

A CLASSIFICATION OF CERTAIN FINITE DOUBLE COSET COLLECTIONS IN THE CLASSICAL GROUPS

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ABSTRACT

Let G be a classical algebraic group, X a maximal rank reductive subgroup and P a parabolic subgroup. This paper classifies when $X \backslash G/P$ is finite. Finiteness is proven using geometric arguments about the action of X on subspaces of the natural module for G . Infiniteness is proven using a dimension criterion which involves root systems.

1. Statement of results

In this paper we classify when the double coset collection $X \backslash G/P$ is finite, where G is a classical algebraic group, X is a maximal rank reductive subgroup, and P is a parabolic subgroup, all defined over an algebraically closed field. The property of finiteness for $X \backslash G/P$ is preserved under taking isogenies, quotients by the center of G , connected components and conjugates (see Lemma 2.1 for a precise statement). Thus, we may state Theorem 1 by specifying only the Lie type of G . Similarly, we can specify only the conjugacy class of X and P ; thus we usually give the Lie type of X and describe P by crossing off nodes from the Dynkin diagram for G . For the purpose of classifying finiteness, it suffices to consider only those X which are defined over \mathbb{Z} , as the others arise via isogenies or graph automorphisms.

A subgroup X is *spherical* if $X \backslash G/B$ is finite for some (or, equivalently, for each) Borel subgroup B . For each classical group we list in Table 1 those maximal rank reductive spherical subgroups which are defined over \mathbb{Z} . We first describe the notation which is used for the list, and for the rest of the paper, and then describe how the list is obtained. We write $X = A_n A_m T_1$ if X is a group of Lie type $A_n + A_m$ which has a 1-dimensional central torus, and we use similar notation for other subgroups. If G equals D_n we adopt a notational convention to distinguish between certain subgroups of the same Lie type which are not conjugate. In $G = \mathrm{SO}(V)$ any factor denoted by D_{n_1} (or SO_{2n_1}) acts as $\mathrm{SO}(V_1)$ for some decomposition $V = V_1 \perp V_2$ and any factors denoted by $A_{n_1} T_1$ (or GL_{n_1+1}) act as GL_{n_1+1} on a pair of totally singular subspaces E and F such that $V = (E \oplus F) \perp V_2$ (in particular

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TABLE 1. *Maximal rank reductive spherical subgroups defined over \mathbb{Z}*

X	\leq	G	X	\leq	G
$A_n A_m T_1$	\leq	A_{n+m+1}	$B_n D_m$	\leq	B_{n+m}
$C_n C_m$	\leq	C_{n+m}	$A_{n-1} T_1$	\leq	B_n
$C_{n-1} T_1$	\leq	C_n	$D_n D_m$	\leq	D_{n+m}
$A_{n-1} T_1$	\leq	C_n	$A_{n-1} T_1$	\leq	D_n

$\dim E = \dim F = n_1 + 1$ and E and F are in duality). We allow the notation A_0, B_0 and C_0 to denote trivial groups and D_1 to denote a group which is a 1-dimensional torus. We now describe how Table 1 is obtained. Krämer [8] classified the reductive spherical subgroups in characteristic 0. The subgroups on Krämer's list were shown to be spherical in all characteristics by Brundan [3] and Lawther [9]. Duckworth [4] showed that this list is complete for maximal rank subgroups.

In Theorem 1 we use the notational conventions just described, as well as the following. We write $X = L_i$ if X is conjugate to a Levi subgroup obtained by crossing off node i from the Dynkin diagram of G (we number the nodes of the Dynkin diagram as in [2]). The meaning of $X = L_{i_1, i_2}$ and $P = P_i$ is similar.

THEOREM 1. *Let G be a simple algebraic group of type A_n, B_n, C_n or D_n , let X be a maximal rank reductive subgroup defined over \mathbb{Z} and let $P \neq G$ be a parabolic subgroup. Then $X \backslash G/P$ is finite if and only if X is spherical or one of the following holds:*

- (i) $G = A_n$,
 - (a) $P \in \{P_1, P_n\}$, or
 - (b) $X = A_{n_1} A_{n_2} A_{n_3} T_2$ and $P = P_i$ for some i ;
- (ii) $G = B_n$, $X = A_{n_1} B_{n_2} T_1$ and $P \in \{P_1, P_n\}$;
- (iii) $G = C_n$,
 - (a) $X \in \{C_{n_1} \cdots C_{n_r}, A_{n_1} C_{n_2} \cdots C_{n_r} T_1\}$ and $P = P_1$, or
 - (b) $X \in \{C_{n_1} C_{n_2} C_{n_3}, C_{n_1} C_{n_2} T_1, A_{n_1} C_{n_2} T_1\}$ and $P = P_n$;
- (iv) $G = D_n$,
 - (a) $G = D_4$, $(X, P) \in \{(L_{2,3}, P_4), (L_{2,4}, P_3)\}$, or
 - (b) $X \in \{A_{n_1} D_{n_2} T_1, A_{n_1} A_{n_2} T_2\}$ and $P = P_1$, or
 - (c) $X \in \{D_{n_1} D_{n_2} D_{n_3}, A_{n_1} D_{n_2} T_1\}$ and $P \in \{P_{n-1}, P_n\}$.

2. An example and preliminaries

The study of spherical subgroups provides one example of a double coset problem. We briefly describe another example here, and refer the reader to [11] for a fuller discussion.

The classification of irreducible finite orbit modules is related to the work in the present paper, and an example of classifying finite double cosets. Let V be a finite dimensional vector space and let X be a closed connected subgroup of $G = \mathrm{GL}(V)$ such that X has finitely many orbits on V and V is irreducible under X . Such modules were classified by Kac [7] in characteristic 0 and by Guralnick, Liebeck, Macpherson and Seitz [6] in positive characteristic. To see the connection between this work and double cosets, note that X is a reductive group and that $X \backslash \mathrm{GL}(V)/P_1$ is finite where P_1 is the stabilizer of a 1-space. The authors of [6] also classified the finiteness of $X \backslash \mathrm{GL}(V)/P_i$ where $1 \leq i \leq \dim V$, and they established Theorem 1 (i), for maximal parabolics, using rather different arguments than those that appear in the present paper.

To summarize, in [6] the group G is of type A_n , X is irreducible and P is a maximal parabolic. In Theorem 1, G is any classical group, X is maximal rank and reductive, and P is any parabolic. Thus Theorem 1 can be viewed as an extension, or a complement, of the work in [6]. In later papers we intend to treat the case where G is of type B_n, C_n , or D_n , and X is irreducible and the case where G is exceptional and X is maximal rank reductive.

The following lemma provides basic reductions in our double coset question.

LEMMA 2.1. *Let G be a group and let X and P be subgroups. Let Z be the center of G , suppose that $Z \leq P$ and let \overline{X} , \overline{G} and \overline{P} be the images of X , G and P , respectively, under the map $G \rightarrow G/Z$. Let K be a finite normal subgroup of G and let \widehat{X} , \widehat{G} and \widehat{P} be the images of X , G and P , respectively, under the map $G \rightarrow G/K$. Let $g, h \in G$. The following are equivalent:*

- (i) $|X \backslash G / P| < \infty$,
- (ii) $|\widehat{X} \backslash \widehat{G} / \widehat{P}| < \infty$,
- (iii) $|\overline{X} \backslash \overline{G} / \overline{P}| < \infty$,
- (iv) $|gXg^{-1} \backslash G / hPh^{-1}| < \infty$.

Let G be an algebraic group with X and P closed subgroups. Denote by X° , G° and P° the identity components of X , G and P respectively. If $X \backslash G / P$ is finite then so is $X^\circ \backslash G^\circ / P^\circ$.

Proof. These statements can all be proven in an elementary fashion. The final statement uses only the fact that X° and P° are normal subgroups of finite index in X and P respectively. \square

This lemma justifies the assumptions and conventions mentioned before Theorem 1. The final statement of the lemma will be applied to recover finiteness results in $\mathrm{SO}(V)$ from arguments made involving $\mathrm{O}(V)$.

REMARKS 2.2. We fix some notation for the rest of the paper. We let G be a classical algebraic group defined over a fixed algebraically closed field. If G is of type B_n , C_n or D_n we will, when convenient, assume that G equals SO_{2n+1} , Sp_{2n} or SO_{2n} , respectively. If G is of type A , we will often take $G = \mathrm{GL}_n$; note that the end-node parabolics are then P_1 and P_{n-1} . In all cases G has rank n . We denote by X a maximal rank reductive subgroup of G and by P a parabolic subgroup of G .

We fix some terminology since usage varies in the literature and refer the reader to [12] or [5] for further details. To each possibility for G we associate a natural module V and a bilinear and quadratic form (we take these forms to be identically zero if $G = \mathrm{GL}_n$). A subspace of V is totally singular if each form is identically zero on the subspace. If P is a parabolic subgroup then it equals the stabilizer of a (partial) flag of totally singular subspaces of V and we identify G/P as a collection of flags of totally singular subspaces. Thus, a spherical subgroup has a finite number of orbits on the set of flags of totally singular subspaces.

Let G equal SO_{2n} . Then two totally singular n -spaces are conjugate under G if and only if their intersection has odd codimension in each space. Thus, there exist two G -classes of totally singular n -spaces which we identify as G/P_{n-1} and G/P_n . If n is odd then L_{n-1} is conjugate to L_n under G , but this is not the case if n is even. These facts are relevant to Theorem 1 (iv).

Let V be the natural module for G . We write $V = V_1 \perp V_2$ if $V = V_1 \oplus V_2$ and each element of V_1 is orthogonal to each element of V_2 . Given such a decomposition we let $\mathrm{Cl}(V_i)$ equal $\mathrm{GL}(V_i)$, $\mathrm{O}(V_i)$ or $\mathrm{Sp}(V_i)$ according as G equals $\mathrm{GL}(V)$, $\mathrm{SO}(V)$ or $\mathrm{Sp}(V)$, respectively. We let $\mathrm{Cl}(V_i)^\circ$ be the connected identity component of $\mathrm{Cl}(V_i)$. If $\mathrm{Cl}(V_i) = \mathrm{O}(V_i)$ then $\mathrm{Cl}(V_i)^\circ = \mathrm{SO}(V_i)$ and otherwise $\mathrm{Cl}(V_i)^\circ = \mathrm{Cl}(V_i)$.

TABLE 2. *Finiteness Cases*

G	Cases	Proof
A_n	all cases	Corollary 3.6
B_n	the case with $P = P_1$	Corollary 3.6
	the case with $P = P_n$	Corollary 3.11
C_n	all cases with $P = P_1$	Corollary 3.6
	all cases with $P = P_n$	Corollary 3.9
D_n	$G = D_4$, all cases	Corollary 3.13
	$(X, P) = (A_{n_1}D_{n_2}T_1, P_1)$	Corollary 3.6
	$(X, P) = (A_{n_1}A_{n_2}T_2, P_1)$	Lemma 3.12
	$(X, P) \in \{D_{n_1}D_{n_2}D_{n_3}, A_{n_1}D_{n_2}T_1\} \times \{P_{n-1}, P_n\}$	Corollary 3.9

3. Finiteness

In this section we will prove those parts of Theorem 1 which assert finiteness. For the convenience of the reader we list in Table 2 the specific result which covers each case.

The next lemma is not used immediately, but we place it here to preserve the line of argument later.

LEMMA 3.1. *Let $G = \mathrm{SO}_{2n}$ and let $X = \mathrm{GL}_n$. Then X acts transitively upon the set of definite 1-spaces in V and has a finite number of orbits upon the set of all 1-spaces in V .*

Proof. Let N_1 be the stabilizer of a definite 1-space in G . Then the first claim is equivalent to having $G = XN_1$. In this form the first claim is proven in [10]. The second claim follows from the facts that GL_n is spherical in G and that every 1-space is either singular or definite. \square

REMARK 3.2. We fix some notation for the next few lemmas. Let G be one of $\mathrm{GL}(V)$, $\mathrm{O}(V)$, $\mathrm{SO}(V)$, or $\mathrm{Sp}(V)$ and fix a decomposition $V = V_1 \perp V_2$ such that $X = X_1X_2$ for subgroups $X_i \leq \mathrm{Cl}(V_i)$. Let P be a maximal parabolic subgroup of G and identify G/P as a collection of totally singular subspaces in V . For $i \in \{1, 2\}$ let $\pi_i : V \rightarrow V_i$ be the natural projection.

LEMMA 3.3. *Let the notation be as in Remark 3.2. In addition let β and φ be, respectively, the bilinear and quadratic forms associated with G .*

- (i) *Let $u, v, x, y \in V$ such that $\beta(u, v) = \beta(x, y) = 0$. Then $\beta(\pi_1 u, \pi_1 v) = \beta(\pi_1 x, \pi_1 y)$ if and only if $\beta(\pi_2 u, \pi_2 v) = \beta(\pi_2 x, \pi_2 y)$.*
- (ii) *Let u and x be two singular vectors with $\varphi(\pi_1 u) = \varphi(\pi_1 x)$. Then $\varphi(\pi_2 u) = \varphi(\pi_2 x)$.*

Proof. Decompose u as $u = \pi_1 u + \pi_2 u$, decompose v, x , and y similarly, and use the fact that V_1 and V_2 are orthogonal to each other. \square

LEMMA 3.4. *Let the notation be as in Remark 3.2 with the additional assumptions that $G = \mathrm{Cl}(V)$ and $X_2 = \mathrm{Cl}(V_2)$. Two totally singular subspaces of the same dimension are conjugate under X if and only if their projections to V_1 and intersections with V_1 are simultaneously conjugate under X_1 .*

The proof of this statement uses Witt's Theorem applied to X_2 , so, if G is an orthogonal group, one cannot replace $\text{Cl}(V_2) = \text{O}(V_2)$ with $\text{Cl}(V_2)^\circ = \text{SO}(V_2)$. We will use Lemma 2.1 to translate finiteness results to SO_n . We note that finiteness results do not always translate between O_n and SO_n in an obvious fashion. For example, the collection $L_{2,3} \backslash \text{SO}_8 / P_4$ is finite whereas $L_{2,3} \backslash \text{O}_8 / P_4$ is infinite.

Proof. It is easy to see that if two subspaces are conjugate under X , then their projections to V_1 and intersections with V_1 are simultaneously conjugate under X_1 .

Conversely, let W and W' be totally singular subspaces of the same dimension such that $x_1(W \cap V_1, \pi_1 W) = (W' \cap V_1, \pi_1 W')$ for some $x_1 \in X_1$. Replacing W with $x_1 W$ we may assume that $(W \cap V_1, \pi_1 W) = (W' \cap V_1, \pi_1 W')$. Note that $\dim W \cap V_2 = \dim W' \cap V_2$. Define the following dimensions:

$$\begin{aligned} a &= \dim W \cap V_1 = \dim W' \cap V_1, \\ c &= \dim W \cap V_2 = \dim W' \cap V_2, \\ b &= \dim W - a - c = \dim W' - a - c. \end{aligned}$$

We will pick bases for W and W' as follows:

$$\begin{aligned} W : & w_1, \dots, w_a, \quad w_{a+1}, \dots, w_{a+b}, \quad w_{a+b+1}, \dots, w_{a+b+c}, \\ W' : & w_1, \dots, w_a, \quad w'_{a+1}, \dots, w'_{a+b}, \quad w'_{a+b+1}, \dots, w'_{a+b+c}. \end{aligned}$$

We start by picking a basis w_1, \dots, w_a of $W \cap V_1 = W' \cap V_1$. Extend this with elements v_{a+1}, \dots, v_{a+b} to a basis of $\pi_1 W = \pi_1 W'$. For each $i \in \{a+1, \dots, a+b\}$ pick $w_i \in W$ and $w'_i \in W'$ such that $\pi_1 w_i = \pi_1 w'_i = v_i$. Let $w_{a+b+1}, \dots, w_{a+b+c}$ and $w'_{a+b+1}, \dots, w'_{a+b+c}$ be bases for $W \cap V_2$ and $W' \cap V_2$ respectively. Then each of $\{w_{a+1}, \dots, w_{a+b+c}\}$ and $\{w'_{a+1}, \dots, w'_{a+b+c}\}$ is a linearly independent set, whence each of $\{\pi_2 w_{a+1}, \dots, \pi_2 w_{a+b+c}\}$ and $\{\pi_2 w'_{a+1}, \dots, \pi_2 w'_{a+b+c}\}$ is a linearly independent set.

Let \tilde{x}_2 be the linear map from the subspace $\langle \pi_2 w_{a+1}, \dots, \pi_2 w_{a+b+c} \rangle$ to the subspace $\langle \pi_2 w'_{a+1}, \dots, \pi_2 w'_{a+b+c} \rangle$ which takes each $\pi_2 w_i$ to $\pi_2 w'_i$.

If $G = \text{GL}(V)$ then one may extend \tilde{x}_2 to an element $x_2 \in \text{GL}(V_2) = X_2$. Note that $x_2 w_i = w'_i$ for each $i \in \{a+1, \dots, a+b+c\}$. This finishes the proof for the case $G = \text{GL}(V)$.

If $G \in \{\text{O}(V), \text{Sp}(V)\}$ we show that \tilde{x}_2 is an isometry from the subspace $\langle \pi_2 w_{a+1}, \dots, \pi_2 w_{a+b+c} \rangle$ to $\langle \pi_2 w'_{a+1}, \dots, \pi_2 w'_{a+b+c} \rangle$. Once this is done, Witt's Theorem implies that we may again extend \tilde{x}_2 to $x_2 \in \text{Cl}(V_2) = X_2$ and we will be finished. If $p \neq 2$ or if G is symplectic, then Lemma 3.3(i) shows that $\beta(\pi_2 w_i, \pi_2 w_j) = \beta(\pi_2 w'_i, \pi_2 w'_j)$ for all $a+1 \leq i, j \leq a+b+c$, whence \tilde{x}_2 is an isometry. If $p = 2$ and G is orthogonal, then Lemma 3.3(ii) shows that $\varphi(\pi_2 w_i) = \varphi(\pi_2 w'_i)$ for $a+1 \leq i \leq a+b+c$, whence \tilde{x}_2 is an isometry. \square

COROLLARY 3.5. *Let the notation be as in Remark 3.2 with the additional assumption that X_2 equals $\text{Cl}(V_2)$ or $\text{Cl}(V_2)^\circ$. Then $X \backslash G / P$ is finite in the following cases:*

- (i) X_1 has a finite number of orbits upon the set of 1-spaces in V_1 and P equals P_1 ;
- (ii) $G = \text{GL}(V)$, X_1 has a finite number of orbits upon the set of all flags in V_1 and P equals P_i for some i ;
- (iii) $G = \text{GL}(V) = \text{GL}_n$, X_1 has a finite number of orbits upon the set of subspaces of V_1 with codimension 1 and P equals P_{n-1} .

Proof. By Lemma 2.1 it suffices to prove Corollary 3.5 with the assumption that $G = \text{Cl}(V)$ and $X = \text{Cl}(V_2)$. By Lemma 3.4 it suffices to show that X_1 has a finite number of orbits upon the set $\{(W \cap V_1, \pi_1 W) \mid W \in G/P\}$. Note that $W \cap V_1 \leq \pi_1 W$ is a flag. It is easy to verify in each case that X_1 has a finite number of orbits on the given set. \square

COROLLARY 3.6. *The double coset collection $X \backslash G/P$ is finite in the following cases:*

- (i) $G = \text{GL}_n$, (a) X is any maximal rank reductive subgroup and $P \in \{P_1, P_{n-1}\}$ or (b) $X = \text{GL}_{n_1} \text{GL}_{n_2} \text{GL}_{n_3}$ and $P = P_i$ for some i ;
- (ii) $(G, X) \in \{(\text{SO}_{2n+1}, \text{GL}_{n_1} \text{SO}_{2n_2+1}), (\text{SO}_{2n}, \text{GL}_{n_1} \text{SO}_{2n_2})\}$ and $P = P_1$;
- (iii) $G = \text{Sp}_{2n}$, $X \in \{\text{GL}_{n_1} \text{Sp}_{2n_2} \cdots \text{Sp}_{2n_r}, \text{Sp}_{2n_1} \cdots \text{Sp}_{2n_r}\}$ and $P = P_1$;

Proof. In each case let V be the natural module for G and fix a decomposition $V = V_1 \perp V_2$ such that $X = X_1 X_2$ with $X_1 \leq \text{Cl}(V_1)$ and $X_2 = \text{Cl}(V_2)^\circ$.

For case (i)(b) note that $X_1 = \text{GL}_{n_1} \text{GL}_{n_2}$ is a spherical subgroup of $\text{GL}_{n_1+n_2} = \text{Cl}(V_1)$ and apply Corollary 3.5 (ii). For case (i)(a) apply Corollary 3.5 part (i) or (iii) according as P equals P_1 or P_{n-1} respectively, and induct on the number of factors in X of the form GL_{n_i} .

Let $G = \text{Sp}_{2n}$. By Corollary 3.5 (i) it suffices to show that X_1 has finitely many orbits on totally singular 1-spaces in V_1 (note that all 1-spaces are totally singular in this case). This is immediate if X_1 equals GL_{n_1} or $\text{Sp}_{2n_1} \text{Sp}_{2n_2}$ since these subgroups are spherical in $\text{Sp}(V_1)$. The general case follows by induction on r .

If G is orthogonal, then $X_1 = \text{GL}_{n_1}$. One applies Corollary 3.5 (i) and Lemma 3.1. \square

LEMMA 3.7. *Let the notation be as in Remark 3.2 with the additional assumption that V is a symplectic or orthogonal space. Let W be a maximal totally singular subspace of V and let $(\pi_i W)^\perp$ be the perpendicular space taken within V_i . If $\dim V$ is even then $\dim(\pi_i W)^\perp / (W \cap V_i)$ equals 0. If $\dim V$ is odd then $\dim(\pi_i W)^\perp / (W \cap V_i)$ equals 0 or 1.*

Proof. Since V_1 is orthogonal to V_2 it is easy to show that $W \cap V_i \leq (\pi_i W)^\perp$. We have that $\dim W$ equals n and $\dim V$ equals $2n$ or $2n+1$. Set $a_i = \dim W \cap V_i$. For i equal to 1 or 2 the inequality $\dim(\pi_i W)^\perp \geq \dim W \cap V_i$ becomes, respectively, $\dim V_1 - (n - a_2) \geq a_1$ or $\dim V_2 - (n - a_1) \geq a_2$. If $\dim V$ is even, then the sum of these last two inequalities is an equality; if $\dim V$ is odd, then the sum of the left sides is 1 greater than the sum of the right sides. \square

COROLLARY 3.8. *Let the notation be as in Remark 3.2 with the additional assumptions that G is not $\text{GL}(V)$, that X_2 equals $\text{Cl}(V_2)^\circ$ or $\text{Cl}(V_2)$ and that P equals P_n . Then $X \backslash G/P$ is finite in the following cases:*

- (i) $\dim V$ is even and X_1 has a finite number of orbits on the set of totally singular subspaces of V_1 ;
- (ii) $\dim V$ is odd and X_1 has a finite number of orbits on the set of pairs of subspaces (W_1, W_2) such that $W_1 \leq W_2 \leq V_1$, W_1 is totally singular, $W_1 \leq W_2^\perp$ and $\dim(W_2/W_1) \leq 1$.

Proof. By Lemma 2.1 we may assume that $G = \text{Cl}(V)$ and $X_2 = \text{Cl}(V)$. By

Lemma 3.4 it suffices to show that X_1 has a finite number of orbits upon the set $\{(W \cap V_1, \pi_1 W) \mid W \in G/P\}$. Given subspaces $W, W' \leq V$ and $x_1 \in X_1$ we have $x_1 \pi_1 W = \pi_1 W'$ if and only if $x_1(\pi_1 W)^\perp = (\pi_1 W')^\perp$, where we take the perpendicular space within V_1 . Thus, it suffices to show that X_1 has a finite number of orbits on the set $\{(W \cap V_1, (\pi_1 W)^\perp) \mid W \in G/P\}$. By Lemma 3.7 we see that $\{(W \cap V_1, (\pi_1 W)^\perp) \mid W \in G/P\}$ is a subset (or may be identified with a subset) of one of the sets given in the statement of Corollary 3.8. \square

COROLLARY 3.9. *The double coset collection $X \backslash G/P$ is finite in the following cases:*

- (i) $G = \mathrm{SO}_{2n}$, $X \in \{\mathrm{SO}_{2n_1} \mathrm{SO}_{2n_2} \mathrm{SO}_{2n_3}, \mathrm{GL}_{n_1} \mathrm{SO}_{2n_2}\}$ and $P \in \{P_{n-1}, P_n\}$;
- (ii) $G = \mathrm{Sp}_{2n}$, $X \in \{\mathrm{Sp}_{2n_1} \mathrm{Sp}_{2n_2} \mathrm{Sp}_{2n_3}, \mathrm{GL}_{n_1} \mathrm{Sp}_{2n_2}, \mathrm{Sp}_{2n_1} \mathrm{Sp}_{2n_2} T_1\}$ and $P = P_n$.

Proof. Let V be the natural module of G and fix a decomposition $V = V_1 \perp V_2$ so that $X = X_1 X_2$ with $X_1 \leq \mathrm{Cl}(V_1)^\circ$ and $X_2 = \mathrm{Cl}(V_2)^\circ$. Then X_1 is a spherical subgroup of $\mathrm{Cl}(V_1)^\circ$ whence the conclusion follows from Corollary 3.8 (i). \square

LEMMA 3.10. *Let $G = \mathrm{SO}_{2n}$, let $X = \mathrm{GL}_n$ and let V be the natural module for G . Let $\langle v \rangle$ be a definite 1-space, let $X_{\langle v \rangle}$ be the stabilizer in X of $\langle v \rangle$ and let $\tilde{X}_{\langle v \rangle}$ be the identity component of the group induced by $X_{\langle v \rangle}$ in $\mathrm{SO}(\langle v \rangle^\perp)$. Then $\tilde{X}_{\langle v \rangle}$ is a spherical subgroup of $\mathrm{SO}(\langle v \rangle^\perp)$.*

Proof. By Lemma 3.1 we may calculate $X_{\langle v \rangle}$ where v is any definite vector. Let $V = E \oplus F$ such that E and F are totally singular and X is the stabilizer in G of this decomposition. Let $\{e_1, \dots, e_n\} \subset E$ and $\{f_1, \dots, f_n\} \subset F$ be dual bases. Let $v = e_1 + f_1$ and let GL_{n-1} denote the subgroup of X which stabilizes the subspaces $\langle e_2, \dots, e_n \rangle$ and $\langle f_2, \dots, f_n \rangle$ and acts trivially upon e_1 and f_1 . Then (an isomorphic image of) GL_{n-1} is a subgroup of $\tilde{X}_{\langle v \rangle}$ which proves the claim since $\mathrm{SO}(\langle v \rangle^\perp) = \mathrm{SO}_{2(n-1)+1}$. \square

COROLLARY 3.11. *Let $G = \mathrm{SO}_{2n+1}$, $X = \mathrm{GL}_{n_1} \mathrm{SO}_{2n_2+1}$, and $P = P_n$. Then $X \backslash G/P$ is finite.*

Proof. Let V be the natural module of G and fix a decomposition $V = V_1 \perp V_2$ so that $X = X_1 X_2$ with $X_1 = \mathrm{GL}_{n_1} \leq \mathrm{SO}(V_1)$ and $X_2 = \mathrm{SO}_{2n_2+1} = \mathrm{SO}(V_2)$. By Corollary 3.8 (ii) it suffices to show that X_1 has a finite number of orbits on the set of pairs of subspaces (W_1, W_2) such that $W_1 \leq W_2 \leq V_1$, W_1 is totally singular, $W_1 \leq W_2^\perp$ and $\dim(W_2/W_1) \leq 1$. We may partition this set into two subsets according as W_2 is, or is not, totally singular. Since X_1 is spherical in $\mathrm{SO}(V_1)$ we see that X_1 has finitely many orbits upon the subset where W_2 is totally singular.

Every pair (W_1, W_2) where W_2 is not totally singular can be rewritten as $(W_1, W_1 \perp \langle v \rangle)$ where v is a definite vector in V_1 . Thus it suffices to show that X_1 has a finite number of orbits upon pairs $(W_1, \langle v \rangle)$ such that v is a definite vector in V_1 and $W_1 \leq \langle v \rangle^\perp$ is totally singular (where this perpendicular space is taken in V_1). By Lemma 3.1, X_1 acts transitively upon definite 1-spaces, whence it suffices to fix v and show that the stabilizer in X_1 of $\langle v \rangle$ has a finite number of orbits on totally singular subspaces in $\langle v \rangle^\perp$. This follows from Lemma 3.10. \square

LEMMA 3.12. *Let $G = \mathrm{SO}_{2n}$, $X = \mathrm{GL}_{n_1} \mathrm{GL}_{n_2}$ and $P = P_1$. Then $X \backslash G/P$ is finite.*

Proof. Let V be the natural module of G and fix a decomposition $V = V_1 \perp V_2$ so that $X = X_1 X_2$ with $X_i = \mathrm{GL}_{n_i} \leq \mathrm{SO}(V_i)$ for each i .

Let $\pi_i : V \rightarrow V_i$ be the natural projection. We wish to show that X has a finite number of orbits on the set $\{\langle v \rangle \mid v \in V \text{ is singular}\}$. By Lemma 3.1 each X_i has finitely many orbits on 1-spaces in V_i . Thus, it suffices to fix an arbitrary singular 1-space $\langle v \rangle$, let $v_i = \pi_i v$ and show that X has a finite number of orbits on the set

$$\{\langle v_1 + \alpha v_2 \rangle \mid \alpha \in k, \alpha \neq 0, v_1 + \alpha v_2 \text{ is singular}\}, \quad (3.1)$$

where k is the ground field. Since $v_1 + \alpha v_2$ is singular, we have $\varphi(v_1) + \alpha^2 \varphi(v_2) = 0$, where φ is the quadratic form. Thus, v_1 is definite if and only if v_2 is. If v_1 and v_2 are both definite then $\alpha^2 = -\varphi(v_1)/\varphi(v_2)$, whence the set in (3.1) contains at most two singular 1-spaces of the form $\langle v_1 + \alpha v_2 \rangle$.

It suffices now to show that X has finitely many orbits on the 1-spaces in set (3.1) where v_1 and v_2 are singular. Let $V_2 = E_2 \oplus F_2$ where E_2 and F_2 are totally singular and X_2 stabilizes E_2 and F_2 . Then $v_2 = e + f$ for some $e \in E_2$, $f \in F_2$. Since v_2 is singular this implies that $\beta(e, f) = 0$, where β is the bilinear form. Easy linear algebra shows that there exists $x_2 \in X_2$ with $x_2 e = \alpha e$ and $x_2 f = \alpha f$, whence $x_2 \langle v_1 + v_2 \rangle = \langle v_1 + \alpha v_2 \rangle$. \square

Recall that $L_{i,2}$ denotes a Levi subgroup as described just before Theorem 1.

COROLLARY 3.13. *Let $G = D_4$ and $(X, P) \in \{(L_{2,3}, P_4), (L_{2,4}, P_3)\}$. Then $X \backslash G/P$ is finite.*

Proof. This follows most easily from applying the graph automorphism of order three to other cases which have been proven finite. For instance $(X, P) = (L_{2,3}, P_4)$ follows from Corollary 3.9 applied to $(X, P) = (L_{1,2}, P_3)$ (with $L_{1,2} = D_1 D_1 D_2$) or from Lemma 3.12 applied to $(X, P) = (L_{2,4}, P_1)$. \square

The reader who wishes for an instructive, though somewhat painful, exercise can prove this corollary using geometric arguments about subspaces of the natural module of D_4 .

4. Infiniteness

Throughout this section, L denotes a Levi factor of P . We begin by stating a result which gives infiniteness in many cases.

THEOREM 2 [4, Theorem 1.3]. *If $X \backslash G/P$ is finite then X or L is a spherical subgroup of G .*

To finish the proof of infiniteness in Theorem 1 it suffices, by Theorem 2, to consider only those P such that L is spherical. Suppose that we have fixed such a P . Then it suffices to prove infiniteness for those X which are maximal subject to the condition that $X \backslash G/P$ is claimed to be infinite in Theorem 1.

TABLE 3. *Infiniteness cases*

G	(X, P)
A_n	$\{A_{n_1}A_{n_2}A_{n_3}A_{n_4}T_3\} \times \{P_i \mid 2 \leq i \leq n-1\}$
B_n	$\{B_{n_1}D_{n_2}D_{n_3}, A_{n_1}A_{n_2}T_2\} \times \{P_1, P_n\}$
C_n	$\{A_{n_1}A_{n_2}C_{n_3}T_2\} \times \{P_1, P_n\}$
	$(C_{n_1}C_{n_2}C_{n_3}C_{n_4} \ (n_i \geq 1), P_n)$
	$(A_{n_1}C_{n_2}C_{n_3}T_1 \ (n_i \geq 1), P_n)$
D_n	$(D_{n_1}D_{n_2}D_{n_3}, P_1)$
	$\{D_{n_1}D_{n_2}D_{n_3}D_{n_4}\} \times \{P_{n-1}, P_n\}$
	$\{A_{n_1}D_{n_2}D_{n_3} \ (n_1 \geq 1)\} \times \{P_{n-1}, P_n\}$
	$\{A_{n_1}A_{n_2}T_2 \ (n_i \geq 1)\} \times \{P_{n-1}, P_n\}$
	with (G, X, P) not as in Theorem 1 (iv)(a)

In Table 3 we list those cases which need to be proven infinite. Recall that we allow the notation A_0, B_0, C_0 and D_1 , but we do not allow D_0 .

For the remainder of this section we assume that X and L contain a common maximal torus, T , which we may do by Lemma 2.1. For a closed subgroup H which contains T we write $\Phi(H)$ for the root system of H defined using T and we write $|\Phi(H)|$ for the number of roots.

THEOREM 3 [4, Theorem 1.1, Lemma 3.3]. *Let L_1 and L_2 be conjugate Levi subgroups, each containing T , with root systems Φ_1 and Φ_2 respectively. If $\frac{1}{2}|\Phi_1| - \text{rank}(\Phi_1) - |\Phi_1 \cap \Phi(X)| - \frac{1}{2}|\Phi_2 \cap \Phi(L)| > 0$ then $X \backslash G/P$ is infinite. In particular infiniteness holds in the following cases:*

- (i) Φ_1 and Φ_2 are of type B_2 , $\Phi_1 \cap \Phi(X) = \emptyset$ and $\Phi_2 \cap \Phi(L)$ is of type A_1 ;
- (ii) Φ_1 and Φ_2 are of type A_3 or D_3 , $\Phi_1 \cap \Phi(X) = \emptyset$ and $\Phi_2 \cap \Phi(L)$ is of type A_1A_1 or D_2 .

We offer comments which help simplify the proof of Lemma 4.1.

All Levi subgroups of type B_2 are conjugate and, unless $G = D_n$, all Levi subgroups of type A_3 are conjugate. Thus, to apply Theorem 3, one often only has to verify that $\Phi_1, \Phi_1 \cap \Phi(X), \Phi_2$ and $\Phi_2 \cap \Phi(L)$ are of the required type. We will construct each Φ_i by giving a base α, β, \dots and setting Φ_i equal to all the \mathbb{Z} -linear combinations of α, β, \dots which are in $\Phi(G)$.

Let $\Delta(G)$ and $\tilde{\Delta}(G)$ be the Dynkin diagram and extended Dynkin diagram of G , respectively. Label the nodes of $\Delta(G)$ using $\alpha_1, \dots, \alpha_n$ as in [2]. In the following lemma we will assume that $\Delta(X)$ has been produced from $\Delta(G)$ by the Borel-de Siebenthal algorithm [1]. Let $X = X_{n_1}X_{n_2} \dots$ with each X_{n_i} equal to D_1, T_1 , or a simple factor of X , as listed in Table 3, with n_i the rank of the factor. We take $\Delta(X_{n_1})$ equal to \emptyset if X_{n_1} equals T_1 or D_1 and otherwise we take $\Delta(X_{n_1})$ equal to the first n_1 nodes of $\Delta(G)$ or $\tilde{\Delta}(G)$ as appropriate. We then repeat this procedure, starting with $\Delta(X_{n_2})$ and the last $n_2 + n_3 + \dots$ nodes of $\Delta(G)$. This procedure determines $\Delta(G) - \Delta(X)$ which, in turn, provides an easy description of $\Phi(X)$. For example, suppose that $G = D_n$ and $X = D_{n_1}D_{n_2}$. Then $\Delta(G) - \Delta(X) = \{\alpha_{n_1}\}$ and $\Phi(X)$ equals all the roots in $\Phi(G)$ which have α_{n_1} -coefficient equal to 0 or ± 2 .

Let $\alpha, \beta \in \Delta(G)$. By the path connecting α to β we mean the unique path in $\Delta(G)$ of minimal length that includes α and β . The sum over this path means the

sum of each element of $\Delta(G)$ which is contained in the path. It is easy to check that such a sum is itself a root.

LEMMA 4.1. *With the notation as in Theorem 3 there exist Φ_1 and Φ_2 as indicated in the following cases.*

- (i) *If $G = A_n$, $X = A_{n_1}A_{n_2}A_{n_3}A_{n_4}T_3$, $P \in \{P_i \mid 2 \leq i \leq n-1\}$, then there exist Φ_1 and Φ_2 of type A_3 as in Theorem 3 (ii).*
- (ii) *If $G = B_n$, $(X, P) \in \{B_{n_1}D_{n_2}D_{n_3}, A_{n_1}A_{n_2}T_2\} \times \{P_1, P_n\}$, then there exist Φ_1 and Φ_2 of type B_2 as in Theorem 3 (i).*
- (iii) *Let $G = C_n$:*
 - (a) *if $(X, P) \in \{A_{n_1}A_{n_2}C_{n_3}T_2\} \times \{P_1, P_n\}$, then there exists $\Phi_1 = \Phi_2$ of type B_2 as in Theorem 3 (i);*
 - (b) *if $X \in \{C_{n_1}C_{n_2}C_{n_3}C_{n_4} \ (n_i \geq 1), A_{n_1}C_{n_2}C_{n_3}T_1 \ (n_i \geq 1)\}$ and $P = P_n$, then there exist Φ_1 and Φ_2 of type A_3 as in Theorem 3 (ii).*
- (iv) *Let $G = D_n$:*
 - (a) *if $X \in \{A_{n_1}D_{n_2}D_{n_3}T_1 \ (n_1 \geq 1), D_{n_1}D_{n_2}D_{n_3}D_{n_4}\}$ and $P \in \{P_{n-1}, P_n\}$, or $(X, P) = (D_{n_1}D_{n_2}D_{n_3}, P_1)$, then there exists $\Phi_1 = \Phi_2$ of type A_3 or D_3 as in Theorem 3 (ii);*
 - (b) *if $X = A_{n_1}A_{n_2}T_2 \ (n_i \geq 1)$, $P \in \{P_{n-1}, P_n\}$, and (X, P) is not one of $(L_{2,3}, P_4)$ or $(L_{2,4}, P_3)$, then there exist Φ_1 and Φ_2 of type A_3 or D_3 as in Theorem 3 (ii).*

Proof. We prove parts (i) and (iv)(b), the other parts being similar and relatively easy.

Part (i). We have $\Delta(G) - \Delta(X) = \{\alpha_{n_1+1}, \alpha_{n_1+n_2+2}, \alpha_{n_1+n_2+n_3+3}\}$. Let Φ_1 have root base α, β, γ with α equal to α_{n_1+1} , β equal to the sum over the path connecting α_{n_1+2} to $\alpha_{n_1+n_2+2}$, and γ equal to the sum over the path connecting $\alpha_{n_1+n_2+3}$ to $\alpha_{n_1+n_2+n_3+3}$. Let Φ_2 have root base given by $\alpha = \alpha_{i-1}$, $\beta = \alpha_i$, and $\gamma = \alpha_{i+1}$.

Part (iv)(b). By our convention with subsystems of type A_{n_i} in D_n , we have that $X \in \{L_{i,n-1}, L_{i,n}\}$ where $i = n_1 + 1$ satisfies $2 \leq i \leq n-2$.

Let $(X, P) = (L_{i,n}, P_n)$. Let $\Phi_1 = \Phi_2$ have root base given by α, β, γ with α equal to the sum over the path connecting α_1 to α_{n-1} , β equal to α_n , and γ equal to the sum over the path connecting α_2 to α_{n-2} . By symmetry the conclusion also holds when $(X, P) = (L_{i,n-1}, P_{n-1})$.

We assume now that $(X, P) \in \{(L_{i,n-1}, P_n), (L_{i,n}, P_{n-1})\}$. If n is odd (in particular if $n = 5$) then $L_{i,n-1}$ is conjugate to $L_{i,n}$, whence the conclusion holds. We assume now that $n \geq 6$. If necessary we replace X by a conjugate to assume that $i \leq n-3$. Let $\Phi_1 = \Phi_2$ have root base given by α, β, γ with α equal to the sum over the path connecting α_i to α_{n-3} , $\beta = \alpha_{n-2} + \alpha_{n-1} + \alpha_n$, and γ equal to the sum over the path connecting α_{i-1} to α_{n-2} . \square

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