Midterm Preview
MA 490: Partial Differential Equations
Loyola College, Spring 2009, W. Ethan Duckworth

Practice each of the following problems, and be prepared for variations. It is possible that the real midterm will have to be shorter.

1. Verify that the following function $u(x,t)$ is a solution of the indicated PDE, for any constants A, B, C, D, $\lambda > 0$

$$\begin{aligned} u_{tt} + ku_{xxxx} &= 0,\ k > 0 \\ u(x,t) &= (A\cos(\lambda x) + B\sin(\lambda x) + Ce^{\lambda x} + De^{-\lambda x})g(t) \\ g(t) &= \text{any solution of the ODE } g''(t) = -k\lambda^4 g(t) \end{aligned}$$

2. (a) Recall that the ODE $2x^2\, y''(x) - 5x\, y'(x) + 3y(x) = 0$ can be solved by making a substitution of the form $y(x) = x^r$, and solving for r. In general you get two values, r_1 and r_2, and the general solution is $Ax^{r_1} + Bx^{r_2}$. Find the general solution of this ODE

   (b) Find the general solution $u(x,t)$ of the following PDE

$$2t^2 u_{tt} - 5tu_t + 3u = 0.$$

3. Solve the following PDE

$$tu_t - u = t^2$$

4. Assume $u(x,t) = f(x)g(t)$, and apply separation of variables to find product solutions of the PDE $u_{xx} + u_{tt} = 0$.

5. State the solution of the following PDE system (you do not need to derive the solution)

$$\begin{aligned} u_t &= 13u_{xx},\ 0 \le x \le 7 \\ u_x(0,t) &= 5,\ u(7,t) = 11 \\ u(x,0) &= 5x - 24 - 32.71\cos\left(\frac{3\pi x}{14}\right) + 61.88\cos\left(\frac{11\pi x}{14}\right) - 0.21\cos\left(\frac{123\pi x}{14}\right) \end{aligned}$$

6. Show that any nonconstant exponential and linear product solutions (in the book's terminology these are case 2 and case 3 solutions respectively) do not satisfy the BC in the following system.

$$\begin{aligned} u_t &= 6u_{xx}, 0 \le x \le L \\ u_x(0,t) &= 0,\ u_x(L,t) = 0 \end{aligned}$$

7. Let $f(x)$ be defined on the interval $[-\pi, \pi]$ as

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x \le 0, \\ \sin(x) & \text{if } 0 \le x \le \pi. \end{cases}$$

   Find the Fourier coefficient a_2, working on the interval $[-\pi, \pi]$.

In addition, for extra practice in 4.1, you may want to do 4.17 #4, 5(a).

1. **Solution:** (Note, my solution is perhaps briefer than some of yours: you should not try to be briefer than is natural.)

   Let $f(x) = A\cos(\lambda x) + B\sin(\lambda x) + Ce^{\lambda x} + De^{-\lambda x}$.

   We have $u_{tt} = g''(t)f(x)$. Since $g(t)$ is a solution of the indicated ODE, we have $g''(t)f(x) = -k\lambda^4 g(t)f(x)$.

   On the other hand, $u_{xxxx} = g(t)f^{(4)}(x) = g(t)(A\lambda^4\cos(\lambda x) + B\lambda^4\sin(\lambda x) + C\lambda^4e^{\lambda x} + D\lambda^4e^{-\lambda x}) = g(t)\lambda^4 f(x)$.

   So $u_{tt} + ku_{xxxx} = -k\lambda^4 g(t)f(x) + k\lambda^4 g(t)f(x) = 0$.

2. **Solution:**

   (a) Plugging $y = x^r$ into the ODE gives

   $$\begin{aligned} 2x^2r(r-1)x^{r-2} - 5xrx^{r-1} + 3x^r &= 0\\ x^r(2r(r-1) - 5r + 3) &= 0\\ 2r^2 - 7r + 3 &= 0\\ r &= 3, 1/2 \end{aligned}$$

   Therefore

   $$y(x) = Ax^3 + Bx^{1/2}.$$

   (b) Since the PDE only involves the derivative with respect to t, we treat it the same way as the previous ODE, using t instead of x. At the end, we replace the unknown constants with unknown functions of x.

   $$u(x,t) = F_1(x)t^3 + F_2(x)t^{1/2}.$$

3. **Solution:** Divide by t

   $$u_t - \frac{1}{t}u = t$$

   use an integrating factor

   $$m(t) = e^{-\int 1/t\,dt} = e^{-\ln|t|} = \frac{1}{t}.$$

   Multiply by $m(t)$,

   $$\begin{aligned} \frac{1}{t}u_t - \frac{1}{t^2}u &= 1\\ \frac{\partial}{\partial t}\frac{1}{t}u &= 1\\ \frac{1}{t}u &= \int 1\,dt + C(x) = t + C(x)\\ u &= t(t + C(x)) \end{aligned}$$

4. **Solution:** We have $u_{xx} = f''(x)g(t)$ and $u_{tt} = f(x)g''(t)$, so the PDE becomes

   $$\begin{aligned} f''(x)g(t) + f(x)g''(t) &= 0\\ \frac{f''(x)}{f(x)} &= -\frac{g''(t)}{g(t)} = r\\ f''(x) = rf(x) &\text{ and } g''(t) = -rg(t) \end{aligned}$$

   If $r > 0$ then

   $$\begin{aligned} f(x) &= Ae^{\sqrt{r}x} + Be^{-\sqrt{r}x}\\ g(t) &= C\sin(\sqrt{r}t) + D\cos(\sqrt{r}t)\\ u(x,t) &= (Ae^{\sqrt{r}x} + Be^{-\sqrt{r}x})(C\sin(\sqrt{r}t) + D\cos(\sqrt{r}t)) \end{aligned}$$

   If $r < 0$ then

   $$\begin{aligned} f(x) &= A\sin(\sqrt{-r}x) + D\cos(\sqrt{-r}x)\\ g(t) &= Ce^{\sqrt{-r}t} + De^{-\sqrt{-r}t}\\ u(x,t) &= (A\sin(\sqrt{-r}x) + D\cos(\sqrt{-r}x))(Ce^{\sqrt{-r}t} + De^{-\sqrt{-r}t}) \end{aligned}$$

   If $r = 0$ then

   $$\begin{aligned} f(x) &= Ax + B\\ g(t) &= Ct + D\\ u(x,t) &= (Ax + B)(Ct + D) \end{aligned}$$

5. **Solution:** The general form of solution for this PDE and BC is

   $$\begin{aligned} u(x,t) &= v(x,t) + u_p(x,t)\\ v(x,t) &= \sum_{n=1}^{N} B_ne^{-\lambda^2kt}\cos(\lambda x),\ \lambda = \frac{(2n+1)\pi}{2L}\\ u_p(x,t) &= ax + (b - aL). \end{aligned}$$

   So we have

   $$\begin{aligned} u(x,t) = 5x - 24 - 32.71e^{-(3\pi/14)^2 13t}\cos\left(\frac{3\pi x}{14}\right)\\ + 61.88e^{-(11\pi/14)^2 13t}\cos\left(\frac{11\pi x}{14}\right) - 0.21e^{-(123\pi/14)^2 13t}\cos\left(\frac{123\pi x}{14}\right). \end{aligned}$$

6. **Solution:** Let $u(x,t) = e^{\lambda^2kt}(Ae^{\lambda x} + Be^{-\lambda x})$. Then $u_x(x,t) = e^{\lambda^2kt}(\lambda Ae^{\lambda x} - \lambda Be^{-\lambda x})$. Then the BC's become

   $$\begin{aligned} u_x(0,t) &= 0 & u_x(L,t) &= 0\\ e^{\lambda^2kt}(\lambda A\cdot 1 - \lambda B\cdot 1) &= 0 & e^{\lambda^2kt}(\lambda Ae^{\lambda L} - \lambda Be^{-\lambda L}) &= 0 \end{aligned}$$

   Factoring out e^{λ^2kt} and λ we get

   $$\begin{aligned} A - B &= 0\\ Ae^{\lambda L} - Be^{-\lambda L} &= 0 \end{aligned}$$

   From the first line we get $A = B$. Plugging this into the second line we get

   $$Be^{\lambda L} - Be^{-\lambda L} = 0 \Rightarrow B(e^{\lambda L} - e^{-\lambda L}) = 0$$

   This implies either $B = 0$ or $e^{\lambda L} - e^{-\lambda L} = 0$. If the second expression is zero then we have $e^{\lambda L} = e^{-\lambda L}$, and so $\lambda L = -\lambda L$: this is impossible unless $\lambda = 0$, which is not allowed for the exponential product solution, or $L = 0$, which is not allowed for the original problem. Therefore $B = 0$. This implies $A = 0$ and so we have only the trivial, constant solution.

   Let $u(x,t) = Ax + B$. Then $u_x(x,t) = A$. Then the initial conditions imply that $A = 0$, and so u is a constant.

7. **Solution:** We have

   $$\begin{aligned} a_2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{2\pi x}{\pi}\right)dx\\ &= \frac{1}{\pi}\int_0^{\pi}\sin(x)\cos(2x)\,dx\\ &= \frac{1}{2}\left(\frac{\cos(3x)}{3} + \cos(x)\right)\Big|_0^{\pi}\\ &= -\frac{2}{3} \end{aligned}$$