply the Lang-Steinberg Theorem [21, 2.7] to find a σ-fixed representative of the G_s-orbit. By Lemma 7 this representative may be written as ${}^{gw}P$ for some $g \in G_s$, $w \in W$. Let $P(\sigma, s)$ denote the set of all such representatives ${}^{gw}P$.

Corollary 9. *With the above notation we have:*

$$1_{P_\sigma}^{G_\sigma}(su) = \sum_{\widetilde{P} \in P(\sigma,s)} 1_{\widetilde{P}_{\sigma,s}}^{G_{\sigma,s}}(u).$$

Proof. By various standard facts about parabolics, we have that $(G/P)_\sigma = G_\sigma/P_\sigma$ and that G/P and G_σ/P_σ may be identified as the set of conjugates of P and P_σ respectively. Applying definitions and Jordan decomposition we get

$$1_{P_\sigma}^{G_\sigma}(su) = |(G_\sigma/P_\sigma)_{su}| = \left|\big((G/P)_\sigma\big)_{s,u}\right| = \left|\big((G/P)_s\big)_{\sigma,u}\right|. \tag{2}$$

The stabilizer in G_s of $\widetilde{P} \in P(\sigma, s)$ equals $G_s \cap \widetilde{P} = \widetilde{P}_s$. Thus, the G_s-orbit of $\widetilde{P}$ may be identified with $G_s/\widetilde{P}_s$, and by Definition 8 we have that

$$\Big((G/P)_s\Big)_\sigma = \Big(\bigcup_{\widetilde{P} \in P(\sigma,s)} G_s/\widetilde{P}_s\Big)_\sigma.$$

Then Equation (2) becomes

$$1_{P_\sigma}^{G_\sigma}(su) = \left|\big(\bigcup_{\widetilde{P}} G_s/\widetilde{P}_s\big)_{\sigma,u}\right| = \left|\bigcup_{\widetilde{P}} \big(G_{\sigma,s}/\widetilde{P}_{\sigma,s}\big)_u\right| = \sum_{\widetilde{P}} 1_{\widetilde{P}_{\sigma,s}}^{G_{\sigma,s}}(u).$$

□

Corollary 10. *With the notation as above we have that*

$$\frac{1}{|W|}\sum \frac{|Z(s,\sigma)|}{|X_{\sigma,s,u}|} 1_{\widetilde{P}_{\sigma,s}}^{G_{\sigma,s}}(u) \quad \leq \quad |X_\sigma \backslash G_\sigma / P_\sigma| \quad \leq \quad \sum \frac{|Z(s,\sigma)|}{|X_{\sigma,s,u}|} 1_{\widetilde{P}_{\sigma,s}}^{G_{\sigma,s}}(u),$$

where each sum is taken over the elements $s \in S_\sigma$, the representatives u of the unipotent classes $[u] \subseteq X_{\sigma,s}$, and $\widetilde{P} \in P(\sigma, s)$.

Lemma 11. *Let G be connected and reductive, and M a maximal rank subgroup. Let W be the Weyl group of G. Then $|M/M^\circ| \leq |W|$.*

Proof. We prove that M/M° is isomorphic to a quotient of a subgroup of W. Let T be a maximal torus contained in M and let $W = N_G(T)/T$.

Claim (a): $M = M^\circ N_M(T)$. Proof: For each $m \in M$ we have that T^m is a maximal torus of M°, and so there exists $m_0 \in M^\circ$ with $T^m = T^{m_0}$. Using this, it is easy to show that $M \leq M^\circ N_M(T)$, and equality holds.

Claim (b): $N_M(T)^\circ = T$. Proof: Since $N_M(T)^\circ$ is a closed, connected subgroup of $N_G(T)$ it is contained in $N_G(T)^\circ$. Therefore $T \leq N_M(T)^\circ \leq N_G(T)^\circ \leq T$.

Using claims (a) and (b), and the Second and Third Isomorphism Theorems, we calculate

$$M/M^\circ \cong \frac{N_M(T)/T}{N_{M^\circ}(T)/T} \leq \frac{W}{N_{M^\circ}(T)/T}.$$

□

Corollary 12. *The number of terms in the sums in Corollary 10 is bounded above by a finite constant C which depends only upon the root system $\Phi(G)$. In particular it does not depend upon p or n.*

Proof. By Lemma 7 we have that $|P(s,\sigma)|$, the number of σ-stable G_s-orbits in $(G/P)_s$, is at most $|W|$. By [15] we have that the number of unipotent classes in X_σ is at most $W_X(\mathrm{rank}(X)+1)$. To finish the proof we show that $|S_\sigma|$ is also bounded.

Claim (a): Let X act on the set $\{G_s \mid s \in X\}$ by conjugation. Then the number of X-orbits is finite and bounded above by a constant C_1 which depends only upon $\Phi(G)$.

Proof: Every semisimple element $s \in X$ is conjugate, via X, to an element of T. Thus every orbit is represented by a group G_s with $s \in T$, whence G_s contains T. Now G_s° is generated by the root groups it contains, and so the number of possible groups G_s°, containing T, is bounded by a constant depending upon $\Phi(G)$. Finally, by Lemma 11, $|G_s/G_s^\circ|$ is bounded by W.

Claim (b): For each $s \in X$, the number of connected components of $N_X(G_s)$, is at most $|W|$.

Proof: Since $N_X(G_s)$ is a maximal rank subgroup we may apply Lemma 11.

Claim (c): $S_\sigma \leq C_1|W|$.

Proof: By the Lang-Steinberg Theorem [21, 2.7], and claims (a) and (b), the number of X_σ orbits on the σ-fixed elements of the set $\{G_s \mid s \in X\}$ is at most $C_1|W|$. □

We now finish the proof of Theorem 3. We assume that $|X_s \backslash G_s/{}^wP_s|$ is finite and bounded independently of p for each $s \in X$, G_s, $w \in W$, and that $\dim X_u \geq \dim(G/P)_u$. By Lemma 4, Corollaries 10 and 12 it suffices to show that each of the quantities

$$\frac{|Z(s,\sigma)|}{|X_{\sigma,s,u}|} 1_{\widetilde{P}_{\sigma,s}}^{G_{\sigma,s}}(u) \tag{3}$$

is bounded above independently of p and n (where σ is a p^n Frobenius map).

Let $s \in S_\sigma$, $u \in X_{\sigma,s}$, $w \in W$, $\widetilde{P} \in P(\sigma, s)$ be fixed. Our argument has two cases: $s \neq 1$ and $s = 1$.

Suppose $s \neq 1$ and let $t \in T$ such that G_t is a maximal centralizer subgroup of G with $G_s \leq G_t$. The basic idea is to find the term given in Expression (3) inside of the same kind of expansion as applied to $X_t \backslash G_t / {}^w P_t$. To (hopefully) simplify the notation, we write $\overline{G} = G_t$, $\overline{X} = X_t$, $\overline{P} = {}^w P_t$, $\overline{s} \in \overline{X}$, $Z_{\overline{G}}(\overline{s}, \sigma)$, etc. Note that $\overline{P}$ is a parabolic in $\overline{G}$. We apply Lemma 4 through Corollary 12 to $\overline{X} \backslash \overline{G} / \overline{P}$ to get

$$\left|\overline{X} \backslash \overline{G} / \overline{P}\right| \geq \frac{1}{\overline{C_1}} \left|\overline{X}_\sigma \backslash \overline{G}_\sigma / \overline{P}_\sigma\right| \geq \frac{1}{\overline{C_1}} \frac{1}{|\overline{W}|} \sum \frac{|Z_{\overline{G}}(\overline{s}, \sigma)|}{|\overline{X}_{\sigma,\overline{s},u}|} 1^{\overline{G}_{\sigma,\overline{s}}}_{\widetilde{\overline{P}}_{\sigma,\overline{s}}}(u)$$

where all the quantities are finite and bounded independently of p and n. One of the terms we have just obtained for $\overline{X} \backslash \overline{G} / \overline{P}$, corresponds to $\overline{s} = s \in \overline{X}$,

$$\frac{1}{\overline{C_1}} \frac{1}{|\overline{W}|} \frac{|Z_{\overline{G}}(s, \sigma)|}{|\overline{X}_{\sigma,s,u}|} 1^{\overline{G}_{\sigma,s}}_{\widetilde{\overline{P}}_{\sigma,s}}(u).$$

Some of the quantities here are the same as the corresponding quantity for the s term in $X \backslash G / P$

$$\overline{G}_{\sigma,s} = G_{\sigma,s},\ \widetilde{\overline{P}}_{\sigma,s} = \widetilde{P}_{\sigma,s},\ \overline{X}_{\sigma,s} = X_{\sigma,s}.$$

Also it is easy to show that $Z_G(s, \sigma) \subset Z_{\overline{G}}(s, \sigma)$. Combining these observations we have

$$\left|\overline{X} \backslash \overline{G} / \overline{P}\right| \geq \frac{1}{\overline{C_1}} \frac{1}{|\overline{W}|} \frac{|Z_{\overline{G}}(s, \sigma)|}{|\overline{X}_{\sigma,s,u}|} 1^{\overline{G}_{\sigma,s}}_{\widetilde{\overline{P}}_{\sigma,s}}(u) \geq \frac{1}{\overline{C_1}} \frac{1}{|\overline{W}|} \frac{|Z_G(s, \sigma)|}{|X_{\sigma,s,u}|} 1^{G_{\sigma,s}}_{\widetilde{P}_{\sigma,s}}(u).$$

We are done since the quantities $\overline{C_1}$ and $|\overline{W}|$ are bounded independently of p and n.

Now, suppose that $s = 1$. Then the term in Expression (3) becomes

$$\frac{|Z(G)_\sigma|}{|X_{\sigma,u}|} 1^{G_\sigma}_{\widetilde{P}_\sigma}(u)$$

where $Z(G)$ is the center of G. Since $Z(G)$ is finite it suffices to show that

$$\frac{1}{|X_{\sigma,u}|} (G/P)_{\sigma,u}$$

is bounded independently of p and n. Let $d_1 = \dim X_u$ and $d_2 = \dim(G/P)_u$ and let $q = p^n$. By examining the formulas for centralizers of unipotent elements, or by using the general bounds given by Nori [18, 3.5], one sees that $|X_{\sigma,u}| \geq (q-1)^{d_1}$. We will be finished once we show that $(G/P)_{\sigma,u} \leq c_1 q^{d_2}$ for some c_1 which does not depend on p or n. For then we will have

$$\frac{1}{|X_{\sigma,u}|} |(G/P)_{\sigma,u}| \leq \frac{1}{(q-1)^{d_1}} c_1 q^{d_2}$$

which is bounded provided $d_1 \geq d_2$.

When p is a good prime (it suffices to assume that $p = 0$ or $p \geq 7$), we have that $|(G/P)_{\sigma,u}|$ is given by a polynomial in q of degree d_2, where the polynomial does not depend upon $p >> 0$ or n [9, 3.10,3.15]. We now discuss how to obtain the desired bound for the remaining, small, characteristics. The projective variety $(G/P)_u$ has a finite cover by affine spaces of dimension d_2 (or less). Each affine space has at most q^{d_2} rational points over $\mathbb{F}_q$. Let c_1 be an upper bound on the number of affine spaces needed for each characteristic up to 7. Then $|(G/P)_{\sigma,u}|$ has at most $c_1 q^{d_2}$ rational points.

This finishes the proof of Theorem 3.

3. Comparing dimensions of fixed points of unipotent elements

The following result records most of the calculations and results we will use for $\dim(G/P)_u$. In the statement of this result all quotients of the form G/B, $(G/P)_u$ etc. are identified with the collection of conjugates of B or P, that contain u if appropriate.

Proposition 13. *Let G be a connected reductive group, B a Borel subgroup, and P a parabolic subgroup containing B, and $u \in G$ a unipotent element.*

1. *Consider the natural map $\varphi : (G/B)_u \to (G/P)_u$.*
   - (a) *φ is surjective.*
   - (b) *Let $\varphi^{-1}({}^gP)$ be a fibre of φ. If this fibre has minimal dimension among all fibres, then $\dim(G/P)_u = \dim(G/B)_u - \dim \varphi^{-1}({}^gP)$.*
   - (c) *$\varphi^{-1}({}^gP) = ({}^gP/{}^gB)_u$.*
2. $\dim(G/B)_u = \frac{1}{2}(\dim G_u - \operatorname{rank} G)$.
3. *Let $u \in {}^gP$, L be a Levi factor of ${}^gP$, B_L be a Borel subgroup of L, and v the projection of u to L. Then we have $\dim({}^gP/{}^gB)_u = \dim(L/B_L)_v$.*
4. *Let $\mathfrak{g}$ be the Lie algebra of G, let u and v be as in part (3) and let $\mathfrak{c}_{\mathfrak{g}}(u)$ and $\mathfrak{c}_g(v)$ be the centralizers of u and v in $\mathfrak{g}$. Then $\dim \mathfrak{c}_{\mathfrak{g}}(u) \leq \dim \mathfrak{c}_{\mathfrak{g}}(v)$.*

Proof. Part (1)(a) is in [22, 2.3].

Part (1)(b) is standard algebraic geometry, once we carefully reduce to irreducible components. The only potential problems here are picking irreducible components of the correct dimension, and making sure that the fibres of φ lie in the irreducible component. Let $X_1, X_2, \ldots$ be the irreducible components of $(G/B)_u$, let $Y_i = \varphi(X_i)$ and note that the irreducible components of $(G/P)_u$ are the maximal Y_i. By [20] all of the X_i have the same dimension, namely $\dim(G/B)_u$. Let Y be a Y_i of maximal dimension, let $I = \{i \mid Y_i = Y\}$, let $U = Y - \bigcup_{i \notin I} Y_i$. Note that U is open and nonempty in Y, so that $\dim U = \dim Y = \dim(G/P)_u$. Let $O = \varphi^{-1}(U)$, note that the fibre of φ over any point in U lies $\bigcup_{i\in I} X_i$, so $O \subset \bigcup_{i \in I} X_i$ and restricting φ to O will not change its fibres (over U). Let $\hat{\varphi} : O \to U$ be the restricted map, note that $\hat{\varphi}$ is surjective and that the irreducible component of any fibre of $\hat{\varphi}$ is a subset of some X_i, $i \in I$. Let $W \subset U$ be the open set such that fibres of $\hat{\varphi}$ over W. Then for such a fibre F we have F is also a fibre of φ, and that $\dim F = \dim X_i - \dim Y$.

Part (1)(c). It is easy to show that the fibre of φ over ${}^gP$ equals the ${}^gP$-conjugates of ${}^gB$ that contain u.

Part (2). This is proven for all characteristics in [20, II.10.15].

Part (3). Let Q be the unipotent radical of ${}^gP$. For each Borel subgroup B' in L, the product $B'Q$ is a Borel subgroup of ${}^gP$. Furthermore $v \in B'$ if and only if $u \in B'Q$. In this way we get an isomorphism from $(L/B_L)_v$ to $({}^gP/{}^gB)_u$.

Part (4). Let Z be the central torus of L. View $\mathfrak{g}$ as a Q module. There is a filtration $0 = F_0 \leq F_1 \leq \cdots \leq F_n = \mathfrak{g}$ with Q acting trivially on each factor F_i/F_{i-1} and Z having distinct weights on distinct factors F_i/F_{i-1} (c.f. [1]). This filtration splits as a Z-module, and since Z is central in L we have that it splits as an L module. We claim $\mathfrak{c}_{\mathfrak{g}}(v) = \sum \mathfrak{c}_{F_i/F_{i-1}}(v) = \sum \mathfrak{c}_{F_i/F_{i-1}}(u) \geq \mathfrak{c}_{\mathfrak{g}}(u)$. The first equality follows from the fact that the filtration splits with respect to v; the second from the fact that $u = vq$ and q acts trivially upon each factor; the last from general principles. □

Comments: To use part (iii) of this result, we need to be able to take a unipotent element $u \in P$, and produce a list of possible projections $v \in L$. Using part (iv), the possibilities for v are those elements that are represented in L and that have at least as many Jordan blocks as u in the action on $\mathfrak{g}$. These possibilities can be easily listed by using the tables in [14].

4. Finiteness in E_6

In this section we will prove all the finiteness assertions made in Theorem 1 about $G = E_6$. We start by verifying that condition (1) in Theorem 3. We have that G_s is a classical group. By [6] we may conclude that Theorem 2 holds for G_s.

Let $G = E_6$, $X \in \{A_5T_1,\ A_1A_4T_1,\ D_4T_2,\ A_2A_2A_2\}$ and $P \in \{P_1,\ P_6\}$, fix $s \in T$, $w \in W$.

If X is a Levi subgroup of G, then X_s is a Levi subgroup of G_s, whence

$$\mathrm{pht}_{G_s}(X_s) = \mathrm{nilp}_{G_s}(X_s) \leq \mathrm{nilp}_G(X) = \mathrm{pht}_G(X) \leq 2.$$

In a similar manner $\mathrm{pht}_{G_s}({}^wP_s) \leq \mathrm{pht}_G(P) = 1$, whence $X_s\backslash G_s/{}^wP_s$ is finite by Theorem 2.

For the rest of the E_6 case we may assume that $X = A_2A_2A_2$. The maximal possibilities for G_s are $A_2A_2A_2$, A_1A_5 and D_5T_1 and we address each of these cases below.

If $G_s = A_2A_2A_2$ then $\mathrm{pht}_{G_s}(T) = 2$, and $\mathrm{pht}_{G_s}({}^wP_s) \leq 1$, whence $T\backslash G_s/{}^wP_s$ is finite by Theorem 2. Since $X \geq T$, this shows that $X_s\backslash G_s/{}^wP_s$ is finite.

Suppose now that $G_s \in \{A_1A_5,\ D_5T_1\}$. We claim the following: (a) $X_s \in \{A_1A_1A_1T_3,\ A_1A_1A_2T_2,\ A_1A_2A_2T_1\}$, (b) if $G = A_1A_5$ and $X = A_1A_1A_1T_3$ then each A_1 factor in X_s is contained in the A_5 factor in G_s, (c) if $G_s = D_5T_1$, then $X \neq A_1A_1A_1T_3$. Suppose for the moment that the characteristic is not 2 or 3. Then G_s is the centralizer of an involution in G, whence X_s is the centralizer of an involution in X, and this gives the possibilities for X_s. If $G_s = A_1A_5$, suppose for contradiction that one of the A_1 factors for X_s equals the A_1 factor for G_s. Then the second and third A_2 factors in X commute with this A_1, and so they are contained in the A_5 factor of G_s. This would show that $X_s = A_1A_2A_2T_1$, contrary to assumption. If $G_s = D_5T_1$, then we use the fact that X is the centralizer in G of a semisimple element of order 3, whence X_s is the centralizer in G_s of a semisimple element of order 3. The only possibility for this which is also a subsystem of X is $A_2T_1D_2$. Finally, if the characteristic is 2 or 3, then the conclusions we have just made about G_s and X_s still hold. This can be verified in more than one way, for instance by showing that the intersection of X with G_s is controlled by double cosets in the Weyl group of G, or by showing that the whole argument is equivalent to intersecting closed subsystems of the root system of G. In any case, in all characteristics, we have that G_s and X_s satisfy our claims.

The claims just proven about G_s and X_s show that in each case we have $\mathrm{pht}_{G_s}(X_s) \leq 2$. We showed above that $\mathrm{pht}_{G_s}({}^wP_s) \leq 1$, whence $X_s\backslash G_s/{}^wP_s$ is finite by Theorem 2 (as applied to the classical groups).

This finishes our verification of (1) of Theorem 3 for all cases of Theorem 1 where $G = E_6$. Now we turn to condition (2) in Theorem 3.

To compare unipotent classes in X and in G we use the Bala-Carter theorem (note that this holds for $G = E_6$ and for $X = D_4T_2$ in all characteristics [7]). For each unipotent class in X (with one exception described presently), the same Bala-Carter theorem gives the same label to the unipotent class in X and the corresponding class in G. The exception is the regular class in $X = A_2A_2A_2$. In X this class is labelled as $A_2 + A_2 + A_2$. Extending this class to a unipotent class in G gives the class labelled as $D_4(a_1)$ in G. To see this note that $N_G(X)$ contains the symmetric group on three letters, S_3, which permutes the factors of X. Thus the centralizer in G of a regular element in X contains S_3 which implies (by examination of the centralizer groups given in [17]) that the class is $D_4(a_1)$.

We note that in many cases we can show that $\dim X_u \geq \dim(G/P)_u$ without using all the parts of Theorem 2. For instance, if $\dim X_u \geq \dim G/P$ or $\dim X_u \geq \dim(G/B)_u$, then $\dim X_u \geq \dim(G/P)_u$ follows.

In Table 2 we list various dimension calculations. We have organized the table by unipotent classes in G, using the Bala-Carter label, but have indicated with the notation "·" that X has no unipotent elements of a given class. The notation "10† " indicates that two possibilities for the dimension existed, the smaller of which is 10. Finally, the quantities in the table need to be verified twice, once for the case $p \neq 2$ and once for the case $p = 2$. We have $\dim G/P = 16$, and so all cases in the first three rows of the table satisfy $\dim X_u \geq \dim G/P \geq \dim(G/P)_u$. Similarly, $d(X,u) \leq 0$ implies that $\dim X_u \geq \dim(G/B)_u \geq \dim(G/P)_u$. For the remaining cases, it suffices to show that $d(X,u) \leq \frac{1}{2}(\dim L_v - 6)$ where v is the projection of v to L (the Levi factor of P) and L_v is the centralizer in L of v.

We illustrate this final step for the unipotent class of type $A_1A_1A_1$. If u is of type $A_1A_1A_1$ then u has 38 or 40 Jordan blocks in its action on $\mathfrak{g}$ [14]. By Proposition 13, and the comments that follow it, we see that v has at least 38 Jordan blocks on $\mathfrak{g}$. Again using the tables in [14], we see that the unipotent class of v has label 1, A_1, A_1A_1 or $A_1A_1A_1$. Calculating $\dim L_v$ we find that $\dim L_v \geq 22$ whence $\frac{1}{2}(\dim L_v - 6) \geq 8$. This shows that for each X_i, we have $d(X,u) \leq \frac{1}{2}(\dim L_v - f)$, which implies condition (2) in Theorem 2.

We leave the remaining cases in E_6 to the reader. (Note that the six cases where $d(X,u) = 1$ are relatively simple: it suffices to show that v is not regular in L).

5. Finiteness in $G = E_7$ and $G = F_4$

In this section we will prove all the finiteness assertions made in Theorem 1 about $G = E_7$ and $G = F_4$. The approach is essentially the same as for $G = E_6$, so we will leave some of the calculations to the reader. We start by verifying that

TABLE 2. Dimension of unipotent fixed points in E_6

$X_1 = A_5T_1,\ X_2 = A_1A_4T_1,\ X_3 = A_2A_2A_2,\ X_4 = D_4T_2$

$d(X,u) = \frac{1}{2}(\dim G_u - 6) - \dim X_u$

Unipotent class in G	dim G_u	$\dim X_u$				$d(X,u)$			
		X_1	X_2	X_3	X_4	X_1	X_2	X_3	X_4
1	78	36	28	24	30	0	8	12	6
A_1	56	26	20†	20	20	−1	5	5	5
A_1A_1	46	20	16†	16	18	0	4	4	2
$A_1A_1A_1$	38	18	14	12	14	−2	2	4	2
A_2	36	18	14	18	12	−3	1	−3	3
A_1A_2	32	14	12†	14	·	−1	1	−1	·
A_2A_2	30	12	·	12	·	0	·	0	·
$A_1A_1A_2$	28	·	10	10	·	·	1	1	·
A_3	26	12	10	·	10	−2	0	·	0
$A_1A_2A_2$	24	·	·	8	·	·	·	1	·
A_1A_3	22	10	8	·	·	−2	0	·	·
$D_4(a_1)$	20	·	·	6	8	·	·	1	0
A_4	18	8	8	·	·	−2	−2	·	·
D_4	18	·	·	·	6	·	·	·	0
A_1A_4	16	·	6	·	·	·	−1	·	·
A_5	14	6	·	·	·	−2	·	·	·

condition (1) in Theorem 3. We have that G_s is a classical group or E_6, so we can use Theorem 2.

Suppose that $G = E_7$, $X \in \{D_6T_1,\ A_1D_5T_1,\ A_6T_1\}$ and $P = P_7$. In each case X is a Levi subgroup, and we have $\mathrm{pht}_{G_s}(X_s) \leq 2$, as in E_6. Similarly, $\mathrm{pht}_{G_s}({}^wP_s) = 1,\ 2$. Finally, ${}^wP_s$ is a parabolic in G_s with abelian unipotent radical. Therefore in all cases $X_s\backslash G_s/{}^wP_s$ is finite by the results already proven in the classical groups and E_6.

Now we verify condition (2) in Theorem 3 for $G = E_7$. In all cases we have $P = P_7$. This gives $\dim G/P = 27$, and we are done with those cases where $\dim X_u \geq 27$ or where $d(X,u) = \frac{1}{2}(\dim G_u - 7) - \dim X_u \leq 0$. For the remaining cases it suffices to show that $\frac{1}{2}(\dim L_v - 7) \geq \dim(X,u)$ where v is the projection of u to L. Again, all the cases where $d(X,u) = 1$ can be handled by the observation that no v can be regular in L. In the end one needs to calculate $\dim L_v$ for only a handful of possibilities for v. The types of these possibilities are: 1, A_1, A_1A_1, $A_1A_1A_1$, A_2, A_1A_2, $A_1A_1A_2$, A_3, A_2A_2, A_1A_3, $D_4(a_1)$, D_4, A_4 and A_5.

We offer a few comments to clarify two cases in $X_4 = A_2A_5$. The regular class A_2A_5 and the class $A_2A_2A_2$ need to be translated into Bala-Carter labels as applied to E_7. We claim that these are the classes $E_7(a_5)$ and $D_4(a_1)$ respectively. The analysis in E_6 shows that $A_2A_2A_2 \sim D_4(a_1)$. Similarly, the component group of the

A_2A_5 class has an element of order 3 (because the center of A_2A_5 has an element of order 3), and $E_7(a_5)$ is the only remaining class in E_7 with the same property.

Now we make a few comments about how these calculations change in characteristic 2. The only changes occur when $p = 2$, as Bala-Carter still holds in all other characteristics [7], and the dimensions of the centralizers don't change. The biggest change in characteristic 2 is the existence of one extra unipotent class in E_7. This extra class has received various labels in the literature: $A_3 + A_2^{(2)}$ in [14]; $A_3 + A_2$ in [17]. As shown in [8, 5.4], it is distinguished in a D_6T_1 Levi subgroup and is not represented in either A_6T_1 or $A_1D_5T_1$. In the same reference one finds information about its Jordan blocks, leading to the calculation $\dim X_u = 17$. In [17] we find that the dimension of G_u is 35. Thus $d(X, u) = \frac{1}{2}(35 - 7) - 17 = -3$ and so we are done with this case.

Besides this change, one must re-check the calculations for the dimension of centralizers. However, for centralizers in E_7 the only change occurs for the A_6 class, whose centralizer structure changes, but which still has the same dimension, 19, as when the characteristic is not 2. The dimensions of X_u don't change for $X = A_6T_1$. For the cases $A_1D_5T_1$ and D_6T_1 one needs to repeat the calculations for $\dim X_u$.

For the remainder of the section we assume that $G = F_4$, (X, P) equals (L_1, P_4) or (L_4, P_1). Notice that the roles of X and the Levi factor of P are symmetric. Therefore, a number of the arguments below will be applied to both of these subgroups.

We wish to apply results in the classical groups to see that $X_s\backslash G_s/{}^wP_s$ is finite. Unlike the cases in E_6 and E_7, here we will need to do more than determine the associated nilpotency classes of X and P. It suffices to work with the maximal possibilities for G_s. The maximal possibilities for G_s are $\{B_4,\ A_1C_3,\ \tilde{A}_1A_3,\ A_2\tilde{A}_2\}$ (we use the convention that $\tilde{A}_i$ has short roots, whereas A_i has long roots).

Lemma 14. *Let $G = F_4$, let $s \in T$ and $w \in W$. The following hold:*

(1) *If $G_s = B_4$ then $({}^wL_1)_s = A_1B_2T_1$ and $({}^wL_4)_s \in \{B_3T_1, A_3T_1\}$ (where the notation A_3T_1 indicates a Levi subgroup of B_4, but a $\mathrm{SO}_6\,\mathrm{SO}_2$ subgroup of B_3T_1).*

(2) *If $G_s = A_1C_3$ then $({}^wL_1)_s \in \{C_3T_1, B_2T_2, A_2T_2\}$ and $({}^wL_4)_s \in \{B_2T_2, A_2T_2,\ A_1A_1\tilde{A}_1T_1\}$.*

(3) *If $G_s = \tilde{A}_1A_3$ then $({}^wL_1)_s$ contains a group of type A_1T_3 and $({}^wL_4)_s$ contains a group of type A_2T_2 or $A_1A_1T_2$.*

(4) *If $G_s = A_2\tilde{A}_2$ then $({}^wL_1)_s$ contains a subgroup of type $\tilde{A}_1T_3$ and $({}^wL_4)_s$ contains a subgroup of type A_1T_3.*

Proof. As was the case for E_6, we sometimes make arguments here that appear to require $p \neq 2$ (namely we will refer to $\mathrm{nilp}(X)$ and $\mathrm{nilp}(X_s)$ and centralizers of involutions). However, for the same reasons as before (e.g. the possibilities for X_s are controlled by the Weyl group), the possibilities that we give do not depend on the characteristic.

Parts (i) and (ii). Since $\mathrm{nilp}(L_i) = 2$ we have $\mathrm{nilp}({}^wL_i)_s \leq 2$. To obtain the groups in (i) and (ii) one lists those Levi subgroups of G_s which are also Levi subgroups of L and that have $\mathrm{nilp}({}^wL_i)_s \leq 2$. The remaining assertions in (i) follow from the fact that A_3T_1 must be the Levi factor of a parabolic in B_4 but in B_3T_1 the only group of type A_3T_1 is $\mathrm{SO}_2\,\mathrm{SO}_6$.

Part (iii). The assertion for $({}^wL_1)_s$ follows from considering the nilpotency class as in parts (i) and (ii). Let $\Phi_\ell(G)$, $\Phi_\ell(G_s)$, and $\Phi_\ell(L_4)$ be the long root subsystems of G, G_s and L_4 respectively. Then we have that $\Phi_\ell(G) = D_4$, $\Phi_\ell(G_s) = A_3$, $\Phi_\ell(L_4) = A_3$. Now consider subgroups of D_4 of these types. Each A_3 is the centralizer of an involution in D_4, so their intersection is the centralizer of an involution in A_3, and must contain an A_2 or A_1A_1.

Part (iv). It suffices to show that $\Phi(L_4)$ intersects every closed subsystem of type A_2, and similarly for $\Phi(L_1)$ and closed subsystems of type $\tilde{A}_2$.

Let $\Phi(A_2)$ be any long, closed, A_2-subsystem of $\Phi(F_4)$. Fix a base β_1, β_2. Since these are long roots, their α_4-coefficients equal -2, 0, or 2. At least one of β_1, β_2 or $\beta_1 + \beta_2$ has α_4-coefficient equal to -2 or 2, whence at least one of these roots has α_4-coefficient equal to 0 and this root is in $\Phi(L_4)$.

A similar argument shows applies to $\Phi(L_1)$, where all the α_1-coefficients equal 0, ± 1. □

Corollary 15. *Let $(X, P) \in \{(L_1, P_4), (P_4, L_1)\}$, $s \in T$ and $w \in W$ as in the Lemma. If $G_s \in \{B_4, A_1C_3, \tilde{A}_1A_3, A_2\tilde{A}_2\}$ then $X_s\backslash G_s/({}^wP)_s$ is finite.*

Proof. It suffices to show that $(L_1)_s\backslash G_s/({}^wP_4)_s$ and $(L_4)_s\backslash G_s/({}^wP_1)_s$ are finite where $({}^wL_4)_s$ and $({}^wL_4)_s$ are minimal among the possibilities listed in Lemma 14.

Suppose $G_s = B_4$. The previous result immediately gives the possibilities for X_s and ${}^wP_s$. It is easy to check that in every case $X_s\backslash G_s/{}^wP_s$ is finite.

In the remaining cases G_s has two simple factors. In these cases it suffices to break the collection $X_s\backslash G_s/{}^wP_s$ into two collections, one for each factor in G_s. Then finiteness holds for $X_s\backslash G_s/{}^wP_s$ if and only if it holds for each of the new collections. For example, when $G = A_1C_3$ then we have

$$|X_s\backslash G_s/({}^wP)_s| < \infty \iff \begin{cases} |(X_s \cap A_1)\backslash A_1/({}^wP_s \cap A_1)| < \infty \\ \qquad \text{and} \\ |(X_s \cap C_3)\backslash C_3/({}^wP_s \cap C_3)| < \infty. \end{cases}$$

If we let R denote one of the factors of G_s, it will sometimes be necessary to describe ${}^wP_s \cap R$. In such a case we will write P_i^R for the parabolic subgroup of R obtained by crossing off node i from the Dynkin diagram of R.

Suppose $G_s = A_1C_3$. Since T is contained in X_s, G_s and ${}^wP_s$ we have that $X_s = (X_s \cap A_1)(X_s \cap C_3)$ and similarly for $({}^wP)_s$. This allows us to decompose the double coset question:

The first collection is always finite since even the maximal torus is spherical in A_1. The possibilities for $(L_1)_s \cap C_3$ are all spherical in C_3 whence the second collection is finite when $X = L_1$. The possibilities for $(L_4)_s \cap C_3$ are $\{B_2T_1,\ A_2T_1,\ A_1\tilde{A}_1T_1\}$. The first two are spherical in C_3 so it remains to treat the last. The possibilities for $({}^wP_1)_s \cap C_3$ are $\{C_3,\ P_1^{C_3},\ P_3^{C_3}\}$. We see that $A_1\tilde{A}_1T_1$ is finite on all of these possibilities by the results in the classical groups (here $\tilde{A}_1 = C_1$).

Suppose that $G = \tilde{A}_1A_3T_1$. Since a maximal torus in $\tilde{A}_1$ is spherical, we have that $X_s \cap \tilde{A}_1\backslash\tilde{A}_1A_3/{}^wP_s \cap \tilde{A}_1$ is finite. By Lemma 14 we have that $P_i^{A_3} := ({}^wP_4)_s \cap$

A_3 is a maximal parabolic in A_3 and that $(L_1)_s \cap A_3$ contains an A_1T_2 subgroup. By results in the classical groups [6], we have that $A_1T_2\backslash A_3/P_i^{A_3}$ is finite. Therefore $L_1\backslash G_s/({}^wP_4)_s$ is finite. By Lemma 14, and [6], $(L_4)_s \cap A_3$ is spherical in A_3, whence $(L_4)_s \cap A_3\backslash A_3/({}^wP_1)_s \cap A_3$ is finite as well.

Suppose now that $G_s = A_2\tilde{A}_2$. It suffices to prove finiteness for the minimal possibilities for X_s and $({}^wP)_s$ so we assume $(L_1)_s = \tilde{A}_1T_3$ and $({}^wL_4)_s = A_1T_3$. As in the previous part, we may decompose the double coset question: $(L_1)_s\backslash G_s/({}^wP_4)_s$ is finite if and only if $\tilde{A}_1T_1\backslash \tilde{A}_2/B^{\tilde{A}_2}$ and $T_2\backslash A_2/P_i^{A_2}$ are both finite. Similarly $(L_4)_s\backslash G_s/({}^wP_1)_s$ is finite if and only if $A_1T_1\backslash A_2/B^{A_2}$ and $T_2\backslash \tilde{A}_2/P_i^{\tilde{A}_2}$ are both finite. It is now easy to check, using the results in A_n, that all the double cosets are finite. □

Now we verify condition (2) in Theorem 3, for $G = F_4$. We have $\dim G/P = 15$, and we are done with those cases where $\dim X_u \geq 15$ or where $d(X,u) = \frac{1}{2}(\dim G_u - 4) - \dim X_u \leq 0$. In the remaining cases one shows that $d(X,u) \leq \frac{1}{2}(\dim L_v - 4)$ where L is the Levi factor of P and v is the projection of u to L. As before, the cases where $d(X,u) = 1$ are relatively easy. This leaves only the case $(X,P) = (L_1,P_4)$, u of type A_1, and $d(X,u) = 2$. We find that v can be the identity, or a unipotent element of type A_1. If $v = 1$ then $\dim L_v = 22$ and $d(X,u) = 2 \leq \frac{1}{2}(\dim L_v - 4) = 9$. If v is of type A_1 then $\dim L_v = 12$ and $d(X,u) \leq \frac{1}{2}(\dim L_4 - 4) = 4$.

Now we offer a few comments about $G = F_4$ in bad characteristics. First, in characteristic 3 there are no substantial changes as Bala-Carter still holds for G, and the dimension formulas in X do not change. In characteristic 2, there is a graph automorphism of G that interchanges short and long roots. Therefore $|L_1\backslash F_4/P_4|$ is finite if and only if $|L_4\backslash F_4/P_1|$ is finite. So we now assume that $X = L_4 = D_3T_1$, $P = P_1$. In this characteristic F_4 has 4 extra classes, but only two of these can be represented in X [8, 5.3]. One of these is distinguished in the B_2T_2 Levi subgroup, and one is distinguished in X. These classes are denoted by $\tilde{A}_1^{(2)}$ and $B_2^{(2)}$ respectively in [14]. Using the information in [8, 5.3] we see that $\dim X_u = 12$, and from [19, Theorem 2.1] we see that $\dim G_u = 30$ whence $d(X,u) = 1$. As before, we let L be the Levi factor of P_1, let v be the projection u to L, and it suffices to show that v is not regular in L. Since u has 31 Jordan blocks it's action on $\mathfrak{g}$ [14], we see that v has at least 31 Jordan blocks as well, whence it is the identity, or of type A_1 or $\tilde{A}_1$ in G. None of these is the regular class in L.

Similarly, if u is of type $B_2^{(2)}$ then we have $\dim X_u = 6$ and $\dim G_u = 16$, so $d(X,u) = 0$ and we are done.

6. Infiniteness

In this section we prove all the infinite cases of double cosets required in Theorem 1.

Throughout this section, L denotes a Levi factor of P. We begin by stating a result which gives infiniteness in many cases.

Theorem 16 ([5, Theorem 1.3])**.** *Let G be a simple algebraic group, X a maximal rank reductive subgroup, P a parabolic subgroup with Levi factor L. If G equals F_4*

suppose that P is not an end node parabolic. If $X\backslash G/P$ is finite then X is spherical or L is spherical.

To finish the proof of infiniteness in Theorem 1 it suffices, by Theorem 16, to consider only those P such that L is spherical. Suppose that we have fixed such a P. Then it suffices to prove infiniteness for those X which are maximal subject to the condition that $X\backslash G/P$ is claimed to be infinite in Theorem 1. It also suffices to prove infiniteness after replacing X or P by a conjugate, or replacing both X and P by $\tau(X)$ and $\tau(P)$ where τ is a graph automorphism.

Combining these observations, it suffices to prove infiniteness in the following cases (in what follows the notation $\widetilde{L}_i$ denotes the pseudo-Levi subgroup obtained by crossing off node i from the extended Dynkin diagram of G).

(1) $G = E_6$, $X \in \{A_4T_2,\ A_1A_1A_3T_1,\ A_1A_2A_2T_1\}$ and $P = P_1$.
(2) $G = E_7$, $X \in \{D_5T_2,\ A_2A_4T_1,\ A_1A_3A_3,\ A_1A_5T_1,\ A_2A_2A_2T_1,\ A_1D_2D_4,\ A_1D_3D_3\}$, $P = P_7$ (the subgroups $A_1D_2D_4$ and $A_1D_3D_3$ are maximal subgroups of $A_1D_6 = \widetilde{L}_1$).
(3) $G = F_4$, $p \neq 2$, (X, P), an element of the following set $\{\widetilde{L}_2, \widetilde{L}_3, A_1A_1C_2, D_4, \widetilde{A}_1D_3\} \times \{P_1, P_4\}$, (L_1, P_1), (L_4, P_4) (the subgroup $A_1A_1C_2$ is a maximal subgroup of $\widetilde{L}_1 = A_1C_3$, the subgroups B_2D_2, D_4 and $\widetilde{A}_1D_3$ are maximal subgroups of $\widetilde{L}_4 = B_4$).
(4) $G = F_4$, $p = 2$ we have the same possibilities as above, and also $X = B_2B_2$, $P \in \{P_1, P_4\}$.

Theorem 17 ([5, Theorem 1.1, Lemma 3.3])**.** *Let L_1 and L_2 be conjugate Levi subgroups, each containing T, with root systems Φ_1 and Φ_2 respectively. If $\frac{1}{2}|\Phi_1| - \mathrm{rank}(\Phi_1) - |\Phi_1 \cap \Phi(X)| - \frac{1}{2}|\Phi_2 \cap \Phi(L)| > 0$ then $X\backslash G/P$ is infinite. In particular infiniteness holds in the following cases:*

(1) *Φ_1 and Φ_2 are each of type A_2, have the same length, $\Phi_1 \cap \Phi(X) = \Phi_2 \cap \Phi(L) = \emptyset$;*
(2) *Φ_1 and Φ_2 are of type B_2, $\Phi_1 \cap \Phi(X) = \emptyset$ and $\Phi_2 \cap \Phi(L)$ is empty or of type A_1;*
(3) *Φ_1 and Φ_2 are of type A_3 or D_3, $\Phi_1 \cap \Phi(X) = \emptyset$ and $\Phi_2 \cap \Phi(L)$ is of type A_1A_1 or D_2.*
(4) *Φ_1 and Φ_2 are of type C_3 with $\Phi_1 \cap \Phi(X) = A_1$ and $\Phi_2 \cap \Phi(L) = A_2$.*

To describe one of the root subsystems Φ_i we will explicitly describe roots β_1, β_2, …, that generate Φ_i. Each β_i will be described by giving its coefficients with respect to the simple roots, arranged as they appear on the Dynkin diagram (labelled as in [2]). For example, if $\alpha_1, \ldots, \alpha_7$ are the simple roots of E_7, then the notation $\begin{smallmatrix}1&1&2&1&1&0\\&&1&&&\end{smallmatrix}$ describes the root $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$.

Let $G = E_6$, $X \in \{A_4T_2,\ A_1A_1A_3T_1,\ A_1A_2A_2T_1\}$ and $P = P_1$. We will construct Φ_1 and Φ_2 satisfying Theorem 17 part 3. Let Φ_2 be the closed subsystem generated by $\beta_1 = \begin{smallmatrix}0&1&1&0&0\\&&1&&\end{smallmatrix}$, $\beta_2 = \begin{smallmatrix}1&2&2&2&0\\&&0&&\end{smallmatrix}$, $\beta_3 = \begin{smallmatrix}0&0&1&1&1\\&&1&&\end{smallmatrix}$. For $X = A_1A_2A_2T_1$, we let $X = L_4$ and let $\Phi_1 = \Phi_2$. For $X = A_4T_2$ let $X = L_{1,3}$ and let Φ_1 be the closed root subsystem generated by $\beta_1 = \begin{smallmatrix}0&1&1&1&0\\&&0&&\end{smallmatrix}$, $\beta_2 = \begin{smallmatrix}1&0&0&0&0\\&&0&&\end{smallmatrix}$, and $\beta_3 = \begin{smallmatrix}0&0&1&1&1\\&&1&&\end{smallmatrix}$. Finally, for $X = A_1A_1A_3T_1$ we let $X = \widetilde{L}_{3,5}$, the pseudo-Levi subgroup obtained by crossing off α_3 and α_5 from the extended Dynkin diagram of E_6. Then $\Phi(X)$ equals all those roots in $\Phi(E_6)$ with α_3-coefficient equal to 0 or ± 2 and α_5-coefficient equal to 0 or ± 2. We let Φ_1 be the closed root subsystem generated by $\beta_1 = \begin{smallmatrix}0&0&1&1&1\\&&0&&\end{smallmatrix}$, $\beta_2 = \begin{smallmatrix}1&1&1&0&0\\&&1&&\end{smallmatrix}$, $\beta_3 = \begin{smallmatrix}0&0&0&1&0\\&&0&&\end{smallmatrix}$.

Let $G = E_7$, $X \in \{D_5T_2,\ A_2A_4T_1,\ A_1A_3A_3,\ A_1A_5T_1,\ A_2A_2A_2T_1,\ A_1D_2D_4,\ A_1D_3D_3\}$. Note that $X = A_2A_2A_2T_1$ gives $|X\backslash G/P| = \infty$ since $\dim X = 25 < \dim G/P$. For the other cases we will construct subsystems of type A_3 satisfying Theorem 17 part (3). Let Φ_2 be generated by $\beta_1 = \begin{smallmatrix}0&0&0&0&1&0\\&&0&&&\end{smallmatrix}$, $\beta_2 = \begin{smallmatrix}0&0&0&0&0&1\\&&0&&&\end{smallmatrix}$, $\beta_3 = \begin{smallmatrix}0&1&2&2&1&0\\&&1&&&\end{smallmatrix}$. For $X = D_5T_2$, let $X = L_{6,7}$ and let $\Phi_1 = \Phi_1$. For $X = A_1A_3A_3$, let $X = \widetilde{L}_4$, and let Φ_1 be generated by $\beta_1 = \begin{smallmatrix}0&1&1&1&0&0\\&&1&&&\end{smallmatrix}$ $\beta_2 = \begin{smallmatrix}0&0&1&0&0&0\\&&0&&&\end{smallmatrix}$, $\beta_3 = \begin{smallmatrix}1&1&1&1&1&0\\&&1&&&\end{smallmatrix}$. For $X = A_1A_5T_1$ let $X = L_3$, let Φ_1 be generated by $\beta_1 = \begin{smallmatrix}1&1&2&1&1&0\\&&1&&&\end{smallmatrix}$, $\beta_2 = \begin{smallmatrix}0&1&1&1&1&1\\&&1&&&\end{smallmatrix}$ and $\beta_3 = \begin{smallmatrix}1&1&1&1&0&0\\&&0&&&\end{smallmatrix}$. For $X = A_2A_4T_1$, let $X = L_5$, let Φ_1 be generated by $\beta_1 = \begin{smallmatrix}1&1&2&1&1&0\\&&1&&&\end{smallmatrix}$, $\beta_2 = \begin{smallmatrix}0&0&0&1&0&0\\&&0&&&\end{smallmatrix}$, $\beta_3 = \begin{smallmatrix}1&2&2&1&1&0\\&&1&&&\end{smallmatrix}$. For X equal to $A_1D_2D_4$ or $A_1D_2D_4$ we let X be a maximal subgroup of $\tilde{L}_1 = A_1D_6$. We will give the same Φ_1 for both cases. If $X = A_1D_2D_4$ then we can describe $\Phi(X)$ as those roots in $\Phi(G)$ with α_1-coefficient equal to 0 or ± 2, and α_6-coefficient equal to 0 or ± 2. For $X = A_1D_3D_3$ we can describe its root system as those roots with α_1-coefficient equal to 0 or ± 2 and α_5-coefficient equal to 0 or ± 2. Let Φ_1 be generated by $\beta_1 = \begin{smallmatrix}0&0&1&1&1&0\\&&0&&&\end{smallmatrix}$, $\beta_2 = \begin{smallmatrix}1&1&0&0&0&0\\&&0&&&\end{smallmatrix}$, $\beta_3 = \begin{smallmatrix}0&0&1&1&1&1\\&&1&&&\end{smallmatrix}$.

Let $G = F_4$ and (X, P), equal to one of the following elements $\{\widetilde{L}_2,\ \widetilde{L}_3,\ A_1A_1C_2,\ D_4,\ \widetilde{A}_1D_3\} \times \{P_1, P_4\}$, (L_1, P_1), (L_4, P_4). If $X = A_1A_1C_2$ then $|X\backslash G/P| = \infty$ since $\dim X = 12 < \dim G/P$. By [5, Prop. 21] if $(X, P) \in \{(\tilde{L}_2, P_1),\ (\widetilde{L}_2, P_4),\ (\widetilde{L}_3, P_4),\ (L_1, P_1),\ (L_4, P_4),\ (A_1A_1C_2, P_4),\ (B_2D_4, P_4),\ (D_4, P_4),\ (\widetilde{A}_1D_3, P_4)\}$, then there exist A_2 subsystems satisfying Theorem 17 part (1). All the remaining cases involve $P = P_1$, and we will be describing root systems satisfying Theorem 17 part (4). In all the remaining cases (except the $p = 2$ case), let $\Phi_1 = \Phi_2$ be the system generated by $\beta_1 = 0\,0\,1\,0$, $\beta_2 = 0\,1\,1\,1$, and $\beta_3 = 1\,0\,0\,0$. Now we discuss how to see that $\Phi(X) \cap \Phi_1$ has type A_1. If $X = \widetilde{L}_3$, then it is easy to identify $\Phi(X)$ and verify the claim. If $X = D_4$, then we note that $\Phi(X) \cap \Phi_1 = A_1$ is a closed subsystem formed of long roots, and the only such subsystem in a C_3 root system has type A_1. If $X = \tilde{A}_1D_3$, then we describe $\Phi(X)$ explicitly. As above, we write roots by giving their coefficients with respect to the root base of F_4. Thus, we let $\alpha_1, \ldots, \alpha_4$ be a base for F_4, labelled as in [2]. Then the notation $0\,1\,2\,2$ describes the root $\alpha_2 + 2\alpha_3 + 2\alpha_4$. Then we can take a base of $\Phi(X)$ given by $0\,0\,1\,0$ (this gives the $\tilde{A}_1$) and $-2\,3\,4\,2$, $1\,0\,0\,0$, $0\,1\,2\,2$.

The final case is $G = F_4$, $p = 2$ and $X = B_2B_2$. We will construct Φ_1 and Φ_2 satisfying Theorem 17 part 2. First we describe X. In characteristic 2 one may extend the Dynkin diagram for F_4 by attaching the (negative) high short root to α_4. Crossing off α_1 gives C_4 subsystem. One can extend C_4 with its high root, which equals the high root of F_4. Crossing off the middle node of the extended C_4 (i.e. α_4 in F_4) gives a base for $\Phi(X)$ consisting of $0\,1\,0\,0$, $0\,0\,1\,0$ and $-1\,2\,3\,2$, $2\,3\,4\,2$. Now we let $\Phi_1 = \Phi_2$ be generated by $\beta_1 = 1\,1\,0\,0$ and $\beta_2 = 0\,1\,2\,1$. Notice that $\Phi_1 \cap \Phi(X) = \emptyset$, $\Phi_2 \cap \Phi(L_4) = \emptyset$ and $\Phi_2 \cap \Phi(L_1) = A_1$.

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