Final Exam Preview
Partial Differential Equations
Spring, 2009

Here's an outline of the methods of solving PDE's we have learned.

1. Reduce to ODE. For us this means reducing to one of the following.
   (a) Separable ODE: separate variables and integrate.
   (b) First order linear equation: use integrating factor.
   (c) 2cnd order linear homogeneous with constant coefficients: use the characteristic equation.
2. Direct integration.
3. Separation of variables.
4. First order linear PDE's: sometimes involves a change of variables to rewrite the equation as one that reduces to an ODE.
5. Heat equation: BC's determine functions of x, either sine or cosine, as well as the type of frequencies. Functions of t are exponential. Initial conditions must be described by a Fourier series (for homogeneous BC's), as well as possibly a few other terms (for inhomogeneous BC's).

   The different cases are all dependent on BC's. We have five homogeneous BC's

$$\begin{array}{c} -L \le x \le L \\ u(-L,t) = u(L,t) \\ u_x(-L,t) = u_x(L,t) \end{array} \qquad \begin{array}{c} 0 \le x \le L \\ u(0,t) = u(L,t) = 0 \end{array} \qquad \begin{array}{c} 0 \le x \le L \\ u_x(0,t) = u(L,t) = 0 \end{array}$$

$$\begin{array}{c} 0 \le x \le L \\ u(0,t) = u_x(L,t) = 0 \end{array} \qquad \begin{array}{c} 0 \le x \le L \\ u_x(0,t) = u_x(L,t) = 0 \end{array}$$

6. Wave equation: BC's determine functions of x, either sine or cosine, as well as the type of frequencies. Functions of t are sine and cosine. Initial conditions must be described by a Fourier series (we only learned about homogeneous BC's).

   The different cases are all dependent on BC's. We have four homogeneous BC's.

$$\begin{array}{c} 0 \le x \le L \\ u(0,t) = u(L,t) = 0 \end{array} \qquad \begin{array}{c} 0 \le x \le L \\ u_x(0,t) = u(L,t) = 0 \end{array} \qquad \begin{array}{c} 0 \le x \le L \\ u(0,t) = u_x(L,t) = 0 \end{array}$$

$$\begin{array}{c} 0 \le x \le L \\ u_x(0,t) = u_x(L,t) = 0 \end{array}$$

The main new mathematical technique we learned was Fourier analysis. In general, given a function $f(x)$, and some sort of basis functions $v_1(x)$, $v_2(x)$, $\ldots$, we try to write $f(x)$ as a linear combination of the $v_i(x)$ as follows

$$f(x) = \sum_{n=1}^{\infty} c_n v_n(x),$$

where the coefficients c_n are given by the inner-product/integral

$$c_n = \# \int_{\#}^{\#} f(x) v_n(x)\, dx$$

Here I am being purposefully vague about the constant numbers in front of the integral and for the endpoints of the integral, since these vary case-by-case.

In the cases we have studied, the functions $v_n(x)$ are always sine or cosine functions. Sometimes we use both, sometimes just one or other. I have been using the word "frequencies" to describe λ where the functions are $\sin(\lambda x)$ and $\cos(\lambda x)$. We have seen different cases that give rise to two different possibilities for λ: $\dfrac{n\pi}{L}$ and $\dfrac{(2n+1)\pi}{2L}$.

As for the midterm, I think you'll want some basic integrals on your note card: $\int e^{ax}\sin(bx)\,dx$, $\int e^{ax}\cos(bx)\,dx$, $\int \sin(ax)\cos(bx)\,dx$, $\int \sin(ax)\sin(bx)\,dx$, $\int \cos(ax)\cos(bx)\,dx$, $\int (ax^2+bx+c)\sin(dx)\,dx$, $\int (ax^2+bx+c)\cos(dx)\,dx$.

1. Retake the quiz.
2. Retake the midterm.
3. Redo half of your homework problems.
4. State the formulas for a_n, b_n, or c_n, as appropriate in each of the following cases

   (a) $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$, $-L \le x \le L$.

   (b) $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$, $0 \le x \le L$.

   (c) $f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$, $0 \le x \le L$

   (d) $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n \sin\left(\frac{(2n+1)\pi x}{2L}\right)$, $0 \le x \le L$

   (e) $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n \cos\left(\frac{(2n+1)\pi x}{2L}\right)$, $0 \le x \le L$.

   (By the way, I found out that the series that appear in the last two parts are called "quarter wave" series. Now that we have a name for these, I could say things like "Find the quarter wave sine series for $f(x)$.")

5. Define $f(x)$ on the interval $[-1,1]$, and $g(x)$ on the interval $[0,1]$ as shown,

$$f(x) = \begin{cases} 1 & \text{if } x = -1 \\ (x-1)^2+1 & \text{if} -1 < x < 0 \\ 1 & \text{if } x = 0 \\ (x-1)^2+1 & \text{if } 0 < x < 1 \\ 2 & \text{if } x = 1 \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } x = 0 \\ (x-1)^2+1 & \text{if } 0 < x < 1 \\ 2 & \text{if } x = 1 \end{cases}$$

In each of the following parts, you will sketch a Fourier series $F(x)$ of the indicated type.

(a) $F(x)$ is the Fourier sine and cosine series of $f(x)$ calculated using $[-1,1]$.

(b) $F(x)$ is the Fourier sine series of $g(x)$ calculated using $[0,1]$.

(c) $F(x)$ is the Fourier cosine series of $g(x)$ calculated using $[0,1]$.

(d) $F(x)$ is the Fourier quarter wave sine series of $g(x)$ calculated using $[0,1]$.

(e) $F(x)$ is the Fourier quarter wave cosine series of $g(x)$ calculated using $[0,1]$.

6. (a) State the BC's for heat flow in a circular ring.

(b) Which of $\sin(\lambda x)$ and $\cos(\lambda x)$ satisfy the BC's? Which values of λ satisfy the BC's?

(c) Suppose the initial heat of a circular ring is given by

$$f(x) = \begin{cases} \frac{1}{L}x + 1 & \text{if } -L \le x \le 0 \\ -\frac{1}{L}x + 1 & \text{if } 0 \le x \le L \end{cases}$$

Find a four term solution of the heat equation.

7. State the formal solution of the following wave equation, defining every letter in an explicit form (this can either be solved for, where appropriate, or in a form suitable for entry into a computer algebra system).

$$u_{tt} = 15u_{xx},\ 0 \le x \le 11$$
$$u(0,t) = 0,\ u(11,t) = 0$$
$$u(x,0) = x^2,\ u_t(x,0) = e^{-x}$$

8. Find and simplify the coefficients of a Fourier Sine Series for the function $y = 3x$ on the interval $[0,2]$.

9. Use separation of variables to solve $u_{tt} + u_{xx} = 0$ (you may assume that the separation constant is a negative number, $-\lambda^2$).

10. Find the general solution $u(x,t)$ of the following PDE

$$xu_x = -3u + \frac{\sin(x)}{x^2}$$