Homework 1

Due Wednesday, January 21.

1. Look up http://en.wikipedia.org/wiki/List_of_non-linear_partial_differential_equations and find your favorite equation and/or the equation that would be worth the most amount of money (if you could solve it, or if you could use it with omniscience to rule the world). Tell me in a few sentences what the equation is, what it describes, and why you like it/think it's worth so much money.

2. Do 1.1#1 from the book (WARNING: there are 9 problems here).

   Find the general solutions of the following equations (all but one can, and should, be found explicitly).

   (a) $\frac{dy}{dx} = xy$

   (b) $\frac{dx}{dt} = x(1-x)$

   (c) $\frac{dy}{dx} = x^2y^2 + x^2 - y^2 - 1$

   (d) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

   (e) $\frac{dx}{dt} + x^2\sin(t) = 0$

   (f) $\frac{dy}{dx} - \frac{x+e^{-x}}{y+e^y} = 0$

   (g) $\frac{dx}{dt} = te^{x+t}$

   (h) $x\frac{dy}{dx} = 1 + y^2$

   (i) $T'(t) + 3T(t) = 0$

3. Do 1.1#4 from the book (WARNING: there are 9 problems here). Solve the following equations, including the initial conditions (all can and should be found explicitly).

   (a) $y'(x) + 2y(x) = e^x$, $y(0) = 1$.

   (b) $x'(t) - \frac{2}{t}x(t) = 1$, $x(1) = 0$.

   (c) $\sin(x)y'(x) - \cos(x)y(x) = \sin(2x)$, $y(\frac{\pi}{2}) = 0$.

   (d) $x'(t) + \frac{x(t)}{t} = t^2$, $x(0) = 0$.

   (e) $3\frac{dy}{dx} + 6xy = 6e^{-x^2}$, $y(0) = 1$.

   (f) $\frac{dy}{dx} = 3y + e^{2x}$, $y(0) = 0$.

   (g) $x'(t) + x(t)\cos(t) = 0$, $x(\pi) = 100$.

   (h) $\frac{dy}{dx} + (1 + 2x + 3x^2)y = e^{-x-x^2-x^3}$, $y(0) = 3$.

   (i) $\frac{dx}{dt} + \frac{3x}{2t+100} = 0$, $x(-49.5) = 1$

Homework 2

Due Wednesday, January 28.

1. 1.1#8

   (a) Show that if $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous linear ODE $a(x)y''+b(x)y'+c(x)y = 0$, then the superposition [i.e. linear combination] $c_1y_1(x) + c_2y_2(x)$ is also a solution [where c_1 and c_2 are real numbers].

   (b) If $a(x)$, $b(x)$ and $c(x)$ are continuous with $a(x)$ never zero, then the ODE in part (a) has a unique solution $y(x)$ with given values for $y(x_0)$ and $y'(x_0)$ (cf. [Simmons, Section 57]). Assuming this, show that no solution of this ODE can have a graph which is tangent to the x-axis at some point, unless the solution is identically zero. [Hint: show that the constant zero function is a solution of the differential equation. Then suppose $y(x)$ is a solution that is tangent to the x-axis at some point. Use uniqueness to show that $y(x)$ is actually the constant zero function.]

2. 1.1#17. Solve each of the following systems subject to the given initial data:

   (a) $x'(t) = 3x(t) - 4y(t), \quad x(0) = 1,$
   $y'(t) = x(t) - y(t), \quad y(0) = 1.$

   (b) $x'(t) = x(t) - 4y(t), \quad x(0) = 1,$
   $y'(t) = x(t) + y(t), \quad y(0) = 1.$

   (c) $x'(t) = x(t) + 2y(t), \quad x(0) = 0,$
   $y'(t) = 3x(t) + 4y(t), \quad y(0) = 1.$

3. 1.2#1. Show that the given functions satisfy the accompanying PDE.

   (a) $u(x, y) = x + y, \quad u_{xx} + u_{yy} = 0.$
   (b) $u(x, y) = f(x) + g(y), \quad u_{xy} = 0$, where the functions f and g are assumed to be in C^2.
   (c) $u(x, y) = f(x + y) + g(x - y), \quad u_{xx} - u_{yy} = 0$, where $f, g \in C^2$.
   (d) $u(x, t) = x^2 + 2t, \quad u_{xx} = u_t$.
   (e) $u(x, y) = \sin(x)\cosh(y), \quad u_{xx} + u_{yy} = 0.$
   (f) $u(x, t) = \sin(x - ct), \quad u_{tt} - c^2u_{xx} = 0$, where c is a real constant.

4. 1.2#9(a),(b). Let $u(x, y, z, t)$ be the solution (11) in Example 3 on wave problems.

   (a) Show that $u_{tt} = -[2\pi\nu(m, n, p)]^2u$, $u_{xx} = -(m\pi/A)^2u$, etc.. Use the facts to deduce that $u(x, y, z, t)$ satisfies the wave equation (9).
   (b) Verify that $u(x, y, z, t)$ meets the B.C. (10).

   Wave equation (9)
   $$u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz})$$

   Solution (11)
   $$u(x, y, z, t) = \sin(2\pi\nu(m, n, p)t)\cos\left(\tfrac{m\pi x}{A}\right)\cos\left(\tfrac{n\pi y}{B}\right)\cos\left(\tfrac{p\pi z}{C}\right)$$
   where this measures air pressure in a box of size $0 \le x \le A$, $0 \le y \le B$, $0 \le z \le C$, and m, n, p are any three integers and
   $$\nu(m, n, p) = \frac{1}{2}a\sqrt{\frac{m^2}{A^2} + \frac{n^2}{B^2} + \frac{p^2}{C^2}}$$

   The B.C. (10)
   $$u_x(0, y, z, t) = 0 \qquad u_y(x, 0, z, t) = 0 \qquad u_z(x, y, 0, t) = 0$$
   $$u_x(A, y, z, t) = 0 \qquad u_y(x, B, z, t) = 0 \qquad u_z(x, y, C, t) = 0$$

Homework 3

Due Friday, February 6.

1. 1.2#5. Give the orders of the following PDEs, and classify them as linear or nonlinear. If the PDE is linear, specify whether it is homogeneous or inhomogeneous.

   (a) $x^2 u_{xxy} + y^2 u_{yy} - \log(1+y^2)u = 0$
   (b) $u_x + u^3 = 1$
   (c) $u_{xxyy} + e^x u_x = y$
   (d) $uu_{xx} + u_{yy} - u = 0$
   (e) $u_{xx} + u_t = 3u$

2. 1.2#7(a). Find a solution of Laplace's equation $u_{xx} + u_{yy} = 0$ of the form $u(x, y) = Ax^2 + Bxy + Cy^2$ $(A^2 + B^2 + C^2 \neq 0)$ which satisfies the boundary condition $u(\cos(\theta), \sin(\theta)) = \cos(2\theta) + \sin(2\theta)$ for all points $(\cos(\theta), \sin(\theta))$ on the circle, $x^2 + y^2 = 1$.

3. 1.3#1. Find the general solution of each of the following PDEs by means of direct integration.

   (a) $u_x = 3x^2 + y^2$, $u = u(x, y)$.
   (b) $u_{xy} = x^2 y$, $u = u(x, y)$.
   (c) $u_{xyz} = 0$, $u = u(x, y, z)$.
   (d) $u_{xtt} = \exp[2x + 3t]$, $u = u(x, t)$.

4. 1.3#2. Find general solutions of the following PDEs for $u = u(x, y)$ by using ODE techniques.

   (a) $u_x - 2u = 0$
   (b) $yu_y + u = x$
   (c) $u_x + 2xu = 4xy$
   (d) $yu_{xy} + 2u_x = x$ (**Hint.** First integrate with respect to x.)
   (e) $u_{yy} - x^2 u = 0$.

5. 1.3#4. Find a nontrivial family of solutions of the PDEs by the method of separation of variables. You need not find the most general solution obtainable in this way.

   (a) $u_t = 2u_{xx}$, $u = u(x, t)$
   (b) $u_x = 4u_y$, $u = u(x, y)$
   (c) $u_{tt} = 16u_{xx}$, $u = u(x, t)$
   (d) $u_t = u_{xx} + u_{yy}$, $u = u(x, y, t)$
   (e) $u_{xx} + u_{yy} + u_{zz} = 0$, $u = u(x, y, z)$

Homework 4

Due Friday, February 13.

1. 1.3#6. In Section 1.2 we have used trial solutions of the form e^{rx} to find particular solutions of certain ODEs. The higher dimensional analogue of this substitution (as, for example, $u(x, y) = \exp(rx + sy)$, where r and s are constants) is called the **exponential substitution**. Use the exponential substitution to find a nontrivial family of solutions of each of the following PDEs.

   (a) $2u_x + 3u_y - 2u = 0, \quad u = u(x, y)$.

   (b) $4u_{xx} - 4u_{xy} + u_{yy} = 0, \quad u = u(x, y)$.

   (c) $u_{xyz} - u = 0, \quad u = u(x, y, z)$.

   (d) $4u_{xx} + u_{yy} = 14 \exp(2x + y), \quad u = u(x, y)$.

   (e) $u_{xx} + u_{yy} = u, \quad u = u(x, y)$.

2. 2.1#1. Find the general solution of each of the following PDEs, where $u = u(x, y)$ in (a) – (d).

   (a) $2u_x - 3u_y = x$.

   (b) $u_x + u_y - u = 0$.

   (c) $u_x + 2u_y - 4u = e^{x+y}$.

   (d) $3u_x - 4u_y = x + e^x$.

   (e) $v_z + 3v_w = 9w^2, \quad v = v(w, z)$.

   (f) $g_t - cg_x = 0, \quad g = g(x, t)$ (c constant).

3. 2.1#2. Find the particular solution of $u_x + 2u_y - 4u = e^{x+y}$ satisfying the following side conditions.
   (a) $u(x, 0) = \sin(x^2)$, (b) $u(0, y) = y^2$, (c) $u(x, -x) = x$.

4. 2.1#12. In Example 8 [the avocado example], now assume that an avocado has a 10% chance of being removed from the shelf on any given day (i.e., more precisely $D(y) = 0.1$, for $y > 0$), regardless of its age.

   (a) Show that in the long run, there will be about $Ce^{-y/10}$ y-day-old avocados on the shelf.
   [In other words, show that $P_\infty(y) = Ce^{-y/10}$.]

   (b) According to part (a), what should the value of C be, if there are still to be about 300 avocados on the shelf in the long run. Does your answer agree with common sense?

5. 2.1#13. Air conditioners are produced at a constant rate of 100 per month beginning on New Years Day. The probability that an air conditioner will break down during a small time interval Δt months, assuming that t months have elapsed since New Years, is $\left(0.2 - 0.1 \cdot \cos\left(\frac{\pi t}{6}\right)\right) \Delta t$, regardless of its age. Approximately how many y-month-old air conditioners will be operational at the end of the year, where $y \leq 12$? How could the total number of operational air conditioners, at the end of the year, be determined? If you have the resources for numerical integration, compute this number. [According to me, you do have the resources for numerical integration. You can use Maple, Mathematica, Matlab, or your TI-84. If you have a Macintosh you can use the built in Grapher application.]

   Hint. When using formula (25) [the formula for $P(a, t)$], remember that $D(y, t) = 0$ for $y < 0$.

Comments

Holy Cow! This was a hard homework, with lots of tricks! Here are some comments to help you out.

1.3#6. The answers here have an r or an s in them (or, in (c) both r and s). The expressions are things like $\frac{1}{2}rx - \frac{3}{2}ry$, for any/all values of r. Sometimes you can write it in more than one way. For instance, if you replace r here with $2r$ (or, if you like, change variables to $r = 2t$ and use t), then you get $rx - 3ry$.

2.1#1. Again, there is more than one way to write the answers here. For instance, in (a) I got $u(x, y) = \frac{1}{9}(-3xy - y^2) + f(3x + 2y)$. But, `<kidding>` as I'm sure everyone noticed `</kidding>`[1], we have $(3x + 2y)^2 = 9x^2 + 12xy + 4y^2$, so if we rewrite $-3xy - y^2$, maybe we can get rid of some of the terms into $(3x + 2y)^2$ and merge this into $f(3x + 2y)$. Here's how to do it

$$\begin{aligned}\frac{1}{9}(-3xy - y^2) = -\frac{1}{9}(3xy + y^2) = -\frac{1}{9}\cdot\frac{1}{4}(12xy + 4y^2) &= -\frac{1}{9}\cdot\frac{1}{4}(-9x^2 + 9x^2 + 12xy + 4y^2)\\ &= -\frac{1}{9}\cdot\frac{1}{4}(-9x^2) - \frac{1}{9}\cdot\frac{1}{4}(3x + 2y)^2 = \frac{1}{4}x^2 - \frac{1}{36}(3x + 2y)^2\end{aligned}$$

Thus, we have

$$u(x, y) = \frac{1}{4}x^2 - \frac{1}{36}(3x + 2y)^2 + f(3x + 2y) = \frac{1}{4}x^2 + f_2(3x + 2y)$$

where we absorbed $-\frac{1}{36}(3x + 2x)^2$ into $f(3x + 2y)$ and called it a new function, $f_2(3x + 2y)$.

In 2.1#1(e) and 2.1#1(f) the book seems to go out if its way to trick you: they've switched the variables, and they've switched the *order* they write them in. The most obvious way to use similar substitutions as before is to write them as

$$\begin{aligned}3v_w + v_z &= 9w^2\\ -cg_x + g_t &= 0\end{aligned}$$

For the first one you can let $x = w - 3z$ and $y = z$ and rewrite this as $u(x, y)$ with u_x etc. For the second you can let $w = x + ct$ and $z = t$ and rewrite this with $v(w, z)$ with v_z etc.

For 2.1#12 and 2.1#13 the book uses y for age, whereas I always used a for age. For 2.1#12(b) you should use an integral to find the total number of avocados, and set this equal to 300 to solve for C.

For 2.1#13 there are two parts. The first part is essentially asking you to find a formula for $P(a, 12)$ where $a < t$. The second part is asking you to set up an integral for the total population from $a = 0$ to $a = 12$, and then calculate this integral. You do not really need a computer to do this integral.

[1] These are html tags included to make it clear where I was being facetious[2].

[2] "Facetious": joking, by saying or writing something that is meant to be clearly false[3].

[3] Ok, forget it, there can be no humor with footnotes!